International Mathematical Forum, Vol. 18, 2023, no. 2, 69 - 73
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2023.912383

A Generalization to Varieties of a Result About Curves

Rita Vincenti

Departement of Mathematics and Computer Science
University of Perugia
Via Vanvitelli 1, 06123, Perugia, Italy

This article is distributed under the Creative Commons by-nc-nd Attribution License. Copyright © 2023 Hikari Ltd.

Abstract

In [1] the author proves that by starting from two projectively equivalent curves in two independent spaces, a 2-dimensional ruled variety can be generated by the lines joining corresponding points of the two curves, the order of the variety being the sum of the orders of them. In this note we prove that result can be extended to any pair of projectively equivalent irreducible varieties of same dimension lying in two complementary spaces.

Mathematics Subject Classification: 51E20, 51A05

Keywords: Finite projective geometry, varieties

1 Introduction

In [5] we introduced varieties arising as partial ruled sets by connecting corresponding points of two projectively equivalent (or, in birational correspondence) varieties lying in two independent spaces. We fix two complementary subspaces $\mathcal{S}$, $\mathcal{S}'$ of $PG(r,q)$ and we choose two projectively equivalent varieties $V \subset \mathcal{S}$ and $V' \subset \mathcal{S}'$. Then consider the partial ruled set $\mathcal{V}$ arising by connecting corresponding points of V and V', respectively.

In [1] the author proves that a ruled variety V_2^{r-1} of a projective r-dimensional space can be generated by the lines joining the points of two curves C^m and C^{r-m-1} of two complementary subspaces and in birational correspondence (cf.

[1], p.290, 7). That is, the variety V_2^{r-1} is a partial ruled set whose order $r-1$ is just the sum of the order of the two curves.

A conjecture has been expressed in [5] (p.2, Introduction), if the result could be extended to the variety partial ruled set arising from any two projectively equivalent varieties lying in two complementary spaces.

In this note we prove the conjecture is true, so that from two projectively equivalent irreducible varieties V_k^m and V_k^n of dimension k and of order m and n, respectively, lying in two complementary spaces, the ruled set arising by connecting corresponding points is a variety $\mathcal{V}_{k+1}^{m+n}$ of dimension $k+1$ and of order $m+n$.

2 Preliminary Notes

Let $F = GF(q)$ be a finite field, $q = p^s$, p prime. Denote F^{n+1}, $n \geq 2$, the $(n+1)$-dimensional vector space over F, $P^n = PrF^{n+1} = PG(n,q)$ the n-dimensional projective space contraction of F^{n+1} over F. Let $\overline{F}$ be the algebraic closure of the field $F = GF(q)$.

Denote S_r with $r \leq n-1$ a subspace of P^n of dimension r. A hyperplane S_{n-1} will be denoted also H.

The geometry P^n is considered a sub-geometry of $\overline{P^n}$, the projective geometry over $\overline{F}$. We refer to the points of P^n as the *rational points* of $\overline{P^n}$.

Choose a coordinate system in P^n so that it is a coordinate system for $\overline{P}^n$ too, denote a point $P \approx (x_0, x_1, ..., x_n) := \overline{F}^*(x_0, x_1, ..., x_n)$, $\overline{F}^* = \overline{F} \setminus \{0\}$.

P is a *rational point* if there exists $(x_0, x_1, ..., x_n) \in F^{n+1}$ such that $P \approx (x_0, x_1, ..., x_n)$.

Definition 2.1 *A variety V_u^v of dimension u and of order v of P^n is the set of the rational points of a projective variety $\overline{V}_u^v$ of $\overline{P}^n$ defined by a finite set of polynomials of $F[x_0, \ldots, x_n]$. If $u = 1$, V_1^v is a curve, if $u = 2$, V_2^v is a surface, if $u > 2$, V_u^v is a hypersurface.*

The dimension u of V_u^v is *the number of coordinates that can be arbitrarily assigned*. In the finite case it is the degree of the polynomial in q representing the number of the points of the variety. The order v of V_u^v is the number of the points that a subspace S_{n-u} of P^n has in common with V_u^v (cf.[1], p.190).

Lemma 2.2 *A subspace S_i of P^n, with $i \geq n-u$ meets V_u^v in a variety V_{i+u-n}^v.*

Proof. See [1], p.191.

Lemma 2.3 *In P^n the intersection of an irreducible variety V_u^v with a subspace S_{n-u+t}, $t > 0$, is an irreducible variety V_t^v.*

Proof. See [1], p.192.

Note - From Lemmas 2.2 and 2.3 follows that a hyperplane H of P^n meets V_u^v in a variety V_{u-1}^v. A subspace S_i of dimension $i = n - u + 1$ meets V_u^v in a curve V_1^v. That means the order of a variety is preserved with respect to the dimension of the subspaces that meet it.

In [1], p.290, 7.- is proved the following result

Lemma 2.4 *The ruled variety generated by the lines connecting the points of two birationally equivalent curves has order the sum of the orders of the curves minus the number of the fixed points if any.*

Definition 2.5 *Two varieties V_u^v and $V_{u'}^{v'}$ are birationally (or, projectively) equivalent if there exist rational bijective functions (or, projectivities) between the points of V_u^v and the points of $V_{u'}^{v'}$. In such a case $u' = u$.*

Definition 2.6 *The incidence hull $\overline{X}$ of a subset X of points of P^n is the set of all lines joining the points of X. The joining line of two points P, Q is $\overline{P, Q} := \overline{\{P, Q\}}$.*

In $P^n = PG(n, q)$ consider two complementary subspaces $\mathcal{S}, \mathcal{S}'$ and two subsets $\mathcal{K} \subset \mathcal{S}$ and $\mathcal{K}' \subset \mathcal{S}'$.

