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On Exact Diagrams and Strict Bounded Group Homomorphisms

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Abstract

Necessary and sufficient conditions for the exactness (in the algebraic sense) of certain sequences of bounded group homomorphisms are established.

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1. Introduction

Two fundamental theorems of Linear Algebra [1, p. 227 and p. 229] assert that the exactness of certain sequences of linear mappings between modules is equivalent to the exactness of group homomorphisms between the corresponding abelian groups of linear mappings. In this work we prove the validity of analogous results in the context of arbitrary bornological groups, thereby extending previous results valid for abelian bornological groups [3]. More precisely, we show that the notion of a strict bounded group homomorphism is the key ingredient which allows us to establish the equivalence between the exactness (in the algebraic sense) of certain sequences of bounded group homomorphisms and the exactness of diagrams of mappings between the corresponding sets of bounded group homomorphisms. It should also be mentioned that the exactness (in the algebraic sense) of sequences of bounded group homomorphisms has already been discussed in [2].

2. Preliminaries

In this work the identity element of any group and a group reduced to its identity element will be denoted by e. For arbitrary groups B and C, the group homomorphism $x \in B \mapsto e \in C$ will also be denoted by e; and, if $u\colon B \to C$ is an arbitrary group homomorphism, its kernel (resp. image) will be represented by $\mathrm{Ker}(u)$ (resp. $\mathrm{Im}(u)$).

A pair $(B, \mathcal{B})$ consisting of a group B and a bornology $\mathcal{B}$ on B is a bornological group if the mapping

$$(x, y) \in (B \times B,\, \mathcal{B} \times \mathcal{B}) \mapsto xy^{-1} \in (B, \mathcal{B})$$

is bounded, where $\mathcal{B} \times \mathcal{B}$ is the product bornology on $B \times B$; examples of bornological groups may be found in [2]. In what follows the set of all bounded group homomorphisms from the bornological group $(B, \mathcal{B})$ into the bornological group $(C, \mathcal{C})$ will be represented by $\mathrm{Hom}_b((B, \mathcal{B}), (C, \mathcal{C}))$; $u \in \mathrm{Hom}_b((B, \mathcal{C}), (C, \mathcal{C}))$ is said to be *strict* if the corresponding bounded group isomorphism $\overline{u}\colon \big(B/\mathrm{Ker}(u), \mathcal{E}\big) \to \big(\mathrm{Im}(u), \mathcal{C}_{\mathrm{Im}(u)}\big)$ is a bornological group isomorphism, where $\mathcal{E}$ is the (group) quotient bornology on the quotient group $B/\mathrm{Ker}(u)$ and $\mathcal{C}_{\mathrm{Im}(u)}$ is the (group) bornology induced by $\mathcal{C}$ on $\mathrm{Im}(u)$. For $u \in \mathrm{Hom}_b((B, \mathcal{B}), (C, \mathcal{C}))$ and a bornological group $(H, \mathcal{H})$, the mapping

$$\varphi \in \mathrm{Hom}_b((H, \mathcal{H}), (B, \mathcal{B})) \mapsto u \circ \varphi \in \mathrm{Hom_b}((\mathrm{H}, \mathcal{H}), (\mathrm{C}, \mathcal{C}))$$

$$\big(\text{resp. } \psi \in \mathrm{Hom}_b((C, \mathcal{C}), (H, \mathcal{H})) \mapsto \psi \circ u \in \mathrm{Hom}_b((B, \mathcal{B}), (H, \mathcal{H}))\big)$$

will be denoted by u^* (resp. u_*).

Let us recall [4, pp. 278-279] that, if E, E', E'' are sets, a diagram

$$E \xrightarrow{u} E' \;\; \underset{v_2}{\overset{v_1}{\rightrightarrows}} \;\; E''$$

of mappings is *exact* (in the sense of Grothendieck) if u is a bijection from E onto the set $\mathrm{Ker}(v_1, v_2) := \{x' \in E'\,; v_1(x') = v_2(x')\}$. In particular, if $(B, \mathcal{B})$, $(C, \mathcal{C})$, $(D, \mathcal{D})$, $(H, \mathcal{H})$ are bornological groups, $u \in \mathrm{Hom}_b\big((B, \mathcal{B}), (C, \mathcal{C})\big)$, $v \in \mathrm{Hom}_b\big((C, \mathcal{C}), (D, \mathcal{D})\big)$, to say that the diagram

