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Inertial Extragradient-Viscosity Algorithms for Nonmonotone and Non-Lipschitzian Equilibrium Problems in Hilbert Spaces

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Abstract

In this paper, combining line-search technique with viscosity method, inertial algorithm and subgradient algorithm, we propose a new iterative algorithm that does not involve in projection operators to solve the nonmonotone and non-Lipschitzian equilibrium problem in Hilbert space and obtain the strong convergence theorem. In addition, we also use our main result to the variational inequality and the convex minimization problem, and obtain the corresponding strong convergence theorems.

Keywords: equilibrium problem; viscosity method; line-search technique; strong convergence; Hilbert space

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1. Introduction

Let C be a nonempty closed convex subset of real Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$, $f : C\times C \to \mathbb{R}$ be an equilibrium bifunction, the equilibrium problem (EP) associated f and C, refer to Blum and Oettli [1] is to find a point $x^* \in C$ such that

$$f(x^*, y) \geq 0, \ \forall y \in C. \tag{1.1}$$

The solution set of $EP(f, C)$ is denoted by S_E. In addition, the Minty equilibrium problem [2,3] consists in finding a point $x^* \in C$ such that

$$f(y, x^*) \leq 0, \ \forall y \in C,$$

which is denoted by $MEP(f, C)$ and its solution set is denoted by S_M. It is well known that the inclusion $S_M \subset S_E$ holds provided that f is jointly weakly continuous on $C \times C$, this means $S_E \neq \emptyset$ if $S_M \neq \emptyset$.

The equilibrium problem is an important problem in the fields of nonlinear analysis and optimization. In recent years, equilibrium problem is widely used to solve optimization problem, variational inequality problem, saddle point problem, Nash equilibrium problem of non-cooperative game theory, fixed point problem, complementary problem and minimax problem, etc (See [1, 4–12]). For example, if the equilibrium bifunction $f(x, y) = \langle A(x), y - x\rangle$ for every $x, y \in C$, where $A : C \to H$ is a continuous mapping, the equilibrium problem is reduced to the variational inequality problem.

It is difficult to obtain the exact solution for most nonlinear problems. Therefore, constructing a reasonable iterative algorithm is the most commonly used method to solve these problems.

It is well known that the solution of $EP(f, C)$ is equivalent to the solution of the following strongly convex programming problem:

$$\min\{\lambda f(x, y) + \frac{1}{2}\|x - y\|^2 : y \in C\}, where\ \lambda > 0.$$

To solve the solution of $EP(f, C)$, when f is strong montone and Lipschitzian, Mastroeni [13] introduced the following iteration method and proved that it converges to a solution of $EP(f, C)$:

$$x_0 \in C, \ x_{k+1} = \arg\min\{\lambda f(x_k, y) + \frac{1}{2}\|x_k - y\|^2 : \ y \in C\}, where\ \lambda > 0.$$

In 2007, Takahashi, et al. [14] introduced the following algorithm combining viscosity algorithm with proximal method to solve the equilibrium problem: give $x_0 \in H$, compute, for all $n \in \mathbb{N}$,

$$\begin{cases} z_n \in C \ such\ that\ f(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0 \ \forall y \in C, \\ x_{n+1} = \theta_n g(x_n) + (1 - \theta_n)S(z_n), \end{cases}$$

where g is a contraction mapping on C and f is monotone. They proved that the sequence $\{x_n\}$ generated by this iteration scheme converges strongly to some $x^* \in S_E$.

Further when f is pseudomonotone and Lipschitzian, Flam and Antipin [15] introduced the following iteration method:

$$\begin{cases} x_0 \in C, \\ y_k = \arg\min\{\lambda_k f(x_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \\ x_{k+1} = \arg\min\{\lambda_k f(y_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \end{cases}$$

where $\lambda_k > 0$, they proved that the sequence $\{x_k\}$ generated by this iteration scheme weakly converges to a solution of $EP(f, C)$.

By using line-search technique and viscosity method, Vuone, et al. [16] proposed the following algorithm 1 to solve the solution of $EP(f, C)$, where f is pseudomonotone and jointly weakly continuous, and proved that the iterative sequence generated by algorithm 1 converges strongly to a solution of $EP(f, C)$:

Algorithm 1.

Step 0. Choose $\alpha \in (0, 2), \gamma \in (0, 1)$ and the sequences $\{\alpha_n\} \subset [0, 1), \{\beta_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, 1]$.

Step 1. Let $x_0 \in C$. Set $n = 0$

Step 2. Solve the strongly convex program $\min\limits_{y \in C} \left\{\lambda_n f(x_n, y) + \frac{1}{2}\|y - x_n\|^2\right\}$ to obtain the unique solution y_n.

Step 3. If $y_n = x_n$, then set $v_n = x_n$ and go to Step 4. Otherwise

Step 3.1. Find m the smallest non-negative integer such that

$$\begin{cases} v_{n,m} = (1 - \gamma_m)x_n + \gamma_m y_n \\ f(v_{n,m}, x_n) - f(v_{n,m}, y_n) \geq \frac{\alpha}{2\lambda_n}\|x_n - y_n\|^2. \end{cases}$$

Step 3.2. Set $\rho_n = \gamma_m, v_n = v_{n,m}$, and go to Step 4.

Step 4. Select $g_n \in \partial_2 f(v_n, x_n)$ and compute $z_n = P_{C \cap H_n}(x_n)$, where $H_n = \{x \in H | \langle g_n, x_n - x \geq f(v_n, x_n)\rangle\}$.

Step 5. Compute $x_{n+1} = (1 - \beta_n)t_n + \beta_n S t_n$, with $t_n = P_C(z_n - \alpha_n F z_n)$.

