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On the Fremlin Projective Tensor Product of Banach d-Algebras and Almost f-Algebras

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Abstract

In this study, we present the Fremlin projective tensor product of Banach d-algebras and Banach almost f-algebras. We prove that the Fremlin projective tensor product of two Banach d-algebras A and B is a Banach d-algebra containing Riesz tensor product $A\overline{\bigotimes}B$ of A, B as a d-algebra. Also, we show that the Fremlin projective tensor product of Banach almost f-algebras A and B is a Banach almost f-algebra containing Riesz tensor product $A\overline{\bigotimes}B$ of A and B as an almost f-algebra.

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1 Introduction

D.H. Fremlin introduced the tensor product of Archimedean Riesz spaces in [6] and projective tensor product of Banach lattices in [7]. A lot of mathematicians such as Grobler, Ben Amor, Buskes and so on studied in this subject. The tensor product of Archimedean ordered vector spaces was introduced by Grobler et al in [8]. Ben Amor proved in [2] that the tensor product of two d-algebras is also a d-algebra.The result that the Fremlin tensor product of two f-algebras is an f-algebra is due to Azouzi et al in [3]. For more information for almost f-algebras and d-algebras, we can refer to the paper in [4]. That the Fremlin tensor product of two f-algebras is an f-algebra was introduced by Buskes et al in [5] by different method.

In [9], Jaber proved that the Fremlin projective tensor product of Banach f-algebras A, B is a Banach f-algebra. Here, we show that the Fremlin projective tensor product of Banach d-algebras A and B is a Banach d-algebra and also the Fremlin projective tensor product of Banach almost f-algebras A and B is a Banach almost f-algebra. Therefore, we have shown that the results of Jaber [9] are also true for Banach d-algebras and Banach almost f-algebras.

We refer to the books [1], [10] for unexplained terminology and notation..

2 Preliminaries

A real vector space E is called a Riesz space (vector lattice) if it is linearly ordered and the infimum and supremum of the set $\{x, y\}$ exist for every $x, y \in E$. We use the notations $\vee, \wedge$ for supremum and infimum, respectively. The set $E^+ = \{x \in E : x \geq 0\}$ denotes the positive cone of a Riesz space E. A Riesz space E is called Dedekind complete if every subset that has an upper bound has a least upper bound(supremum). Dedekind complete Riesz spaces are certainly Archimedean. The absolute value(modulus) of $x \in E$ is defined by the formula : $|x| = x \vee -x$. A norm $\|.\|$ on a vector lattice E is called a lattice norm if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in E$. A vector lattice is called a normed vector lattice if it has a lattice norm. We say that a Banach lattice is a normed complete vector lattice.

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Definition 2.1 *A linear operator $T : E \to F$ between vector lattices E, F is called Riesz homomorphism (lattice homomorphism) if $|T(x)| = T(|x|)$ holds for every $x \in E$. A linear operator $T : E \to F$ between vector lattices E, F is said to be positive if $Tx \in F^+$ whenever $x \in E^+$.*

Notice that a lattice homomorphism is positive.

Definition 2.2 *Let E, F, G be Archimedean vector lattices. A bilinear map $\Psi : E \times F \to G$ is called positive if $\Psi(x, y) \in G^+$ for every $x \in E^+, y \in F^+$. A bilinear map $\Psi : E \times F \to G$ is said to be lattice(Riesz) bimorphism if $\Psi(|x|, |y|) = |\Psi(x, y)|$ holds for all $x \in E, y \in F$.*

Assume that any Archimedean vector lattices E and F are given. Then,it can be constructed an Archimedean vector lattice $E\overline{\bigotimes}F$ called Fremlin tensor product and a map $\bigotimes : E \times F \to E\overline{\bigotimes}F$ such that the following properties hold in [6]:

1. $\otimes$ is a Riesz bimorphism and represents algebraic tensor product $E \otimes F$ as a linear subspace of $E\overline{\otimes}F$.

2.If G is any Archimedean vector lattice, then there is a one-one correspondence between Riesz bimorphisms $\Psi : E \times F \to G$ and lattice homomorphism $\tau : E\overline{\otimes}F \to G$ given by $\Psi = \tau \otimes$.