$$\mathrm{Hom}_b((H, \mathcal{H}), (B, \mathcal{B})) \xrightarrow{u^*} \mathrm{Hom}_b((H, \mathcal{H}), (C, \mathcal{C})) \;\; \underset{\mathbf{1}}{\overset{v^*}{\rightrightarrows}} \;\; \mathrm{Hom_b}((\mathrm{H}, \mathcal{H}), (\mathrm{D}, \mathcal{D}))$$

$\big($where $\mathbf{1}(\psi) = e$ for $\psi \in \mathrm{Hom}_b\big((H, \mathcal{H}), (C, \mathcal{C})\big)\big)$

$$\left(\text{resp. } \mathrm{Hom}_b\big((D, \mathcal{D}), (H, \mathcal{H})\big) \xrightarrow{v_*} \mathrm{Hom}_b\big((C, \mathcal{C}), (H, \mathcal{H})\big) \;\; \underset{\mathbf{1}}{\overset{u_*}{\rightrightarrows}} \;\; \mathrm{Hom}_b\big((B, \mathcal{B}), (H, \mathcal{H})\big)\right)$$

(where $\mathbf{1}(\psi) = e$ for $\psi \in \mathrm{Hom}_b\big((C,\mathcal{C}),(H,\mathcal{H})\big)$) of mappings is exact is equivalent to saying that u^* is a bijection from $\mathrm{Hom}_b\big((H,\mathcal{H}),(B,\mathcal{B})\big)$ onto $\mathrm{Ker}(v^*,\mathbf{1})$ (resp. v_* is a bijection from $\mathrm{Hom}_b\big((D,\mathcal{D}),(H,\mathcal{H})\big)$ onto $\mathrm{Ker}(u_*,\mathbf{1})$).

If $(B,\mathcal{B})$, $(C,\mathcal{C})$ are abelian bornological groups, $\mathrm{Hom}_b\big((B,\mathcal{B}),(C,\mathcal{C})\big)$ is an abelian group and u^*, u_*, v^*, v_* are group homomorphisms. In this context, the exactness of the diagram

$$\mathrm{Hom}_b((H,\mathcal{H}),(B,\mathcal{B})) \xrightarrow{u^*} \mathrm{Hom}_b((H,\mathcal{H}),(C,\mathcal{C})) \underset{\mathbf{1}}{\overset{v^*}{\rightrightarrows}} \mathrm{Hom_b}((\mathrm{H},\mathcal{H}),(\mathrm{D},\mathcal{D}))$$

(resp. $\mathrm{Hom}_b((D,\mathcal{D}),(H,\mathcal{H})) \xrightarrow{v_*} \mathrm{Hom}_b((C,\mathcal{C}),(H,\mathcal{H})) \underset{\mathbf{1}}{\overset{u^*}{\rightrightarrows}} \mathrm{Hom_b}((\mathrm{B},\mathcal{B}),(\mathrm{H},\mathcal{H}))\big)$ of mappings is equivalent to the exactness of the sequence

$$e \longrightarrow \mathrm{Hom}_b((H,\mathcal{H}),(B,\mathcal{B})) \xrightarrow{u^*} \mathrm{Hom}_b((H,\mathcal{H}),(C,\mathcal{C})) \xrightarrow{v^*} \mathrm{Hom}_b((H,\mathcal{H}),(D,\mathcal{D}))$$

(resp. $e \longrightarrow \mathrm{Hom}_b((D,\mathcal{D}),(H,\mathcal{H})) \xrightarrow{v_*} \mathrm{Hom}_b((C,\mathcal{C}),(H,\mathcal{H})) \xrightarrow{u_*} \mathrm{Hom}_b((B,\mathcal{B}),(H,\mathcal{H}))\big)$ of group homomorphisms.