Step 6. Set $n := n + 1$, and go to Step 2.

In 2015, Dinh, et al. [17] proposed a projection algorithm to solve the solution of $EP(f, C)$ which is based on Bregman distance and Armijo-linesearch technique in Euclidean space and obtained the convergence of the iterative algorithm. In 2016, using Armijo-linesearch technique and projection algorithm, Dinh, et al. [18] introduced a new iterative algorithm for nonmonotone function f. In 2020, Deng and Fang [19] introducted the following algorithm 2 with the assumption that f is nonmontone and non-Lipschitzian:

Algorithm 2.

Step 0. Take $x_0 \in C$, choose parameters $\theta \in (0, 1)$ and $\rho > 0$, and set $k = 0$ and $B_0 = C$.

Step 1. Solve the strongly convex program

$$\min\{f(x_k, y) + \frac{\rho}{2}\|y - x_k\|^2 : y \in C\}.$$

to obtain its unique solution y_k. If $y_k = x_k$, stop. x_k is a solution of $EP(f, C)$. Otherwise, go to Step 2.

Step 2. (Armijo-linesearch) Find m_k as the smallest positive integer m satisfying

$$\begin{cases} z_{k,m} = (1-\theta_m)x_k + \theta_m y_k, \\ g_{k,m} \in \partial_2 f(z_{k,m}, z_{k,m}), \\ \langle g_{k,m}, x_k - y_k\rangle \geq \frac{\rho}{2}\|x_k - y_k\|^2. \end{cases}$$

Step 3. Set $\theta_k = \theta_{m_k}, z_k = z_{k,m_k}, g_k = g_{k,m_k}$. Take $\omega_k \in \partial_2 f(z_k, x_k)$ and

$$H_k = \{x \in \mathbb{H} : \langle \omega_k, x_k - x\rangle \geq f(z_k, x_k)\}.$$

Step 4. Compute $\sigma_k = \frac{f(z_k,x_k)}{\|\omega_k\|^2}$ and

$$u_k = P_{C\cap H_k}(x_k - \gamma_k \sigma_k \omega_k).$$

Step 5. Take $B_{k+1} = \{x \in B_k : \|x - u_k\| \leq \|x - x_k\|\}$ and compute

$$x_{k+1} = P_{B_{k+1}}(x_0).$$

Set $n := n + 1$ and go to Step 1.

Although the algorithm 2 does not depend on monotonicity and Lipschitz-type property of the equilibrium function f and has strong convergence, it needs to compute two projections at each iteration step. Therefore, we focus on constructing a new iterative algorithm which does not involve projections for the equilibrium function without monotonicity and Lipschitzian property.

2. Preliminaries

In this section, we show some basic definitions and common lemmas that will be used in subsequent chapters.

Let H be a real Hilbert space and C be a non-empty closed convex subset of H. We use $\rightharpoonup$ and $\to$ to describe weak convergence and strong convergence respectively.

For any $x, y \in H$ and $\alpha \in \mathbb{R}$, clearly,

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \tag{2.1}$$

and

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2. \tag{2.2}$$

Lemma 2.1. *(Lemma 1 in [19]) The following properties hold:*

(i) $z = P_C(x)$ *if and only if* $\langle x - z, y - z\rangle \leq 0,\ \forall y \in C$;

(ii) $\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2 - \|P_C(x) - x + y - P_C(y)\|^2,\ \forall x, y \in C.$

Definition 2.2. (Definition 1 in [19]) A function $f : \mathbb{H} \to [-\infty, +\infty]$ is said to lower semicontinuous at $x \in \mathbb{H}$ if for any sequence $\{x_k\}$, which converges to x, the $f(x) \leq \liminf_{k\to\infty} f(x_k)$ holds. A function f said to be upper semicontinuous at $x \in \mathbb{H}$ if $-f$ is lower semicontinuous at x. If f is lower semicontinuous and upper semicontinuous at $x \in \mathbb{H}$, then it is continuous at $x \in \mathbb{H}$. Furthermore, f is continuous on C if it is continuous at each $x \in C \subset \mathbb{H}$.

Definition 2.3. (Definition 2.1 in [18]) A bifunction $f : C \times C \to \mathbb{R}$ is said to be jointly weakly continuous on $C \times C$ if for all $x, y \in C$ and sequences $\{x_k\}, \{y_k\}$ in C weakly converging to x and y, respectively, then $f(x_k, y_k)$ converges to $f(x, y)$ as $k \to \infty$.

To solve the solution of $EP(f, C)$, we need to make the following assumptions:
(A1) $f(\cdot, y)$ is convex on C for each $y \in C$;
(A2) f is jointly weakly continuous on $C \times C$;
(A3) f is nonmonotone and non-Lipschitzian;
(A4) $S_M \neq \emptyset$.

