International Mathematical Forum, Vol. 18, 2023, no. 3, 121 - 123
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2023.912392

Quasicomponents of Elements and the Partition of a Finite Disconnected Topological Space

Olufemi Sunday Olusa

Department of Mathematical Sciences
Adekunle Ajasin University
Akungba-Akoko
Ondo State, Nigeria

Abstract

It is proved that the number of elements in the quasicomponents in a certain finite disconnected topological space divides the number of elements in the space.

Mathematics Subject Classification: 54D05, 03E02

Keywords: Quasicomponent, Connected space, Disconnected space

1 Introduction

The quasicomponent of a point x in a metric space X is defined as the intersection of all clopen (open and closed) subsets of X which contain x([2]). The objective of this paper is to see how this concept is broadened to general topological spaces with a view of getting a result regarding the number of elements in a certain finite topological space.

It is known that a topological space is connected if and only if the only subsets which are clopen are the empty set and the whole space (page 148 of [1]).

In order to ensure a supply of clopen sets therefore, an equivalence relation is defined on a disconnected space. It turns out that the equivalence classes emanating from this relation, which gives a partition of the space, coincides with the quasicomponent $Q(x)$ for each $x \in X$.

We conclude that the number of elements in the quasicomponent for a certain finite disconnected topological space divides the number of elements in it.

2 The Quasicomponent as an Equivalence Class

Definition 2.1 *Let X be a topological space. The quasicomponent of $x \in X$ is defined by*

$$Q(x) = \bigcap_{\substack{C \subset X \ \text{clopen} \\ x \in C}} C$$

Lemma 2.2 *Let X be a disconnected topological space. Define a relation $\sim$ on X by $x \sim y \iff$ whenever C is a clopen set containing x, it also contains y. Then $\sim$ is an equivalence relation.*

Proof: Let $x \in X$ and let $x \in C$, C a clopen set
$\Rightarrow x \sim x \ \forall \ x \in C$
$\Rightarrow \sim$ is reflexive
Suppose $x \sim y$
$\Rightarrow$ Any clopen set C containing x contains y.
Let C be a clopen set containing y
We show that $x \in C$
but suppose that $x \notin C$
$\Rightarrow \ x \in C' = X - C$
Now C' is closed (C is open)
and C' is open (C is closed)
$\Rightarrow C'$ is also clopen which contain x
$\Rightarrow y \in C'$ (since $x \sim y$)
This is a contradiction since $y \in C$
The contradiction arose from the assumption that $x \notin C$
Thus any clopen set which contains y will also contain x
Therefore $x \sim y \ \Rightarrow \ y \sim x \ \forall \ x, y \in C$
$\Rightarrow \sim$ is symmetric
Let $x \sim y$ and $y \sim z$
Let C be a clopen set $\ni \ x \in C$
$\Rightarrow y \in C$ $(x \sim y)$
$\Rightarrow z \in C$ $(y \sim z)$
Therefore $x \sim y, \ y \sim z \ \Rightarrow \ x \sim z$
$\Rightarrow \sim$ is transitive.
Hence $\sim$ is an equivalence relation.

Remark 2.3 *(i)* $[x] = \{t \in X | x \sim t\}$ *is the equivalence class of* $x \in X$ *under the relation* $\sim$.

(ii) *The set of all equivalence classes under the relation* $\sim$ *above is denoted by* $X/\sim$ *and the number of equivalence classes by* $o(X/\sim)$

(iii) *The number of elements in an equivalence class for* $x \in X$ *will be denoted by* $o([x])$. *Likewise,* $o(X)$ *and* $o(Q(x))$ *will denote the number of elements in* X *and the quasicomponent of* $x \in X$ *respectively.*

Lemma 2.4 $[x] = Q(x) \ \forall \ x \in X$

Proof: Let $y \in [x]$
$\iff x \sim y$
$\iff$ Any clopen set $C \subset X$ which contains x must also contain y

$$\iff \ y \in \left(\bigcap_{\substack{C \subset X \text{ clopen} \\ x \in C}} \right)$$

$\iff y \in Q(x)$

Hence $[x] = Q(x)$.

Theorem 2.5 *If* X *is a finite disconnected topological space with* $o(X) = n$ *and* $o(Q(x)) = o(Q(y)) = k \ \forall \ x, y \in X$. *Then* k *divides* $o(X)$.

Proof: Since $\sim$ is an equivalence relation on X (Lemma 2.2) which is finite with $o(X) = n$, and $Q(x) = [x] \ \forall \ x \in X$ (Lemma 2.4), it follows that the equivalence classes partitions X. Suppose that $o(X/\sim) = m$; we will have that $n = km$.

References

[1] J. Munkres, *Topology,* Pearson New International Edition, 2014.

[2] L.G. Oversteegen, and E.D Tymchatyn, On the dimension of certain totally disconnected spaces, *Proc. Amer. Math. Soc.,* **122** (1994), 885 - 891. https://doi.org/10.1090/s0002-9939-1994-1273515-1

Received: September 12, 2023; Published: October 11, 2023