3. The results

We shall first establish the following

Theorem 3.1. *Let $(B,\mathcal{B})$, $(C,\mathcal{C})$, $(D,\mathcal{D})$ be bornological groups and consider $u \in \mathrm{Hom}_b((B,\mathcal{B}),(C,\mathcal{C}))$ and $v \in \mathrm{Hom}_b((C,\mathcal{C}),(D,\mathcal{D}))$. Then the following conditions are equivalent*:

(a) *u is strict and the sequence*

$$e \to B \xrightarrow{u} C \xrightarrow{v} D$$

of group homomorphisms is exact;

(b) *for each bornological group $(H,\mathcal{H})$, the diagram*

$$\mathrm{Hom}_b((H,\mathcal{H}),(B,\mathcal{B})) \xrightarrow{u^*} \mathrm{Hom}_b((H,\mathcal{H}),(C,\mathcal{C})) \underset{\mathbf{1}}{\overset{v^*}{\rightrightarrows}} \mathrm{Hom_b}((\mathrm{H},\mathcal{H}),(\mathrm{D},\mathcal{D}))$$

of mappings is exact.

Proof. (a) $\Rightarrow$ (b): The exactness of the sequence

$$e \to B \xrightarrow{u} C$$

is equivalent to the injectivity of u.

Let $(H,\mathcal{H})$ be an arbitrary bornological group, and let us show the exactness of the diagram

$$\mathrm{Hom}_b((H,\mathcal{H}),(B,\mathcal{B})) \xrightarrow{u^*} \mathrm{Hom}_b((H,\mathcal{H}),(C,\mathcal{C})) \underset{\mathbf{1}}{\overset{v^*}{\rightrightarrows}} \mathrm{Hom_b}((\mathrm{H},\mathcal{H}),(\mathrm{D},\mathcal{D})).$$

It is obvious that the injectivity of u^* follows from the injectivity of u. Now let us prove that $\mathrm{Im}(u^*) = \mathrm{Ker}(v^*, \mathbf{1})$.

Indeed, since $\mathrm{Im}(u) \subset \mathrm{Ker}(v)$, it follows that $\mathrm{Im}(u^*) \subset \mathrm{Ker}(v^*, \mathbf{1})$. On the other hand, if $\psi \in \mathrm{Ker}(v^*, \mathbf{1})$, $v \circ \psi = e$, and hence $\mathrm{Im}(\psi) \subset \mathrm{Ker}(v) = \mathrm{Im}(u)$. Consequently, there is a unique mapping $\varphi\colon H \to B$ such that $\psi = u \circ \varphi$. We claim that $\varphi \in \mathrm{Hom_b}((\mathrm{H}, \mathcal{H}), (\mathrm{B}, \mathcal{B}))$. In fact, it is clear that φ is a group homomorphism. By the strictness of u, the group isomorphism $u(x) \in \mathrm{Im}(u) \mapsto x \in B$ belongs to $\mathrm{Hom}_b((\mathrm{Im}(u), \mathcal{C}_{\mathrm{Im}(u)}), (B, \mathcal{B}))$; moreover, if we view ψ as a group homomorphism from H into $\mathrm{Im}(u)$, it follows that $\psi \in \mathrm{Hom}_b((H, \mathcal{H}), (\mathrm{Im}(u), \mathcal{C}_{\mathrm{Im}(u)}))$. Thus $\varphi \in \mathrm{Hom}_b((H, \mathcal{H}), (B, \mathcal{B}))$, and hence $\psi = u^*(\varphi) \in \mathrm{Im}(u^*)$. Therefore $\mathrm{Ker}(v^*, \mathbf{1}) \subset \mathrm{Im}(u^*)$, and the equality $\mathrm{Im}(u^*) = \mathrm{Ker}(v^*, \mathbf{1})$ is established.