Definition 2.4. (Definition 3 in [19]) Let $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be a function such that $f(x, \cdot)$ is convex for all $x \in \mathbb{H}$. For $x, y \in \mathbb{H}$, the subdifferential $\partial_2 f(x, y)$ of $f(x, \cdot)$ at y is defined by

$$\partial_2 f(x, y) = \{\beta \in \mathbb{H} : f(x, z) - f(x, y) \geq \langle \beta, z - y\rangle, \forall z \in \mathbb{H}\}.$$

Lemma 2.5. *(Lemma 2 in [19]) Let $f(x, y)$ be a convex differentiable function with respect to y at $x = x^* \in C$ and $\rho > 0$. Then x^* is a solution of $EP(f, C)$ if and only if it is a solution of the auxiliary equilibrium problem:*

$$\textit{Find} \;\; x^* \in C : \;\; f(x^*, y) + \frac{\rho}{2}\|y - x^*\|^2 \geq 0, \quad \forall y \in C.$$

Lemma 2.6. *(Lemma 2.5 in [18]) With assumptions (A1) and (A2), if $\{z_k\} \subset C$ is a sequence converging strongly to $\overline{z}$ and the sequence $\{w_k\} \subseteq \partial_2 f(z_k, z_k)$, converges weakly to $\overline{w}$, then $\overline{w} \in \partial_2 f(\overline{z}, \overline{z})$.*

Lemma 2.7. *(Proposition 2.1 in [20]) With assumptions (A1) and (A2), for $\overline{x}, \overline{y} \in C$ and sequence $\{x_k\}, \{y_k\}$ in C converging weakly to $\overline{x}$ and $\overline{y}$ respectively, it follows that for any $\varepsilon > 0$, there exist $\xi > 0$ and $k_\xi \in \mathbb{N}$ such that*

$$\partial_2 f(x_k, y_k) \subset \partial_2 f(\overline{x}, \overline{y}) + \frac{\varepsilon}{\xi} B,$$

for every $k \geq k_\varepsilon$, where B denotes the closed unit ball in $\mathbb{H}$.

Lemma 2.8. *(Lemma 2.6 in [19]) With assumptions (A1) and (A2), if $\{x_k\} \subset C$ is bounded, $\rho > 0$, and $\{y_k\}$ is a sequence such that*

$$y_k = \arg\min\{f(x_k, y) + \frac{\rho}{2}\|y - x_k\|^2 : y \in C\},$$

then $\{y_k\}$ is bounded.

Lemma 2.9. *(Lemma 1 in [21]) Assume that* $\{s_n\}_{n=0}^{\infty}$ *is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n\delta_n, \forall n \geq 0,$$
$$s_{n+1} \leq s_n - \mu_n + \phi_n, \forall n \geq 0,$$

where $\{\lambda_n\}$ *is a sequence in (0,1),* $\{\phi_n\}_{n=0}^{\infty}$ *is a sequence of nonnegative real numbers, and* $\{\delta\}_{n=0}^{\infty}$*,* $\{\phi_n\}_{n=0}^{\infty}$ *are two real sequences such that*

(i) $\sum_{n=0}^{\infty}\lambda_n = \infty$*;*

(ii) $\lim_{n\to\infty}\phi_n = 0$*;*

(iii) $\lim_{k\to\infty}\mu_{n_k} = 0$ *implies* $\limsup_{n\to\infty}\delta_{n_k} \leq 0$ *for any subsequence* $\{n_k\}_{k=0}^{\infty} \subset \{n\}_{n=0}^{\infty}$. *Then* $\lim_{n\to\infty} s_n = 0$.

3. Main results

Motivated by the work of Deng and Fang [19], when equilibrium bifunction f is nonmonotone and non-Lipschitzian in Hilbert spaces, we combine the viscosity algorithm with the subgradient algorithm to construct a new iterative algorithm 3 to approximate a solution of the equilibrium problem. The strong convergence of the algorithm 3 is proved by referring to the proof methods in Thong [21] and Xie [22]. The new algorithm does not depend on the projection operator.

Before giving the algorithm, we need to make some assumptions:

(B1) $T : C \to H$ is a contraction mapping with a constant $c \in [0, 1)$;

(B2) $\{\alpha_n\}, \{\eta_n\}$ are two sequencesa in [0,1) statisfying the following conditions:

$$\lim_{n\to\infty}\alpha_n = 0, \ \lim_{n\to\infty}\eta_n = 0, \ \sum_{n=1}^{\infty}\eta_n = \infty.$$

(B3) $\{\gamma_n\}$ is a sequence in (0,2) and statisfies $\liminf_{n\to\infty}\gamma_n(2-\gamma_n) > 0$;

(B4) $\{\tau_n\} \subset [0, \theta)$ for some $\theta > 0$ is a positive sequence such that $\lim_{n\to\infty}\frac{\tau_n}{\eta_n} = 0$.

(B5) $f(\cdot, \cdot)$ is a equilibrium bifunction and statisfies assumption A1-A3.

Algorithm 3.

Initialization. Give $\eta > 0$. arbitrarily chosen $x_0, x_1 \in C$.

Iterative Steps. Calculate x_{n+1} as follows:

Step 1. Give the sequence x_{n-1} and $x_n (n \geq 0)$, choose θ_n such that $0 \leq \theta_n \leq \overline{\theta_n}$, where

$$\overline{\theta_n} = \begin{cases} \min\{\theta, \frac{\tau_n}{\|x_n - x_{n-1}\|}\} & if \quad x_n \neq x_{n-1}, \\ \theta & if \quad otherwise. \end{cases} \tag{3.1}$$

Step 2. Set $w_n = x_n + \theta_n(x_n - x_{n-1})$.

Step 3. Solve the strongly convex program:

$$y_n = \arg\min\{f(w_n, y) + \frac{\rho}{2}\|y - w_n\|^2 : y \in C\}. \tag{3.2}$$

Get the solution y_n. If $y_n = w_n$, stop. w_n is an element of S_E. Otherwise, go to **Step 4**.