3. $E \otimes F$ is dense in $E\overline{\otimes}F$ in the sense that for any $u \in E\overline{\otimes}F$ there exist $x_0 \in E^+, y_0 \in F^+$ such that for every $\delta > 0$ there is a $v \in E\overline{\otimes}F$ with $|u - v| \leq \delta x_0 \otimes y_0$.

4. If $u \in E\overline{\otimes}F$, then there exist $x_0 \in E^+$ and $y_0 \in F^+$ such that $|u| \leq x_0 \otimes y_0$.

5. $E \otimes F$ is order dense in $E\overline{\otimes}F$ in the sense that for any $0 < u \in E\overline{\otimes}F$ there exist $x > 0$ in E , $y > 0$ in F such that $0 < x \otimes y \leq u$.

6. If G is any Archimedean vector lattice and $\Phi : E \times F \to G$ is a Riesz bimorphism such that $\Phi(x, y) > 0$ whenever $x > 0$ in E and $y > 0$ in F, then $E\overline{\otimes}F$ may be identified with the Riesz subspace of G generated by $\Phi[E \times F]$.

7. If G is a uniformly complete Archimedean vector lattice, then there is a $1 - 1$ correspondence between positive bilinear maps $\Phi : E \times F \to G$ and increasing linear maps $\tau : E\overline{\otimes}F \to G$ given by $\Phi = \tau\otimes$.

Let us take E, F as Banach lattices, [7].

8. $E \otimes F$ is dense in the Riesz tensor product $E\overline{\otimes}F$ for any topology on $E\overline{\otimes}F$ defined by a Riesz norm.

9. If G is any Banach lattice and $\Phi : E \times F \to G$ is a Riesz bimorphism and $\tau : E\overline{\otimes}F \to G$ is the corresponding Riesz homomorphism, then the closure of $\tau(E \otimes F)$ in G is equal to the closure of $\tau(E\overline{\otimes}F)$, so is a closed Riesz subspace of G and is a Banach lattice.

10. If E and F are Banach lattices, the positive projective norm $\|.\|_{|\pi|}$ on $E \otimes F$ is defined by

$$\|u\|_{|\pi|} = sup\{\|\widehat{\varphi}\| : \varphi \quad a \quad positive \quad bilinear \quad function \quad on \quad E{\times}F, \|\varphi\| \leq 1\}$$

where φ runs over all positive bilinear maps from $E \times F$ to all Banach lattices G , $\widehat{\varphi} : E \otimes F \to G$ is in each case the linear map corresponding to φ and

$$\|\varphi\| = sup\{|\varphi(x, y)| : \|x\| \leq 1, \|y\| \leq 1\}$$

for each φ .

11. If E, F are Banach lattices, $E\widehat{\otimes}F$ is the completion of $E \otimes F$ under the norm $\|.\|_{|\pi|}$.

The Riesz space structure of $E\widehat{\otimes}F$:

Let E, F be Banach lattices. Then $\|.\|_{|\pi|}$ is a norm on $E \otimes F$, so $E\widehat{\otimes}F$ is a Banach space, and there is a unique Riesz space structure on $E\widehat{\otimes}F$ such that :

a.$E\widehat{\bigotimes}F$ is a Banach lattice, and $\bigotimes : E\times F \to E\widehat{\bigotimes}F$ is a Riesz bimorphism,

b. The positive cone in $E\widehat{\bigotimes}F$ is the closure in $E\widehat{\bigotimes}F$ of the cone $P \subseteq E\bigotimes F$ generated by $\{x\otimes y : x\in E^+, y\in F^+\}$,

c. For any Banach lattice G , there is a $1-1$ norm preserving correspondence between continuous positive bilinear maps $\varphi : E\bigotimes F \to G$ and continuous increasing linear maps $\tau : E\widehat{\bigotimes}F \to G$ given by $\varphi = \tau\bigotimes$

d. τ is a Riesz homomorphism if and only if φ is a Riesz bimorphism,

e. $\|x\otimes y\|_{|\pi|} = \|x\|\|y\|$ for every $x\in E, y\in F$,

f. $E\overline{\bigotimes}F$ is naturally embedded as a norm-dense Riesz subspace of $E\widehat{\bigotimes}F$ and for $u\ inE\overline{\bigotimes}F$,

$$\|u\|_{|\pi|} = inf\{\sum_{i\leq n}\|x_i\|\|y_i\| : x_i\in E^+, y_i\in F^+\forall i\leq n, |u|\leq \sum_{i\leq n}x_i\otimes y_i\}$$

g. For any $u\in E\widehat{\bigotimes}F$,

$$\|u\|_{|\pi|} = inf\{\sum_{i\in N}\|x_i\|\|y_i\| : x_i\in E^+, y_i\in F^+\forall i\in N, |u|\leq \sum_{i\in N}x_i\otimes y_i\}$$