(b) $\Rightarrow$ (a): By taking the inclusion mapping $\varphi\colon \mathrm{Ker}(u) \to B$, we have that $\varphi \in \mathrm{Hom}_b\big((\mathrm{Ker}(u), \mathcal{B}_{\mathrm{Ker}(u)}), (B, \mathcal{B})\big)$ and $u^*(\varphi) = u^*(e)$. Since, by hypothesis, u^* is injective, it follows that $\varphi = e$, which implies the exactness of the sequence

$$e \to B \xrightarrow{u} C.$$

On the other hand, by hypothesis, the diagram

$$\mathrm{Hom}_b((B, \mathcal{B}), (B, \mathcal{B})) \xrightarrow{u^*} \mathrm{Hom}_b((B, \mathcal{B}), (C, \mathcal{C})) \;\underset{\mathbf{1}}{\overset{v^*}{\rightrightarrows}}\; \mathrm{Hom}_b((B, \mathcal{B}), (D, \mathcal{D}))$$

is exact. Moreover, $(v^* \circ u^*)(\mathbf{1}_B) = v \circ u$. Consequently, $\mathrm{Im}(u) \subset \mathrm{Ker}(v)$. Since, by hypothesis, the diagram

$$\begin{aligned}\mathrm{Hom}_b((\mathrm{Ker}(v), \mathcal{C}_{\mathrm{Ker}(v)}), (B, \mathcal{B})) \xrightarrow{u^*}\;& \mathrm{Hom}_b((\mathrm{Ker}(v), \mathcal{C}_{\mathrm{Ker}(v)}), (C, \mathcal{C}))\\ \underset{\mathbf{1}}{\overset{v^*}{\rightrightarrows}}\;& \mathrm{Hom}_b((\mathrm{Ker}(v), \mathcal{C}_{\mathrm{Ker}(v)}), (D, \mathcal{D}))\end{aligned}$$

is exact, and since the inclusion mapping $\psi\colon \mathrm{Ker}(v) \to C$ belongs to

$$\mathrm{Ker}(v^*, \mathbf{1}) = \mathrm{Im}(u^*),$$

there is a $\varphi \in \mathrm{Hom}_b((\mathrm{Ker}(v), \mathcal{C}_{\mathrm{Ker}(v)}), (B, \mathcal{B}))$ so that $\psi = u \circ \varphi$. Therefore $\mathrm{Ker}(v) \subset \mathrm{Im}(u)$, and the sequence

$$B \xrightarrow{u} C \xrightarrow{v} D$$

is exact.

Finally, let us show that u is strict. Indeed, by hypothesis, the diagram

$$\begin{aligned}\mathrm{Hom}_b((\mathrm{Im}(u), \mathcal{C}_{\mathrm{Im}(u)}), (B, \mathcal{B})) \xrightarrow{u^*}\;& \mathrm{Hom}_b((\mathrm{Im}(u), \mathcal{C}_{\mathrm{Im}(u)}), (C, \mathcal{C}))\\ \underset{\mathbf{1}}{\overset{v^*}{\rightrightarrows}}\;& \mathrm{Hom}_b((\mathrm{Im}(u), \mathcal{C}_{\mathrm{Im}(u)}), (D, \mathcal{D}))\end{aligned}$$

is exact. Since the inclusion mapping $\psi\colon \mathrm{Im}(u) \to C$ belongs to $\mathrm{Ker}(v^*, \mathbf{1}) = \mathrm{Im}(u^*)$, there is a $\varphi \in \mathrm{Hom}_b((\mathrm{Im}(u), \mathcal{C}_{\mathrm{Im}(u)}), (B, \mathcal{B}))$ so that $\psi = u \circ \varphi$. Consequently

$$u(x) = \psi(u(x)) = u(\varphi(u(x)))$$

for all $x \in B$, and the injectivity of u implies $\varphi(u(x)) = x$ for all $x \in B$. Thus the mapping $x \in (B, \mathcal{B}) \mapsto u(x) \in (\mathrm{Im}(u), \mathcal{C}_{\mathrm{Im}(u)})$ is a bornological group isomorphism. Hence u is strict, thereby concluding the proof.