Step 4. (Line-search) Find the smallest positive integer m to satisfy:

$$\begin{cases} u_{n,m} = (1-\alpha_m)w_n + \alpha_m y_n, \\ f(u_{n,m}, w_n) - f(u_{n,m}, y_n) \geq \frac{\rho}{2}\|w_n - y_n\|^2. \end{cases} \tag{3.3}$$

Step 5. Set $\alpha_n = \alpha_{n,m}, u_n = u_{n,m}$, Let $d_n \in \partial_2 f(u_n, w_n)$ and satisfy:

$$\langle w_n - y, d_n\rangle \geq f(u_n, w_n), \forall y \in C. \tag{3.4}$$

Step 6. Compute $z_n = w_n - \gamma_n e_n d_n$, where $e_n = \frac{f(u_n,w_n)}{\|d_n\|^2}$.

Step 7. Compute $x_{n+1} = (1-\eta_n)z_n + \eta_n T(z_n)$.

Set $n := n+1$ and go to **Step 1**.

Lemma 3.1. *The line-search rule* (3.3) *is well defined when assume A1-A4, B2 hold and $w_n \neq y_n$.*

Proof. Assume that (3.3) is not true. So there exists non-negative integer m such that:

$$\begin{cases} u_{n,m} = (1-\alpha_m)w_n + \alpha_m y_n, \\ f(u_{n,m}, w_n) - f(u_{n,m}, y_n) < \frac{\rho}{2}\|w_n - y_n\|^2. \end{cases}$$

Since $\lim\limits_{m\to\infty} \alpha_m = 0$, we have $u_{n,m} \to w_n (m \to \infty)$. Set $u_{n,m} = w_n$. Since f is jointly weakly continuous and $f(w_n, w_n) = 0$, we have

$$f(w_n, y_n) + \frac{\rho}{2}\|w_n - y_n\|^2 \geq 0.$$

On the other hand, since y_n is a solution of the strongly convex programming problem, it can be obtained

$$f(w_n, y_n) + \frac{\rho}{2}\|w_n - y_n\|^2 \leq f(w_n, y) + \frac{\rho}{2}\|w_n - y\|^2, \forall y \in C.$$

we set $y = w_n$ and get $f(w_n, y_n) + \frac{\rho}{2}\|w_n - y_n\|^2 \leq 0$, namely, $\frac{\rho}{2}\|w_n - y_n\|^2 = 0$, which implies that $w_n = y_n$, we get contradiction. □

Lemma 3.2. *If the algorithm stops at some iteration points w_n, Then w_n is an element in S_E.*

Proof. When the algorithm 3 stops at some iteration points w_n, which means $y_n = w_n$. Due to $f(w_n, w_n) = 0$, so

$$f(w_n, y_n) + \frac{\rho}{2}\|y_n - w_n\|^2 = 0.$$

Since y_n is a solution of the strongly convex programming problem, the following holds:

$$f(w_n, y_n) + \frac{\rho}{2}\|w_n - y_n\|^2 \leq f(w_n, y) + \frac{\rho}{2}\|w_n - y\|^2, \forall y \in C.$$

Thus we have $f(w_n, y) + \frac{\rho}{2}\|y - w_n\|^2 \geq 0$. By Lemma 2.5, we have $w_n \in S_E$. □

Lemma 3.3. *If the assumption A1-A4 hold, then the d_n in algorithm 3 exists.*

Proof. If $S_M \neq \emptyset$, it means that there exists $x^* \in S_M$, such that $f(y, x^*) \leq 0$ for any $y \in C$. Set $y = u_n$, we have $f(u_n, x^*) \leq 0$.

Taking $e \in \partial_2 f(u_n, w_n)$, from the convexity of $f(u_n, \cdot)$, we get

$$f(u_n, y) - f(u_n, w_n) \geq \langle e, y - w_n\rangle,\ \forall y \in C.$$

Set $y = x^*$ and $e = d_n$, then we have $\langle d_n, w_n - x^*\rangle \geq f(u_n, w_n) - f(u_n, x^*) \geq f(u_n, w_n)$. Hence, the d_n in algorithm 3 exists. □

Lemma 3.4. *Let $\{w_n\}$ and $\{y_n\}$ be the two sequences generated by algorithm 3. If $w_{n_k} \rightharpoonup w$ as $k \to \infty$ and $\lim_{k\to\infty} \|y_{n_k} - w_{n_k}\| = 0$, then $w \in S_E$.*

Proof. Let i_C be the indicator function of C, i.e.

$$i_C y = \begin{cases} 0 & y \in C, \\ \infty & y \notin C. \end{cases}$$

From the definition of y_n in algorithm 3, we get

$$y_n = \arg\min\{f(w_n, y) + \frac{\rho}{2}\|y - w_n\|^2 + i_C y\}.$$

According to generalized Fermat's theorem, we have

$$0 \in \partial_2 f(w_n, y_n) + \rho(y_n - w_n) + N_C(y_n),$$

where $N_C(y_n)$ is the norm cone of C at y_n, which is defined by

$$N_C(y_n) = \{\zeta \in H : \langle \zeta, y - y_n\rangle \leq 0,\ \forall y \in C\}.$$

Namely, there exists $\beta \in \partial_2 f(w_n, y_n)$ such that $0 = \beta + \rho(y_n - w_n) + \zeta$, so $\beta = \rho(w_n - y_n) - \zeta$.

