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Group Theory in Pairwise Comparison Matrices PCM

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Abstract

This study examines why elements of pairwise comparisons matrix should create a group.

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1 Introduction

The first use of abelian group theory in pairwise comparisons method could be traced to [7]. In [1], provided a necessary condition (the torsion free abelian group) for the pairwise comparisons matrix elements to create a group. For it, two Levi's theorems of 1942 were used. The necessary condition for a PC matrix elements to create a general group was published in analyzed in [1] but the sufficiency condition has not been analyzed.

From [3], we know that rating scale values (usually represented by positive real numbers)cannot be used in PC matrices.

The Hadamard product has Lie group structure but at the PC matrix (not its elements) level.

2 The problem formulation

A factorization method of PC matrices was introduced in [10] for the coordinate-wise multiplication. One of the factors is an approximation PC matrix and the other one is the orthogonal component of the approximation. The height h function is defined and used ans inconsistency measure. Examples are provided to illustrate the obtained results.

Lie theory was used to analyze PC matrices in [10]. In this study, we use group theory at the PC matrix element level. We will attempt to examine whether or not torsion-free abelian groups (as found in [1] are sufficient or only necessary condition.)

3 Preliminaries to pairwise comparisons

Pairwise comparisons are divided into two types: additive (e.g., x is greater than y by 7 units) and multiplicative (e.g. x is t times greater than y), $x, y, t > 0$

Probably the easiest and the most compelling case for using pairwise comparisons in academia is an application to grading final exams. For simplicity, let us assume that we have four problems to solve; A, B, C, and D. Evidently, hardly ever all problems are of equal level of difficulty. In such case, it is fair to compare A to B, A to C, A to D, B to C, B to D, and C to D. We assume the reciprocity of PC matrix M: $m_{ji} = 1/m_{ij}$ which is reasonable (when comparing B to A, we expect to get the inverse of A to B). The exam is hence represented by the following PC matrix M:

$$M = [m_{ij}] = 1A/BA/CA/DB/A1B/CB/DC/AC/B1C/DD/AD/BD/C1 \tag{1}$$

As previously stated, A/B reads "the ratio between A and B" and it may not be a result of the division. In the case of abstract concepts, such as software quality and software safety, the division operation makes no sense to use but the ratio estimation between them does.

Ratios of three entities in a cycle create a triad $(A/B, A/C, B/C)$, which is said to be *consistent* providing $A/B * B/C = A/C$. [A/B] reflects the assessed ratio of lengths. randomly displayed segments on the screen is hard to measure since the size of the screen and its resolution may differ. For this reason, we need to rely on the expert opinion, hence the use of pairwise comparisons is useful.

Symbolically, in a PC matrix M, each triad (or a cycle) is defined by (m_{ik}, m_{ij}, m_{kj}). It is consistent if and only if $(m_{ik} * m_{kj} = m_{ij})$. When all triads are consistent (known as the *consistency condition* or *transitivity condition*), the entire PC matrix is considered *consistent*.

The inconsistency occurs if we have at least three entities to compare and conduct all three comparisons instead of the sufficient two comparisons and computing the third compassion. Axiomatization for inconsistency is one of challenges for pairwise comparisons. The convergence of inconsistency algorithms (examined in [12]).

Example 3.1 *Let us take three randomly generated segments:*
a: ———————————-
b: ————
c: ——

giving (by "eye length assessment") ratios $T = (x, y, z) = (a/b, a/c, b/c) = (2, 5, 3)$ *since* $y = a/c = (a/b) * (b/c) = x * z$, *in our case "should be" 6 (not 5) but we do not know which or three ratios are correct or incorrect. In practice, all assessments can be incorrect. All ratios of segments are compiled in the following table:*

	a	b	c
a	*1*	*2*	*5*
b	$\frac{1}{2}$	*1*	*3*
c	$\frac{1}{5}$	$\frac{1}{3}$	*1*

It seems that a trivial mistake took place: 6 should be in place of 5 since $2 * 3$ gives this value. However, it unreasonably assumes that 2 and 3 are accurate assessments. We simply do not know which of the three assessments are, or are not, accurate.