The strictness of u is essential for the validity of the implication (a) $\Rightarrow$ (b) in Theorem 3.1, as we shall see in the following

Example 3.2. Let $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) be the discrete (resp. trivial) bornology on the additive group $\mathbb{R}$ of real numbers. Let $u \in \mathrm{Hom}_b\big((\mathbb{R}, \mathcal{B}_1), (\mathbb{R}, \mathcal{B}_2)\big)$ be given by $u(x) = x$ for all $x \in \mathbb{R}$, and let $v \in \mathrm{Hom}_b\big((\mathbb{R}, \mathcal{B}_2), (\mathbb{R}, \mathcal{B}_2)\big)$ be given by $v(x) = e$ for all $x \in \mathbb{R}$. It is obvious that u is not strict and that sequence

$$e \to \mathbb{R} \xrightarrow{u} \mathbb{R} \xrightarrow{v} \mathbb{R}$$

is exact. Nevertheless, the sequence

$$\mathrm{Hom}_b\big((\mathbb{R}, \mathcal{B}_2), (\mathbb{R}, \mathcal{B}_1)\big) \xrightarrow{u^*} \mathrm{Hom}_b\big((\mathbb{R}, \mathcal{B}_2), (\mathbb{R}, \mathcal{B}_2)\big) \xrightarrow{v^*} \mathrm{Hom}_b\big((\mathbb{R}, \mathcal{B}_2), (\mathbb{R}, \mathcal{B}_2)\big)$$

is not exact. For, if it were, since $1_{\mathbb{R}} \in \mathrm{Ker}(v^*) = \mathrm{Im}(u^*)$, there would exist a $\varphi \in \mathrm{Hom}_b((\mathbb{R}, \mathcal{B}_2), (\mathbb{R}, \mathcal{B}_1))$ so that $u^*(\varphi) = u \circ \varphi = 1_{\mathbb{R}}$; but this would imply $\varphi(x) = x$ for $x \in \mathbb{R}$, which is not bounded as a mapping from $(\mathbb{R}, \mathcal{B}_2)$ into $(\mathbb{R}, \mathcal{B}_1)$.

If B, C are arbitrary groups, $\mathrm{Hom}(B, C)$ will denote the set of all group homomorphisms from B into C.

Corollary 3.3. *Let B, C, D be groups, $u \in \mathrm{Hom}(B, C)$ and $v \in \mathrm{Hom}(C, D)$. Then the following conditions are equivalent*:

(a) *the sequence*

$$e \to B \xrightarrow{u} C \xrightarrow{v} D$$

of group homomorphisms is exact;

(b) *for each group H, the diagram*

$$\mathrm{Hom}(H, B) \xrightarrow{u^+} \mathrm{Hom}(H, C) \underset{\mathbf{1}}{\overset{v^+}{\rightrightarrows}} \mathrm{Hom}(H, D)$$

of mappings is exact, where $u^+(\varphi) = u \circ \varphi$ for $\varphi \in \mathrm{Hom}(H, B)$, $v^+(\psi) = v \circ \psi$ for $\psi \in \mathrm{Hom}(H, C)$ and $\mathbf{1}(\psi) = e$ for $\psi \in \mathrm{Hom}(H, C)$.

Proof. For each group G let t_G be the trivial bornology on G. Then

$$\mathrm{Hom}(H, G) = \mathrm{Hom}_b((H, \mathcal{H}), (G, t_G))$$

for every bornological group $(H, \mathcal{H})$; in particular,

$$\mathrm{Hom}(H, G) = \mathrm{Hom}_b((H, t_H), (G, t_G))$$

for every group H.

In the proof we will consider B (resp. C, D) endowed with t_B (resp. t_C, t_D).

(a) $\Rightarrow$ (b): It is obvious that the bounded group homomorphism

$$u\colon (B, t_B) \to (C, t_C)$$

is strict. Therefore, by (a) $\Rightarrow$ (b) of Theorem 3.1, the diagram

$$\mathrm{Hom}(H, B) = \mathrm{Hom}_b((H, t_H), (B, t_B)) \xrightarrow{u^+} \mathrm{Hom}(H, C) = \mathrm{Hom}_b((H, t_H), (C, t_C)) \underset{\mathbf{1}}{\overset{v^+}{\rightrightarrows}} \mathrm{Hom}(H, D) = \mathrm{Hom}_b((H, t_H), (D, t_D))$$

is exact for every group H.