On the other hand, due to $\beta \in \partial_2 f(w_n, y_n)$, we have

$$\begin{aligned} f(w_n, y) - f(w_n, y_n) &\geq \langle \beta, y - y_n\rangle \\ &= \langle \rho(w_n - y_n - \zeta), y - y_n\rangle \\ &= \langle \rho(w_n - y_n), y - y_n\rangle - \langle \zeta, y - y_n\rangle. \end{aligned}$$

Since $w_{n_k} \rightharpoonup w$ and $\lim_{k\to\infty} \|y_{n_k} - w_{n_k}\| = 0$, we get $y_{n_k} \rightharpoonup w$. Taking the limit as $k \to \infty$, since f be jointly weakly continuous, we obtain

$$f(w, y) - f(w, w) \geq \rho\langle w - w, y - w\rangle = 0,$$

this implies that

$$f(w, y) \geq 0,\ \forall y \in C.$$

Therefore $w \in S_E$. □

Theorem 3.5. *Let $\{x_n\}$ be the sequence generated by algorithm 3, assume that A1-A4 and B1-B4 hold. Then the sequence $\{x_n\}$ converges strongly to an element $q \in S_E$, where $q = P_{S_E}(T(q))$.*

Proof. We divide the proof into six claims.

Claim 1. For any certain point $p \in S_E$, the following inequality holds:

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \frac{2-\gamma_n}{\gamma_n}\|z_n - w_n\|^2.$$

Indeed, by (3.4), we have

$$\langle w_n - p, d_n\rangle \geq f(u_n, w_n) = e_n\|d_n\|^2. \tag{3.5}$$

It follows from the definition of z_n and (3.5) that

$$\begin{aligned}\|z_n - p\|^2 &= \|w_n - \gamma_n e_n d_n - p\|^2\\ &= \|w_n - p\|^2 - 2\gamma_n e_n\langle w_n - p, d_n\rangle + \gamma_n^2 e_n^2\|d_n\|^2\\ &\leq \|w_n - p\|^2 - \gamma_n(2-\gamma_n)(e_n\|d_n\|)^2\\ &= \|w_n - p\|^2 - \frac{2-\gamma_n}{\gamma_n}\|z_n - w_n\|^2.\end{aligned} \tag{3.6}$$

Therefore, the claim 1 holds.

Claim 2. We prove that the inequality $\|w_n - y_n\|^2 \leq \frac{2}{\rho(\gamma_n e_n)^2}\|z_n - w_n\|^2$ holds.

By line-search technique (3.3), we get

$$\begin{aligned}\|w_n - y_n\|^2 &\leq \frac{2}{\rho}(f(u_n, w_n) - f(u_n, y_n))\\ &\leq \frac{2}{\rho}\langle w_n - y_n, d_n\rangle \leq \|w_n - y_n\|\|d_n\|.\end{aligned} \tag{3.7}$$

Thus, we have

$$\frac{\rho}{2}\|w_n - y_n\| \leq \|d_n\|. \tag{3.8}$$

From the definition of z_n and (3.8), we get

$$\|z_n - w_n\|^2 = \gamma_n^2 e_n^2\|d_n\|^2 \geq \frac{\rho\gamma_n^2 e_n^2}{2}\|w_n - y_n\|^2. \tag{3.9}$$

It follow from (3.9) that $\|w_n - y_n\|^2 \leq \frac{2}{\rho(\gamma_n e_n)^2}\|z_n - w_n\|^2$, so the claim 2 is proved.

Claim 3. Next, we prove that the $\{x_n\}$ is bounded.
Indeed, according to claim 1, we have

$$\|z_n - p\| \leq \|w_n - p\|. \tag{3.10}$$

Thus

$$\begin{aligned}
\|x_{n+1}-p\| &= \|(1-\eta_n)z_n+\eta_n T(z_n)\| \\
&= \|(1-\eta_n)(z_n-p)+\eta_n(T(z_n-p))\| \\
&\le (1-\eta_n)\|z_n-p\|+\eta_n\|T(z_n)-p\| \\
&\le (1-\eta_n)\|z_n-p\|+\eta_n\|T(z_n)-T(p)\|+\eta_n\|T(p)-p\| \\
&\le \big(1-\eta_n(1-c)\big)\|z_n-p\|+\eta_n\|T(p)-p\| \\
&\le \big(1-\eta_n(1-c)\big)\|w_n-p\|+\eta_n\|T(p)-p\|.
\end{aligned} \tag{3.11}$$

By (3.1), we have $\theta_n\|x_n-x_{n-1}\|\le\tau_n$ for all $n\ge 1$. In addition, since $\lim\limits_{n\to\infty}\frac{\tau_n}{\eta_n}=0$, we have $\lim\limits_{n\to\infty}\frac{\theta_n}{\eta_n}\|x_n-x_{n-1}\|\le\lim\limits_{n\to\infty}\frac{\tau_n}{\eta_n}=0$. Therefore, there exists a contant $M_1\ge 0$, such that

$$\frac{\theta_n}{\eta_n}\|x_n-x_{n-1}\|\le M_1, \forall n\ge 1. \tag{3.12}$$

By the definition of w_n and (3.12), we get

$$\begin{aligned}
\|w_n-p\| &= \|x_n+\theta_n(x_n-x_{n-1})-p\| \\
&\le \|x_n-p\|+\theta_n\|x_n-x_{n-1}\| \\
&= \|x_n-p\|+\eta_n\cdot\frac{\theta_n}{\eta_n}\|x_n-x_{n-1}\| \\
&\le \|x_n-p\|+\eta_n M_1.
\end{aligned} \tag{3.13}$$