Let us recall that a Banach lattice is a Banach lattice algebra if it is a Banach algebra where the multiplication of positive elements is positive.

In [9], for any given Banach lattice algebras E and F, the Fremlin projective tensor product $E\widehat{\bigotimes}_{|\pi|}F$ is a Banach lattice algebra with the following universal property:

For every Banach lattice algebra G and for every positive (continuous), multiplicative bilinear map $\Psi : E\times F \to G$ there exists a unique positive algebra homomorphism $\tau : E\widehat{\bigotimes}_{|\pi|}F \to E\widehat{\bigotimes}_{|\pi|}F$ for which $\Psi(x,y) = \tau(x\otimes y)$ for every $x\in E, y\in F$. In [9], Jaber proved that for given Banach f-algebras E and F, the Fremlin projective tensor product $E\widehat{\bigotimes}_{|\pi|}F$ is a Banach f-algebra. We denote by $E\widehat{\bigotimes}_{|\pi|}F$ the completion of $E\bigotimes F$ with respect to the positive projective norm $\|.\|_{|\pi|}$. Fremlin in [7] proved that $E\widehat{\bigotimes}_{|\pi|}F$ is a Banach lattice called the Fremlin projective tensor product of E and F.

3 Results and Discussion

Definition 3.1 *A Banach lattice algebra A is called a Banach f-algebra if $x\wedge y = 0$ and $z\in A^+$ imply $zx\wedge y = xz\wedge y = 0$.*

For example, the algebra $C(K)$ of all real-valued continuous functions on a compact Hausdorff space K is a Banach f-algebra.

Definition 3.2 *A Banach lattice algebra A is said to be a Banach d-algebra if $x \wedge y = 0$ and $z \in A^+$ imply $zx \wedge zy = xz \wedge yz = 0$.*

Definition 3.3 *A Banach lattice algebra A is called a Banach almost f-algebra if $x \wedge y = 0$ in A implies $x.y = 0$.*

Theorem 3.4 [9] *Let A be a normed f-algebra. Then, the completion $\widehat{A}$ of A is a Banach f-algebra containing A as an f-subalgebra.*

Theorem 3.5 *Assume that A is a normed d-algebra. Then, the completion $\widehat{A}$ of A is a Banach d-algebra containing A as a d-subalgebra.*

Proof. It is clear that the norm completion $\widehat{A}$ of A is a Banach lattice and a Banach algebra.Let $0 \leq x, y \in \widehat{A}$. So, there are sequences $(x_n), (y_n)$ in A satisfying (x_n) converging to x and (y_n) converging to y. Since x, y are positive, we can assume that sequences $(x_n), (y_n)$ are positive.Since $(x_n y_n)$ converges to xy and the multiplication of x_n and y_n is positive , we get that xy is positive. Our claim is that $\widehat{A}$ is a d-algebra. Let us take elements $x, y \in \widehat{A}$ satisfying $x \wedge y = 0$ and $0 \leq z \in \widehat{A}$. Firstly, let $z \in A$. There are sequences $(x_n), (y_n)$ in A satisfying (x_n) converging to x and (y_n) converging to y. By using the continuity of lattice operations, we get

$$x_n \wedge y \to x \wedge y = 0.$$

From the following inequality,

$$|x_n \wedge y_n| = |x_n \wedge y_n - x_n \wedge y + x_n \wedge y| \leq |x_n \wedge y_n - x_n \wedge y| + |x_n \wedge y|$$

$$\leq |y_n - y| + |x_n \wedge y|$$

for every natural number n, we have

$$\|x_n \wedge y_n\| \leq \|y_n - y\| + \|x_n \wedge y\|$$

converging to zero. Assume that $u_n = (x_n - y_n)^+$ and $v_n = (x_n - y_n)^-$ for every natural number n. Again, by using the continuity of lattice operations, we obtain $u_n \to x$ and $v_n \to y$. since A is a d-algebra, it follows from $u_n \wedge v_n = 0$ that

$$zu_n \wedge zv_n = u_n z \wedge v_n z = 0$$

for every natural number n. Since $zu_n \wedge zv_n$ converges to $zx \wedge zy = 0$ and $u_n z \wedge v_n z$ converges to $xz \wedge yz = 0$