(b) $\Rightarrow$ (a): Let $(H, \mathcal{H})$ be an arbitrary bornological group. By hypothesis, the diagram

$$\mathrm{Hom}(H, B) = \mathrm{Hom}_b((H, \mathcal{H}), (B, t_B)) \xrightarrow{u^*} \mathrm{Hom}(H, C) = \mathrm{Hom}_b((H, \mathcal{H}), (C, t_C)) \underset{\mathbf{1}}{\overset{v^*}{\rightrightarrows}} \mathrm{Hom}(H, D) = \mathrm{Hom}_b((H, \mathcal{H}), (D, t_D))$$

is exact. Therefore, by (b) $\Rightarrow$ (a) of Theorem 3.1, the sequence

$$e \longrightarrow B \xrightarrow{u} C \xrightarrow{v} D$$

is exact.

Now let us prove the following

Theorem 3.4. *Let* $(B, \mathcal{B})$, $(C, \mathcal{C})$, $(D, \mathcal{D})$ *be bornological groups and consider* $u \in \mathrm{Hom}_b((B, \mathcal{B}), (C, \mathcal{C}))$ *and* $v \in \mathrm{Hom}_b((C, \mathcal{C}), (D, \mathcal{D}))$. *Then the following conditions are equivalent*:

(a) v *is strict and the sequence*

$$B \xrightarrow{u} C \xrightarrow{v} D \longrightarrow e$$

of group homomorphisms is exact;

(b) $\mathrm{Im}(u)$ *is a normal subgroup of* C, $\mathrm{Im}(v)$ *is a normal subgroup of* D *and, for each bornological group* $(H, \mathcal{H})$, *the diagram*

$$\mathrm{Hom_b}((\mathrm{D}, \mathcal{D}), (\mathrm{H}, \mathcal{H})) \xrightarrow{v_*} \mathrm{Hom_b}((\mathrm{C}, \mathcal{C}), (\mathrm{H}, \mathcal{H})) \underset{\mathbf{1}}{\overset{u_*}{\rightrightarrows}} \mathrm{Hom_b}((\mathrm{B}, \mathcal{B}), (\mathrm{H}, \mathcal{H}))$$

of mappings is exact.

Proof. (a) $\Rightarrow$ (b): First of all, to say that the sequence

$$B \xrightarrow{u} C \xrightarrow{v} D \longrightarrow e$$

of group homomorphisms is exact is equivalent to saying that $\mathrm{Im}(u) = \mathrm{Ker}(v)$ and v is surjective; hence, in this case, Im(u) is a normal subgroup of C and $\mathrm{Im}(v)$ $(= D)$ is a normal subgroup of D.

Let $(H, \mathcal{H})$ be an arbitrary bornological group, and let us show the exactness of the diagram

$$\mathrm{Hom_b((D,\mathcal{D}),(H,\mathcal{H}))} \xrightarrow{v_*} \quad \mathrm{Hom_b((C,\mathcal{C}),(H,\mathcal{H}))} \quad \underset{\mathbf{1}}{\overset{u_*}{\rightrightarrows}} \quad \mathrm{Hom_b((B,\mathcal{B}),(H,\mathcal{H}))}.$$

The injectivity of v_* follows from the surjectivity of v. Now let us prove that $\mathrm{Im}(v_*) = \mathrm{Ker}(u_*, \mathbf{1})$.

Indeed, since $\mathrm{Im}(u) \subset \mathrm{Ker}(v)$, we get $\mathrm{Im}(v_*) \subset \mathrm{Ker}(u_*, \mathbf{1})$. On the other hand, let $w \in \mathrm{Ker}(u_*, \mathbf{1})$. Hence $w \circ u = e$, which implies

$$\mathrm{Ker}(v) = \mathrm{Im}(u) \subset \mathrm{Ker}(w).$$

Put $w'(v(y)) = w(y)$ for $y \in C$; w' is well defined. We claim that w' is a bounded group homomorphism from $(D, \mathcal{D})$ into $H, \mathcal{H})$. In fact, it is clear that w' is a group homomorphism. Let $R \in \mathcal{D}$ be arbitrary. Since v is strict, Theorem 3.16 of [3] ensures the existence of an $S \in \mathcal{C}$ so that $v^{-1}(R) \subset S\,\mathrm{Ker}(v)$. Consequently

$$\begin{aligned} w'(R) &= w'(v(v^{-1}(R))) = w(v^{-1}(R)) \subset w(S\,\mathrm{Ker}(v)) \\ &= w(S)w(\mathrm{Ker}(v)) \subset w(S)w(\mathrm{Ker}(w)) = w(S). \end{aligned}$$