Substitute (3.13) into (3.11), we have

$$\begin{aligned}
\|x_{n+1}-p\| &\le \big(1-\eta_n(1-c)\big)\|w_n-p\|+\eta_n\|T(p)-p\| \\
&\le \big(1-\eta_n(1-c)\big)(\|x_n-p\|+\eta_n M_1)+\eta_n\|T(p)-p\| \\
&= \big(1-\eta_n(1-c)\big)\|x_n-p\|+(1-\eta_n(1-c))\eta_n M_1+\eta_n\|T(p)-p\| \\
&\le \big(1-\eta_n(1-c)\big)\|x_n-p\|+\eta_n M_1+\eta_n\|T(p)-p\| \\
&= \big(1-\eta_n(1-c)\big)\|x_n-p\|+\eta_n(1-c)\cdot\frac{M_1+\|T(p)-p\|}{1-c} \\
&\le \max\left\{\|x_n-p\|,\frac{M_1+\|T(p)-p\|}{1-c}\right\} \\
&\le \max\left\{\|x_{n-1}-p\|,\frac{M_1+\|T(p)-p\|}{1-c}\right\} \\
&\le \cdots \\
&\le \max\left\{\|x_0-p\|,\frac{M_1+\|T(p)-p\|}{1-c}\right\}.
\end{aligned} \tag{3.14}$$

It implies that the sequence $\{x_n\}$ is bounded.

Claim 4. We prove that

$$\frac{2-\gamma_n}{\gamma_n}\|z_n - w_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \eta_n M_4. \tag{3.15}$$

Indeed, from the definition of x_{n+1}, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \|(1-\eta_n)z_n + \eta_n T(z_n) - p\|^2 \\
&= \|(1-\eta_n)(z_n - p) + \eta_n(T(z_n) - p)\|^2 \\
&= (1-\eta_n)\|z_n - p\|^2 + \eta_n\|T(z_n) - p\|^2 - \eta_n(1-\eta_n)\|z_n - T(z_n)\|^2 \\
&\le (1-\eta_n)\|z_n - p\|^2 + \eta_n\|T(z_n) - p\|^2 \\
&\le (1-\eta_n)\|z_n - p\|^2 + \eta_n(\|T(z_n - T(p))\| + \|T(p) - p\|)^2 \\
&\le (1-\eta_n)\|z_n - p\|^2 + \eta_n\big((1-c)\|z_n - p\| + \|T(p) - p\|\big)^2 \\
&\le (1-\eta_n)\|z_n - p\|^2 + \eta_n(\|z_n - p\| + \|T(p) - p\|)^2 \\
&= \|z_n - p\|^2 + \eta_n(2\|z_n - p\|\|T(p) - p\| + \|T(p) - p\|^2).
\end{aligned} \tag{3.16}$$

From claim 3, we can get

$$\|z_n - p\| \le \|w_n - p\| \le \|x_n - p\| + \eta_n M_1.$$

By the assumption B3, we have $\lim_{n\to\infty} \eta_n = 0$. So $\|z_n - p\|$ is bouned, i.e. there exists constant N such that $\|z_n - p\| \le N$, Further we get

$$\|x_{n+1} - p\|^2 \le \|z_n - p\|^2 + \eta_n M_2,$$

where $M_2 := 2N\|T(p) - p\| + \|T(p) - p\|^2$.
Form claim 1 and (3.13), we get

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \frac{2-\gamma_n}{\gamma_n}\|z_n - w_n\|^2, \tag{3.17}$$

and

$$\begin{aligned}
\|w_n - p\|^2 &\le (\|x_n - p\| + \eta_n M_1)^2 \\
&= \|x_n - p\|^2 + \eta_n(2M_1\|x_n - p\| + \eta_n M_1^2) \\
&\le \|x_n - p\|^2 + \eta_n M_3,
\end{aligned} \tag{3.18}$$

where $M_3 := 2M_1\|x_n - p\| + \eta_n M_1^2$.
It follows from (3.16), (3.17) and (3.18) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \|w_n - p\|^2 - \frac{2-\gamma_n}{\gamma_n}\|z_n - w_n\|^2 + \eta_n M_2 \\
&\le \|x_n - p\|^2 - \frac{2-\gamma_n}{\gamma_n}\|z_n - w_n\|^2 + \eta_n M_4,
\end{aligned} \tag{3.19}$$

where $M_4 := M_2 + M_3$. The proof of claim 4 is completed.

Claim 5. We prove that

$$\|x_{n+1}-p\|^2 \leq \big(1-\eta_n(1-c)\big)\|x_n-p\|^2 + \eta_n(1-c)\Big[\frac{2}{1-c}\langle T(p)-p, x_{n+1}-p\rangle + \frac{3M}{1-c}\cdot\frac{\theta_n}{\eta_n}\|x_n-x_{n+1}\|\Big].$$

By the definition of x_{n+1} and (3.10), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\eta_n)z_n+\eta_n T(z_n)-p\|^2\\
&= \|(1-\eta_n)(z_n-p)+\eta_n(T(z_n)-T(p))+\eta_n(T(p)-p)\|^2\\
&\leq \|(1-\eta_n)(z_n-p)+\eta_n(T(z_n)-T(p))\|^2+2\eta_n\langle T(p)-p, x_{n+1}-p\rangle\\
&\leq (1-\eta_n)\|z_n-p\|^2+\eta_n\|T(z_n)-T(p)\|^2+2\eta_n\langle T(p)-p, x_{n+1}-p\rangle\\
&\leq (1-\eta_n)\|z_n-p\|^2+\eta_n c^2\|z_n-p\|^2+2\eta_n\langle T(p)-p, x_{n+1}-p\rangle\\
&\leq (1-\eta_n(1-c))\|z_n-p\|^2+2\eta_n\langle T(p)-p, x_{n+1}-p\rangle\\
&\leq (1-\eta_n(1-c))\|w_n-p\|^2+2\eta_n\langle T(p)-p, x_{n+1}-p\rangle.
\end{aligned}\tag{3.20}$$