Set $\mathcal{R} = \{X \in l = \overline{P, P'} \mid P \in \mathcal{K},\, P' \in \mathcal{K}'\}$, that is, the set of the points lying in the lines $l = \overline{P, P'}$. Then $\mathcal{X} := \bigcup_{X \in \mathcal{R}} X$ is called a *ruled set*.

Definition 2.7 *Any subset of $\mathcal{X}$ is a partial ruled set.*

3 Main Result

Let $P^{r+r'+1} = PG(r + r' + 1, q)$ the $(r + r' + 1)$-dimensional projective space over $F = GF(q)$.

In two complementary subspaces S_r and $S_{r'}$ choose and fix two projectively equivalent and irreducible varieties $V_k^m \subset S_r$ and $V_k^n \subset S_{r'}$ of dimension k and of order m and n, respectively.

Let $f(q)$ be the polynomial representing $|V_k^m| = |V_k^n|$, that is, the number of the points of each variety, so that $k = deg\, f(q)$.

Denote $\mathcal{V}$ the variety partial ruled set obtained by connecting the corresponding points of V_k^m and V_k^n.

Theorem 3.1 *The variety $\mathcal{V}$ has dimension $k+1$ and order $m+n$.*

Proof. Let $\mathcal{V}$ be the partial ruled set arising by connecting the corresponding points of V_k^m and V_k^n respectively. The number of the points of the ruled variety $\mathcal{V}$ is $(q+1)f(q)$ hence the dimension of $\mathcal{V}$ is $k+1$.

Let S_h with $h = r + r' - k + 2$ be a subspace of $P^{r+r'+1}$. As from $(r + r' - k + 2) + r = (r + r' + 1) + x$ follows $x = r - k + 1$, the intersection $S_h \cap S_r$ is a subspace S_{r-k+1}. From Lemmas 2.2 and 2.3 we get that S_{r-k+1} meets V_k^m in a variety $V^m_{(r-k+1)+k-r} = V_1^m$, that is, a curve of order m.

Analogously we get that the intersection $S_h \cap S_{r'}$ is a subspace $S_{r'-k+1}$ (as $(r + r' - k + 2) + r' = (r + r' + 1) + x$). From Lemmas 2.2 and 2.3 follows that $S_{r'-k+1}$ meets V_k^n in a variety $V^n_{(r'-k+1)+k-r'} = V_1^n$ that is a curve of order n.

Denote V the ruled variety $S_{r+r'-k+2} \cap \mathcal{V}$ consisting of the lines connecting the points of V_1^m and V_1^n. It has dimension 2 and, from Lemma 2.4, order $m + n$. Hence, from Lemmas 2.2, 2.3 and the Note, follows that $m + n$ is also the order of $\mathcal{V}$.

Example Let S_4 and S_3 be two complementary subspaces of $PG(8, q)$. In S_4 choose and fix the variety V_2^3, in S_3 choose and fix the hyperbolic quadric V_2^2. It is known from [4], Section 3.5, that V_2^3 is the partial ruled set of the lines connecting the corresponding points of two projectively equivalent directrices, a conic and a line lying in two complementary subspaces (cf. [1], p.290). Moreover V_2^3 consists of $(q+1)^2$ points and a 3-dimensional subspace meets V_2^3 in a cubic curve (cf. [4], Section 3.5, pp. 90–94). The quadric V_2^2 consists of $(q+1)^2$ points and a plane meets it in a conic (cf. [2], pp. 23–26). Let S_7 be a hyperplane of $PG(8, q)$. It is $S_7 \cap S_4 = S_3$. Denote C^3 the cubic curve $S_3 \cap V_2^3$. It is $S_7 \cap S_3 = S_2$, such a plane S_2 meets V_2^2 in a conic C^2. The partial ruled variety $\mathcal{V}$ of the lines connecting corresponding points of C^3 and C^2 consists of $(q+1)^3$ points, that is, the dimension of $\mathcal{V}$ is 3. From Theorem 3.1 the order of $\mathcal{V}$ is $3 + 2 = 5$.

References

[1] E. Bertini, *Introduzione alla geometria proiettiva degli iperspazi*, Enrico Spoerri Editore, Pisa, 1907.

[2] J.W.P. Hirschfeld, *Finite projective spaces of three dimensions*, Clarendon Press, Oxford, 1985.

[3] J.W.P. Hirschfeld, J.A. Thas, *General Galois geometry*, Oxford University Press, Oxford, 1991.

[4] R. Vincenti, *Finite fields, projective geometry and related topics*, Morlacchi Editore, 2021, ISBN 978-88-9392-259.

[5] R. Vincenti, Varieties and codes from partial ruled sets, *International Mathematical Forum* **18**, (2023), no. 1, 1-14.
https://doi.org/doi.org/10.12988/imf.2023.912353

Received: April 23, 2023; Published: May 15, 2023