Since $w(S) \in \mathcal{H}$, it follows that $w'(R) \in \mathcal{H}$; thus $w' \in \mathrm{Hom}_b((D, \mathcal{D}), (H, \mathcal{H}))$. Finally, $w = v_*(w') \in \mathrm{Im}(v_*)$, and $\mathrm{Ker}(u_*, \mathbf{1}) \subset \mathrm{Im}(v_*)$. Thus $\mathrm{Im}(v_*) = \mathrm{Ker}(u_*, \mathbf{1})$, and the proof of (b) is concluded.

(b) $\Rightarrow$ (a): Let $\mathcal{E}$ be the quotient bornology on the quotient group $D/\mathrm{Im}(v)$. By hypothesis, the mapping

$$\mathrm{Hom}_b((D, \mathcal{D}), (D/\mathrm{Im}(v), \mathcal{E})) \xrightarrow{v_*} \mathrm{Hom}_b((C, \mathcal{C}), (D/\mathrm{Im}(v), \mathcal{E}))$$

is injective. But this implies the exactness of the sequence

$$C \xrightarrow{v} D \longrightarrow e\,,$$

that is, the surjectivity of v, because the canonical surjection

$$\pi\colon (D, \mathcal{D}) \to (D/\mathrm{Im}(v), \mathcal{E})$$

is bounded and $v_*(\pi) = e$.

Now, let us prove the exactness of the sequence

$$B \xrightarrow{u} C \xrightarrow{v} D.$$

Indeed, by hypothesis, the diagram

$$\mathrm{Hom}_b((D,\mathcal{D}),(D,\mathcal{D})) \xrightarrow{v_*} \quad \mathrm{Hom}_b((C,\mathcal{C}),(D,\mathcal{D})) \quad \overset{u_*}{\underset{\mathbf{1}}{\rightrightarrows}} \quad \mathrm{Hom}_b((B,\mathcal{B}),(D,\mathcal{D}))$$

is exact, which furnishes

$$e = (u_* \circ v_*)(1_D) = v \circ u.$$

Consequently, $\mathrm{Im}(u) \subset \mathrm{Ker}(v)$. On the other hand, by hypothesis, the diagram

$$\begin{aligned}\mathrm{Hom}_b((D,\mathcal{D}),(C/\mathrm{Im}(u),\mathcal{E})) \xrightarrow{v_*} \quad & \mathrm{Hom}_b((C,\mathcal{C}),(C/\mathrm{Im}(u),\mathcal{E})) \\ \overset{u_*}{\underset{\mathbf{1}}{\rightrightarrows}} \quad & \mathrm{Hom}_b((B,\mathcal{B}),(C/\mathrm{Im}(u),\mathcal{E}))\end{aligned}$$

is exact, where $\mathcal{E}$ is the quotient bornology on the quotient group $C/\mathrm{Im}(u)$. Since the canonical surjection $\xi\colon C \to C/\mathrm{Im}(u)$ belongs to $\mathrm{Ker}(u_*, \mathbf{1}) = \mathrm{Im}(v_*)$, there is a $\psi \in \mathrm{Hom}_b((D,\mathcal{D}),(C/\mathrm{Im}(u),\mathcal{E}))$ so that $\xi = v_*(\psi) = \psi \circ v$. Consequently,

$$\xi(y) = \psi(v(y)) = \psi(e) = e$$

for all $y \in \mathrm{K}er(v)$, that is, $\mathrm{Ker}(v) \subset \mathrm{Ker}(\xi) = \mathrm{Im}(u)$. Therefore $\mathrm{Im}(u) = \mathrm{Ker}(v)$. Finally,

$$\psi(v(y)) = \xi(y) = y\,\mathrm{Ker}(v)$$

for all $y \in C$, that is, ψ is the inverse of the bounded group isomorphism

$$y\,\mathrm{Ker}(v) \in (C/\mathrm{Ker}(v),\mathcal{E}) \mapsto v(y) \in (D,\mathcal{D}).$$

Thus v is strict, thereby concluding the proof of the theorem.