In addition, from the definition of w_n, we have

$$\begin{aligned}
\|w_n\|^2 &= \|x_n+\theta_n(x_n-x_{n-1})-p\|^2\\
&= \|x_n-p\|^2+2\theta_n\langle x_n-p, x_n-x_{n-1}\rangle+\theta_n^2\|x_n-x_{n-1}\|^2\\
&\leq \|x_n-p\|^2+2\theta_n\|x_n-p\|\|x_n-x_{n-1}\|+\theta_n^2\|x_n-x_{n-1}\|^2.
\end{aligned}\tag{3.21}$$

So, combining (3.20) with (3.21), we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\eta_n(1-c))\big[\|x_n-p\|^2+2\theta_n\|x_n-p\|\|x_n-x_{n-1}\|\\
&\quad+\theta_n^2\|x_n-x_{n-1}\|^2\big]+2\eta_n\langle T(p)-p, x_{n+1}-p\rangle\\
&= (1-\eta_n(1-c))\|x_n-p\|^2+2\theta_n(1-\eta_n(1-c))\|x_{n-p}\|\|x_n-x_{n-1}\|\\
&\quad+\theta_n^2(1-\eta_n(1-c))\|x_n-x_{n-1}\|^2+2\eta_n\langle T(p)-p, x_{n+1}-p\rangle\\
&\leq (1-\eta_n(1-c))\|x_n-p\|^2+2\theta_n\|x_n-p\|\|x_n-x_{n-1}\|\\
&\quad+\theta_n^2\|x_n-x_{n-1}\|^2+2\eta_n\langle T(p)-p, x_{n+1}-p\rangle\\
&\leq (1-\eta_n(1-c))\|x_n-p\|^2+\eta_n(1-c)\big[\frac{2}{1-c}\langle T(p)-p, x_{n+1}-p\rangle\big]\\
&\quad+3M\theta_n\|x_n-x_{n-1}\|\\
&\leq (1-\eta_n(1-c))\|x_n-p\|^2+\eta_n(1-c)\big[\frac{2}{1-c}\langle T(p)-p, x_{n+1}-p\rangle\\
&\quad+\frac{3M}{1-c}\cdot\frac{\theta_n}{\eta_n}\|x_n-x_{n-1}\|\big],
\end{aligned}\tag{3.22}$$

where $M=\sup\limits_{n\in N}\{\|x_n-p\|, \theta\|x_n-x_{n-1}\|\}$.

Claim 6. We prove that $\{x_n\}$ converges strongly to q, where $q=P_{S_E}(T(q))$.

Indeed, we set

$$\begin{aligned} s_n &= \|x_n - q\|, \quad \lambda_n = \eta_n(1-c), \\ \delta_n &= \frac{2}{1-c}\langle T(q) - q, x_{n+1} - q\rangle + \frac{3M}{1-c}\cdot\frac{\theta_n}{\eta_n}\|x_n - x_{n-1}\|, \\ \mu_n &= \frac{2-\gamma_n}{\gamma_n}\|z_n - w_n\|, \quad \phi_n = \eta_n M_4. \end{aligned}$$

By combining Lemma 2.9 with claim 4 and claim 5, we have

$$\begin{aligned} s_{n+1} &\le (1-\lambda_n)s_n + \lambda_n\delta_n, \\ s_{n+1} &\le s_n - \mu_n + \phi_n. \end{aligned}$$

It is easy to prove that condition *(i)* and *(ii)* of Lemma 2.9 are satisfied. By (B1), we have $\sum\limits_{n=0}^{\infty}\lambda_n = \infty$ and

$$\lim_{n\to\infty}\phi_n = 0.$$

Next, we just have to prove $\limsup\limits_{k\to\infty}\delta_{n_k} \le 0$.

From $\lim\limits_{k\to\infty}\mu_{n_k} = 0$, we get

$$\lim_{k\to\infty}\|z_{n_k} - w_{n_k}\| = 0. \tag{3.23}$$

By claim 2, we have

$$\lim_{k\to\infty}\|w_{n_k} - y_{n_k}\| = 0. \tag{3.24}$$

By using $\lim\limits_{n\to\infty}\eta_n = 0$ and $x_{n+1} = (1-\eta_n)z_n + \eta_n T(z_n)$, we get

$$\lim_{k\to\infty}\|x_{n_k+1} - z_{n_k}\| = \lim_{k\to\infty}\eta_{n_k}\|z_{n_k} - T(z_{n_k})\| = 0, \tag{3.25}$$

and

$$\lim_{k\to\infty}\|x_{n_k} - w_{n_k}\| = \lim_{k\to\infty}\theta_{n_k}\|x_{n_k} - x_{n_k-1}\| = \lim_{k\to\infty}\eta_{n_k}\cdot\frac{\theta_{n_k}}{\eta_{n_k}}\|x_{n_k} - x_{n_k-1}\| = 0. \tag{3.26}$$

Furthermore, we get

$$\|x_{n_k+1} - x_{n_k}\| \le \|x_{n_k+1} - z_{n_k}\| + \|z_{n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\| \to 0. \tag{3.27}$$

From claim 3, we known that $\{x_{n_k}\}$ is bounded. So there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$, which converges weakly to some $w \in H$, such that

$$\limsup_{k\to\infty}\langle T(q) - q, x_{n+1} - q\rangle = \lim_{j\to\infty}\langle T(q) - q, x_{n_{k_j}} - q\rangle = \langle T(q) - q, w - q\rangle. \tag{3.28}$$

From (3.26), we known that $w_{n_{k_j}} \rightharpoonup w$. Therefore it follows from (3.24) and Lemma 3.4 that $w \in S_E$. Further from (3.28), we have

$$\limsup_{k\to\infty}\langle T(q) - q, x_{n_k}\rangle = \langle T(q) - q, w - q\rangle \le 0. \tag{3.29}$$