Let $0 \leq z \in \widehat{A}$. There is a sequence $0 \leq (z_n)$ in A converging to z. By before, we have

$$z_n x \wedge z_n y = x z_n \wedge y z_n = 0$$

for all natural number n. As $n \to \infty$, we get that $zx \wedge zy = xz \wedge yz = 0$.

Theorem 3.6 *Assume that A is a normed almost f-algebra. Then, the completion $\widehat{A}$ of A is a Banach almost f-algebra containing A as an almost f-subalgebra.*

Proof. It is clear that the norm completion $\widehat{A}$ of A is a Banach lattice and a Banach algebra.Let $0 \leq x, y \in \widehat{A}$. So, there are sequences $(x_n), (y_n)$ in A satisfying (x_n) converging to x and (y_n) converging to y. Since x, y are positive, we can assume that sequences $(x_n), (y_n)$ are positive.Since $(x_n y_n)$ converges to xy and the multiplication of x_n and y_n is positive , we get that xy is positive. Our claim is that $\widehat{A}$ is an almost f-algebra. Let us take elements $x, y \in \widehat{A}$ satisfying $x \wedge y = 0$. There are sequences $(x_n), (y_n)$ in A satisfying (x_n) converging to x and (y_n) converging to y. By using the continuity of lattice operations, we get

$$x_n \wedge y \to x \wedge y = 0.$$

From the following inequality,

$$|x_n \wedge y_n| = |x_n \wedge y_n - x_n \wedge y + x_n \wedge y| \leq |x_n \wedge y_n - x_n \wedge y| + |x_n \wedge y|$$

$$\leq |y_n - y| + |x_n \wedge y|$$

for every natural number n, we have

$$\|x_n \wedge y_n\| \leq \|y_n - y\| + \|x_n \wedge y\|$$

converging to zero. Assume that $u_n = (x_n - y_n)^+$ and $v_n = (x_n - y_n)^-$ for every natural number n. Again, by using the continuity of lattice operations, we obtain $u_n \to x$ and $v_n \to y$. Since A is an almost f-algebra, it follows from $u_n \wedge v_n = 0$ for every natural number n that

$$u_n \wedge v_n = u_n . v_n = 0$$

converges to 0.

Since $u_n . v_n$ converges to 0 and $u_n . v_n$ converges to $x.y$ by the uniqueness of limits, $x.y = 0$. It shows that $\widehat{A}$ is a Banach almost f-algebra.

Let Banach lattice algebras E and F be given. The algebraic tensor product $E \bigotimes F$ can be given by a canonical algebra product satisfying $(a \otimes b).(c \otimes d) = ac \otimes bd$ for every $a, c \in E$ and $b, d \in F$.

We can extend this multiplication to a Banach lattice algebra product on the Fremlin projective tensor product $E\widehat{\bigotimes}_{|\pi|}F$, [9]. We need the following information about multiplication in [9].

Define for a given $a \in A$ and $b \in B$ the bilinear map

$$L_{a,b} : A \times B \to A\widehat{\bigotimes}_{|\pi|}B$$

by $L_{a,b}(x, y) = ax \otimes by$ for every $x \in A, y \in B$. Let us assume that $a \in A^+, b \in B^+$. So, $L_{a,b}$ is a positive bilinear map. By using the equality,

$$\|x \otimes y\|_{|\pi|} = \|x\|\|y\|$$

for all $x \in A$, $y \in B$, we get

$$\|L_{a,b}(x, y)\|_{|\pi|} = \|ax \otimes by\|_{|\pi|} = \|ax\|\|by\|$$

,

$$\leq \|a\|\|x\|\|b\|\|y\|,$$

This shows that $L_{a,b}$ is continuous and $\|L_{a,b}\| \leq \|x\|\|y\|$.