Evidently,$T1 = (2, 6, 3)$ is consistent since $6 = 2 * 3$. For the same reason $T2 = (4, 12, 3)$ is consistent but only one of them may consist correct ration assessments. Often, none of them may have correct ratios.

The *solution* to the PC matrix is a vector of weights which are geometric means of rows. In our example, the weights computed in this fashion (say: $[30, 20, 10, 40]$), are now being used. By looking at the example results, we can conclude that problem D is the most difficult with the weight 40. The easiest problem is C giving one of the pairwise comparisons $D/C = 4$.

Inconsistency exists if we compare the minimum of three objects (including abstract concepts). In case of two objects, we only have inaccuracy which is s different concept. In particular, three objects can be inaccurate but comparing them in pairs may have 0 inconsistency making a new meaning of "he was consistently wrong"

4 Group theory for PCs

In [2], the classification of abelian groups was proposed.

4.1 Rough set theory

It looks like non-trivial finite abelian groups cannot be used for PCs. By "trivial", we understand $\{0\}$ for additive and $\{1\}$ for multiplicative PCs. In the case of multiplicative PCs, for $x > 0$ and $x \neq 1$, triad (x, x^2, x) is consistent (since $x * x = x^2$), there is also triad (x^2, x^4, x^2) and we can assume $x > 1$ hence for $xinR^+$, the infinity is evident.

If the following is used, move to bibliography:

Torres, GM, On Rough Approximations of Languages under Infinite Index Indiscernibility Relations, 179 (3), pp.275-293, 2021.

In the paper [13] Paun, Polkowski and Skowron introduce several indiscernibility relations among strings that are infinite index equivalence or tolerance relations, and study lower and upper rough approximations of languages defined by them. In this paper we develop a further study of some of these indiscernibility relations among strings. We characterize the classes defined by them, and the rough approximations of general and context free languages under them. We also compare some of the rough approximations these relations produce to the ones given by the congruences defining testable, reverse testable, locally testable, piecewise testable and commutative languages. Those yield languages belonging to that families. Next, we modify some of the relations to obtain congruences, and study the families of languages the rough approximations under them give rise to. One of these modificated relations turns out to be the k-abelian congruence, that was defined by J. Karhumaki in [7], in the context of combinatorics on words. We show that it defines a pseudo-principal +-variety, a term defined in [9]. Our results in that work are then applied to determine when a given language has a best upper approximation in that family. Finally, we make some comments on the accuracy of the rough approximations obtained in each case.

Usually, rough set theory compresses R^+ into a finite set of "slots" hence we see a limited way of application to PCs.

4.2 Complex numbers

It is true that the group $(C, +)$ of complex numbers under addition is abelian but by finding in [1], it must be torsion-free.

The lack of "natural" total order on complex numbers precludes them from being used for PC matrix elements. So we can give the following result.

Theorem 4.1 *There is no a finite non-trivial complex torsion-free abelian group.*

Proof. Suppose for the sake of contradiction that there exists a finite complex torsion-free abelian group G. Let G have n elements, where n is a positive integer greater than 1.

Then, since G is torsion-free, it follows that for every element g in G, the order of g (i.e. the least positive integer k such that $g^k = e$, where e is the identity element of G) is infinite.

Now consider the subgroup H of G generated by all of the elements of G. Since G is finite, H must also be finite. However, since the order of every element in H is infinite, it follows that H must be an infinite group, which is a contradiction.

Therefore, there cannot exist a finite non-trivial complex torsion-free abelian group. ■

5 Conclusions

The use of group theory in PCs is non-trivial and limited. It seems that only real positive numbers (R^2) and subsets (e.g., intervals) of (R^2) are the only candidates for PC matrix elements.

However, we know that fuzzy "numbers" (which are functions) do not create a group.

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