As before (recall Example 3.2) it is easily seen that the strictness of v is essential for the validity of the implication (a) $\Rightarrow$ (b) in Theorem 3.4.

Corollary 3.5. *Let B, C, D be groups, $u \in \mathrm{Hom(B,C)}$ and $v \in \mathrm{Hom(C,D)}$. Then the following conditions are equivalent*:

(a) *the sequence*

$$B \xrightarrow{u} C \xrightarrow{v} D \longrightarrow e$$

of group homomorphisms is exact;

(b) $\mathrm{Im}(u)$ *is a normal subgroup of C,* $\mathrm{Im}(v)$ *is a normal subgroup of D and, for each group H, the diagram*

$$\mathrm{Hom}(D,H) \xrightarrow{v_+} \mathrm{Hom}(C,H) \overset{u_+}{\underset{\mathbf{1}}{\rightrightarrows}} \quad \mathrm{Hom}(B,H)$$

of mappings is exact, where $v_+(\varphi) = \varphi \circ v$ *for* $\varphi \in \mathrm{Hom}(D, H)$*,* $u_+(\psi) = \psi \circ u$ *for* $\psi \in \mathrm{Hom}(C, H)$ *and* $\mathbf{1}(\psi) = e$ *for* $\psi \in \mathrm{Hom}(C, H)$*.*

Proof. For each group G let d_G be the discrete bornology on G. Then

$$\mathrm{Hom}(G, H) = \mathrm{Hom}_b((G, d_G), (H, \mathcal{H}))$$

for every bornological group $(H, \mathcal{H})$; in particular,

$$\mathrm{Hom}(G, H) = \mathrm{Hom}_b((G, d_G), (H, d_H))$$

for every group H.

In the proof we will consider B (resp. C, D) endowed with d_B (resp. d_C, d_D).

(a) $\Rightarrow$ (b): It is obvious that the bounded group homomorphism

$$v \colon (C, d_C) \to (D, d_D)$$

is strict. Therefore, by (a) $\Rightarrow$ (b) of Theorem 3.4, $\mathrm{Im}(u)$ (resp. $\mathrm{Im}(v)$) is a normal subgroup of C (resp. D) and the diagram

$$\begin{array}{lcl} \mathrm{Hom}(D, H) = \mathrm{Hom}_b((D, d_D), (H, d_H)) & \xrightarrow{v_+} & \mathrm{Hom}(C, H) = \mathrm{Hom}_b((C, d_C), (H, d_H)) \\ & \overset{u_+}{\underset{\mathbf{1}}{\rightrightarrows}} & \mathrm{Hom}(B, H) = \mathrm{Hom}_b((B, d_B), (H, d_H)) \end{array}$$

is exact for every group H.

(b) $\Rightarrow$ (a): Let $(H, \mathcal{H})$ be an arbitrary bornological group. By hypothesis, the diagram

$$\begin{array}{lcl} \mathrm{Hom}(D, H) = \mathrm{Hom}_b((D, d_D), (H, \mathcal{H})) & \xrightarrow{v_*} & \mathrm{Hom}(C, H) = \mathrm{Hom}_b((C, d_C), (H, \mathcal{H})) \\ & \overset{u_*}{\underset{\mathbf{1}}{\rightrightarrows}} & \mathrm{Hom}(B, H) = \mathrm{Hom}_b((B, d_B), (H, \mathcal{H})) \end{array}$$

is exact. Therefore, by (b) $\Rightarrow$ (a) of Theorem 3.4, the sequence

$$B \xrightarrow{u} C \xrightarrow{v} D \longrightarrow e$$

is exact.

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[In 2.6 (and in 2.7), 2nd line, to read "the set" instead of "the group"; 3rd line, to insert ". If H is an abelian group, $\mathrm{Hom_b}((\mathrm{G}, \mathcal{B}), (\mathrm{H}, \mathcal{C}))$ is an abelian group" after the first "$(H, \mathcal{C})$"; 4th line, to read "In this case" instead of "Then". In 3.19, 2nd line, to read "the set" instead of "the group"; 3.6, 4.4, 4.9, 4.10, 4.12, 4.13, 4.14, 5.11, 5.12, 5.14 are valid, for example, under the additional assumption that the groups under consideration be abelian.]

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