By the universal property of the Banach lattice tensor product, there is a unique continuous linear map

$\lambda_{a,b} : A\widehat{\bigotimes}_{|\pi|}B \to A\widehat{\bigotimes}_{|\pi|}B$ such that $(a \otimes b).(x \otimes y) = L_{a,b}(x, y) = \lambda_{a,b}(x \otimes y)$ for all $x \in A, y \in B$ and $\|\lambda_{a,b}\| \leq \|a\|\|b\|$. Notice that $\lambda_{a,b}$ is the unique continuous linear map on $A\widehat{\bigotimes}_{|\pi|}B$ satisfying

$\lambda_{a,b}(x \otimes y) = (a \otimes b) \star (x \otimes y)$ and then the map $R : A \times B \to A\widehat{\bigotimes}_{|\pi|}B$ is a positive bilinear map. Let $v \in A\widehat{\bigotimes}_{|\pi|}B$. Let us consider the bilinear map $R_v : A \times B \to A\widehat{\bigotimes}_{|\pi|}B$ by $R_v(a, b) = \lambda_{a,b}(v)$ for all $(a, b) \in A \times B$. R_v is continuous . For every $0 \leq v \in A\widehat{\bigotimes}_{|\pi|}B$ there is a unique positive linear map $\rho_v : A\widehat{\bigotimes}_{|\pi|}B \to A\widehat{\bigotimes}_{|\pi|}B$ such that $\rho_v(a \otimes b) = R_v(a, b)$ for all $(a, b) \in A \times B$. Hence, ρ_v is a positive map for every $0 \leq v \in A\widehat{\bigotimes}_{|\pi|}B$ and the map $v \to \rho_v$ is linear.

We introduce a multiplication $\star$ on $A\widehat{\bigotimes}_{|\pi|}B$. For every $u, v \in A\widehat{\bigotimes}_{|\pi|}B$, we define $u \star v$ by $\rho_u(v) = v \star u$. This multiplication extends the canonical multiplication on $A \bigotimes B$.

Theorem 3.7 [9] *Let A and B be Banach lattice algebras. Then, the Fremlin projective tensor product $A\widehat{\bigotimes}_{|\pi|}B$ is a Banach lattice algebra which contains $A \bigotimes B$ as a subalgebra with respect to the multiplication $\star$.*

Theorem 3.8 [9] *Assume that A and B are Banach f-algebras. Then, the Fremlin projective tensor product $A\widehat{\bigotimes}_{|\pi|}B$ is a Banach f-algebra that contains $A\overline{\bigotimes}B$ as an f-algebra.*

Theorem 3.9 *The Riesz tensor product $A\overline{\bigotimes}B$ of Banach d-algebras A, B is a normed d-algebra with respect to the positive projective norm $\|.\|_{|\pi|}$.*

Proof. Since A and B are Archimedean d-algebras, the Riesz tensor product $A\overline{\bigotimes}B$ can be endowed with a d-algebra product denoted by . which extends the algebraic multiplication. So, it is enough to show that $u.v = u\star v$ for every $u, v \in A\overline{\bigotimes}B$. We claim that this multiplication is true for all $u, v \in A\overline{\bigotimes}B$. Clearly, this multiplication is true for all $u, v \in A \otimes B$. Let $u \in A\overline{\bigotimes}B$. By [7] , there exist $0 \leq a \in A$ and $0 \leq b \in B$ and a sequence u_n in $A \otimes B$ such that $|u - u_n| \leq \frac{1}{n} a \otimes b$ for all natural numbers $n = 1, 2, 3, ...$ So, for all $v \in A \otimes B$, we have

$|u.v - u_n \star v| = |u.v - u_n.v| \leq \frac{1}{n}(a \otimes b)|v|$ for all natural numbers $n = 1, 2, 3, ...$ It implies that the sequence $(u_n \star v)$ converges in $A\widehat{\bigotimes}_{|\pi|}B$ to $u.v$. Also, $(u_n \star v)$ converges to $(u \star v)$ in $A\widehat{\bigotimes}_{|\pi|}B$. By the uniqueness of limits, we have $u.v = u\star v$ for all $u \in A\overline{\bigotimes}B$ and $v \in A \otimes B$. We conclude that $u.v = u\star v$ true for all $u, v \in A\overline{\bigotimes}B$.