Combining (3.27) with (3.29), we get

$$\begin{aligned}
&\limsup_{k\to\infty}\langle T(q)-q, x_{n_k+1}-q\rangle \\
(3.30)\quad &\le \limsup_{k\to\infty}\langle T(q)-q, x_{n_k+1}-x_{n_k}\rangle + \limsup_{k\to\infty}\langle T(q)-q, x_{n_k}-q\rangle \\
&\le 0.
\end{aligned}$$

Hence, by Lemma 2.9, we conclude that

$$\lim_{n\to\infty}\|x_n-q\|=0.$$

The proof of the Theorem 3.5 is completed. □

4. Applications

In this section, we apply the main results to solve variational inequality problem and convex minimization problem in Hilbert spaces.

(I) Application to the variational inequality problem

Let C be a nonempty closed convex subset of real Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Assume that $A : C \to H$ is a mapping. Set $f(x,y) = \langle A(x), y-x\rangle$, the equilibrium problem reduces to the variational inequality problem: find $x^* \in C$ such that

$$\langle A(x^*), y-x^*\rangle \ge 0 \text{ for every } y \in C.$$

The set of solution of the variational inequality problem is denoted S_{VI}. In order to obtain the corresponding algorithm 4 and Theorem 4.1, we assume that the following assumptions on A hold:

(C1) A is convex and subdifferentiable on H;

(C2) A is weakly continuous for on C, i.e if $\{x_n\}$ is a sequences in C and $x_n \rightharpoonup x$, then $A(x_n) \to A(x)$;

(C3) The set solution of $\langle Ay, x^*-y\rangle \le 0$ is non-empty.

Algorithm 4.

Initialization. Give $\eta > 0$.Let $x_0, x_1 \in C$ be arbitrary.

Iterative Steps. Calculate x_{n+1} as follows:

Step 1. Give the iterates x_{n-1} and $x_n (n \ge 0)$, choose θ_n such that $0 \le \theta_n \le \overline{\theta_n}$, where

$$\overline{\theta_n} = \begin{cases} \min\{\theta, \frac{\tau_n}{\|x_n-x_{n-1}\|}\} & if \quad x_n \ne x_{n-1}, \\ \theta & if \quad otherwise. \end{cases} \tag{4.1}$$

Step 2. Set $w_n = x_n + \theta_n(x_n - x_{n-1})$.

Step 3. Compute

$$y_n = P_C(w_n - \frac{1}{\rho}A(w_n)).$$

Get the solution y_n. If $y_n = w_n$, stop. w_n is a element of S_{VI}. Otherwise, go to Step4.

Step4. (line-search) Find m_n in the smallest positive integer m to satisfy:

$$\begin{cases} u_{n,m} = (1-\alpha_m)w_n + \alpha_m y_n, \\ \langle A(u_{n,m}), w_n - y_n\rangle \geq \frac{\varrho}{2}\|w_n - y_n\|^2. \end{cases}$$

Step5. Set $\alpha_n = \alpha_{n,m}, u_n = u_{n,m}$, Let $d_n \in \partial_2\langle A(u_n), w_n - u_n\rangle$ and satisfy the following condition:

$$\langle w_n - y, d_n\rangle \geq \langle A(u_n), w_n - u_n\rangle, \forall y \in C.$$

Step6. Compute $z_n = w_n - \gamma_n e_n d_n$, where $e_n = \frac{\langle A(u_n), w_n - u_n\rangle}{\|d_n\|^2}$.

Step7. Compute $x_{n+1} = (1-\eta_n)z_n + \eta_n T(z_n)$.

Set $n := n+1$ and go to **Step1**.

Theorem 4.1. *Let $\{x_n\}$ be the sequence generated by algorithm 4 and assume that C1-C2 and B1-B4 hold. The sequence $\{x_n\}$ converges strongly to an element $p \in S_{VI}$, where $p = P_{S_{VI}}(T(p))$.*

(II) Application to the convex minimization problem

Let $\varphi : C \to \mathbb{R}$ be a convex and differentiable function. The convex minimization problem is as follows:

$$\text{minimize } \varphi(x), \text{ where } x \in C.$$

The set of solution of the convex minimization problem is denoted by S_{CMP}. It is well known that a point $x^* \in C$ is a solution of the convex minimization problem if and only if it is a solution of the following variational inequality problem (see [23]):

$$\langle \nabla\varphi(x^*), y - x^*\rangle \geq 0, \ \forall y \in C.$$

Obviously, set $A(x) = \nabla\varphi(x)$, the convex minimization problem reduces to the variational inequality problem. In order to obtain the corresponding algorithm 5 and Theorem 4.2, we assume that the following assumptions on φ hold:

(D1) $\nabla\varphi$ is convex and subdifferentiable on $\mathbb{R}$;

(D2) $\nabla\varphi$ is weakly continuous for on $\mathbb{R}$, i.e if $\{x_n\}$ are sequences in C and convergs weakly to x, then $\nabla\varphi(x_n) \to \nabla\varphi(x)$;

(D3) The set solution of $\langle \nabla\varphi(y), x^* - y\rangle \leq 0$ is non-empty.

Therefore, we get the following Theorem 4.2.

Theorem 4.2. *Let $\{x_n\}$ be the sequence generated by algorithm 4 and assume B1-B4 and D1-D3 hold. The sequence $\{x_n\}$ converges strongly to an element $p \in S_{CMP}$, where $p = P_{S_{CMP}}(T(p))$.*

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