Theorem 3.10 *Assume that A and B are Banach d-algebras.Then, the Fremlin projective tensor product $A\widehat{\bigotimes}_{|\pi|}B$ is a Banach d-algebra containing Riesz tensor product $A\overline{\bigotimes}B$ of d-algebras A and B as a d-algebra.*

Proof. It is known that the vector lattice tensor product $A\overline{\bigotimes}B$ of A and B is a d-algebra, [2]. So,the norm completion $A\widehat{\bigotimes}_{|\pi|}B$ of $A\overline{\bigotimes}B$ is a Banach d-algebra by Theorem 3.5 .

A vector lattice is called laterally complete whenever every set of pairwise disjoint positive elements has a supremum. A Riesz space that is both laterally complete and Dedekind complete is called a universally complete Riesz space.If E is an Archimedean Riesz space, then there exists a unique (up to a lattice isomorphism) universally complete Riesz space E^u called the universal completion of E such that E is Riesz isomorphic to an order dense Riesz subspace of E^u. Identifying E with its copy in E^u, we have the Riesz subspace inclusion $E \subseteq E^u$ with E order dense in E^u. The Dedekind completion E^δ of E can be identified with the ideal generated by E in E^u, and we get the Riesz subspace inclusions $E \subseteq E^\delta \subseteq E^u$with E order dense in E^u. The vector lattice E^u is of the form $C^\infty(X)$ for some Hausdorff , extremally disconnected, compact topological space X. $C^\infty(X)$, with X extremally disconnected, under the pointwise multiplication is an Archimedean f-algebra with unit element the constant function one. That is, $E^u = C^\infty(X)$.

Definition 3.11 *Let E and F be Archimedean Riesz spaces and*

$$\Phi : E \times E \to F$$

be a bilinear mapping Φ is called an orthosymmetric mapping if $x \perp y$ implies $\Phi(x, y) = 0$ for every $x, y \in E$, where $x \perp y$ means $|x| \wedge |y| = 0$.

It is known that every symmetric mapping is an orthosymmetric mapping. In the commutative case, they are identical.

Let A and B be almost f-algebras. We denote by A^u and B^u the universally completion of A and B. Here, A^u and B^u are f-algebras. So, $A^u \overline{\otimes} B^u$ is an f-algebra, [3], [5]. Consider the bilinear mapping $\Psi : A\overline{\otimes}B \times A\overline{\otimes}B \to A^u\overline{\otimes}B^u$ defined by $\Psi(u, v) = u.v$ for every $u, v \in A\overline{\otimes}B$.

Theorem 3.12 *The Riesz tensor product of almost f-algebras A and B is an almost f-algebra.*

Proof. Every Archimedean f-algebra is commutative, [10]. In this reason , the mapping Ψ is symmetric and so it is an orthosymmetric mapping. Since $A\overline{\otimes}B$ is a subspace of the $A^u\overline{\otimes}B^u$, it follows that $\Psi(u, v) = u.v = 0$ for every $u, v \in A\overline{\otimes}B$ with $u \perp v$. Hence, $A\overline{\otimes}B$ is an almost f-algebra.

Theorem 3.13 *The Riesz tensor product $A\overline{\otimes}B$ of Banach almost f-algebras A, B is a normed almost f-algebra with respect to the positive projection norm $\|.\|_{|\pi|}$.*

Proof. The proof of this result is similar to that of Theorem 3.9. Hence, we omit it.

Theorem 3.14 *Assume that A and B are Banach almost f-algebras. Then, the Fremlin projective tensor product of A and B is a Banach almost f-algebra and contains the Riesz tensor product $A\overline{\otimes}B$ as an almost f-algebra.*

Proof. The Riesz tensor product $A\overline{\otimes}B$ of almost f-algebras A, B is an almost f-algebra by Theorem 3.12 . So, the norm completion $A\widehat{\otimes}_{|\pi|}B$ of $A\overline{\otimes}B$ is a Banach almost f-algebra by Theorem 3.6 .

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