International Mathematical Forum, Vol. 18, 2023, no. 2, 83 - 88
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2023.912384

On Unbounded Order Convergence of Operator Nets

Ebru Aydogan [1]

Yildiz Technical University
Graduate School of Science and Engineering
Mathematics Department,
34220, Davutpasa, Istanbul, Turkey

Elif Demir

Yildiz Technical University
Faculty of Arts and Sciences
Mathematics Department
34220, Davutpasa, Istanbul, Turkey

Abstract

In this study, we investigate uo-convergence of nets of positive continuous operators defined on the topological dual of a completely regular Hausdorff topological space. Firstly, we make the definition of uo-convergence on this class and then we present some characterizations of it.

Mathematics Subject Classification: 46A40, 47B38, 47B60

Keywords: uo-convergence, topological dual, completely regular Hausdorff space, nets of positive operators

[1]The author was granted by TUBITAK (The Scientific and Technological Research Council of Türkiye

1 Introduction

The notion of unbounded order convergence, simply uo-convergence, is firstly studied in the article [7] by a mathematician Hidegoro Nakano. The aim of Nakano was to define almost everywhere convergence in terms of lattice operations without direct use of measure theory and he defined "individual convergence". Then, it was named as "unboundedly order convergence" in the paper [3] by Ralph DeMaar. The development of this subject continued with [4], [5], [8], [9] and other researchers. We refer the reader to [2] for a survey of some convergence types on vector lattices.

In section 2, we give some basic definitions of Riesz spaces and some convergences on it. Let X be a set with an order relation " $\leq$ ". X is said to be a lattice if any two elements $x, y \in X$ have a least upper bound and a greatest lower bound such that $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$. A lattice $(X, \leq)$ is called a vector lattice if the order and vector space structure are compossible. A net is a function from a directed set A to an arbitrary set (a vector lattice) X. It is denoted by $(x_\alpha)_{\alpha \in A}$ or simply (x_α) if the index set is clear from a context. We refer the reader to [6] to for detailed information about vector lattices.

In section 3, we characterize uo-convergence of nets of operators which are defined on continuous functionals. In the paper [1], order and uo-convergence on $C(\Omega)$ were characterized where Ω is a Baire space and $C(\Omega)$ is the space of all real valued functions. We study the uo-convergence of operator nets defined on the space of continuous functionals satisfying the same conditions in [1].

2 Preliminaries

Definition 2.1 *Let X be a vector lattice. The positive cone X_+ consists of all $x \in X$ such that $x \geq 0$. Furthermore, for every $x \in X$ let*

$$x^+ = x \vee 0,\ x^- = (-x) \vee 0, \quad \text{and} \ \ |x| = x \vee (-x)$$

be the positive part, the negative part and the absolute value of x, respectively.

Definition 2.2 *Let (x_α) be a net indexed by a directed set A. For $\alpha_0 \in A$ fixed, let $A_0 = \{\alpha \in A : \alpha \geq \alpha_0\}$ which is again a directed set under the pre-order induced from A. The restriction of the function x to A_0 is called a tail of (x_α), and it is denoted by $(x_\alpha)_{\alpha \geq \alpha_0}$.*

Definition 2.3 *Let $(x_\alpha)_{\alpha \in A}$ be a net in X. (x_α) is said to be order convergent to x if there exists a net $(y_\alpha)_{\alpha \in A}$ such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for all $\alpha \in A$. It is denoted as $x_\alpha \xrightarrow{o} x$.*

Lemma 2.4 (Lemma 2.1, [1]) *For a net (x_α) in a vector lattice X, $x_\alpha \xrightarrow{o} x$ if and only if there exists a set $G \subseteq X_+$ such that $\inf G = 0$ and every element of G dominates a tail of (x_α), that is, for every $g \in G$ there exists α_0 such that $|x_\alpha| \leq g$ for all $\alpha \geq \alpha_0$.*

Definition 2.5 *A net (x_α) is said to be unbounded order converges to x if $|x_\alpha - x| \wedge y \xrightarrow{o} 0$ for every $y \geq 0$. We will use the notation (x_α) uo-converges to x, for short, and we will denote it as $x_\alpha \xrightarrow{uo} x$.*

Order and uo-convergences agree for order bounded nets. If $w \geq 0$ is a weak unit then $x_\alpha \xrightarrow{uo} x$ if and only if $|x_\alpha - x| \wedge w \xrightarrow{o} 0$ [1].

Definition 2.6 *i. A subspace Y of X is called a sublattice of Y if $x \vee y \in Y$ and $x \wedge y \in Y$ for all $x, y \in Y$.*

ii. A sublattice Y of X is called order dense if $0 < x \in X$ implies that there exists $y \in Y$ with $0 < y \leq x$.

iii. A sublattice is regular if the inclusion map is order continuous, that is, it preserves order convergence of nets.

Every order dense sublattice is regular. For a net (x_α) in a regular sublattice Y of X, $x_\alpha \xrightarrow{uo} 0$ in X if and only if $x_\alpha \xrightarrow{uo} 0$ in Y.

Definition 2.7 *A net $(x_\alpha)_{\alpha \in A}$ is called order Cauchy (or simply o-Cauchy) if the double net $(x_\alpha - x_\beta)_{A^2}$ is order null. Unbounded order Cauchy (simply uo-Cauchy) is defined in the same way.*

Before giving the Uryson's lemma, recall that a normal space is a topological space that satisfies the Axiom T_4, that is, every disjoint closed sets of a topological space have disjoint open neighborhoods. In addition, a normal Hausdorff space is called T_4 space.

Theorem 2.8 (Uryson's Lemma) *A topological space X is normal if and only if for any two disjoint nonempty closed subsets $Y, Z \subseteq X$ there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in Y$ and $f(x) = 1$ for all $x \in Z$.*

Definition 2.9 *A topological space X is completely regular if the points can be seperated from closed sets via continuous real-valued functions. That is, for any closed set $A \subseteq X$ and any point $x \in X \backslash A$, there exists a real-valued continuous function $f : X \to \mathbb{R}$ such that*

$$f(x) = 1$$

and

$$f|_A = 0.$$

Throughout the paper, X stands for a locally solid Riesz space and also completely regular Hausdorff topological space which is exactly the class of Hausdorff spaces where the conclusion of Uryson's Lemma holds. As a consequence of Uryson's Lemma, we can say that every locally compact Hausdorff space or every normal space is completely regular.

Let X' be a topological dual of X which we have mentioned above. $C(X)$ denotes the space of all real-valued continuous functions on X, and $\mathbb{1}$ for the constant one function. Let $B(X')_+$ be a family of all continuous positive operators defined as follows:

$$B(X')_+ = \{T : X' \to X' \mid Tf \geq 0 \text{ for } f \in X'\}. \tag{1}$$

3 Main Results

The following lemma essentially Lemma 3.1 in [1], we rewrite it for our main goal:

Lemma 3.1 *Suppose that X is a Hausdorff locally solid vector lattice (and Tychonoff), X' be the topological dual of it and $H \subseteq B(X')_+$. The following are equivalent:*

i. $\inf H = 0$;

ii. for every non-empty open set U, $g \in X'$ and every $\varepsilon > 0$, there exists an element $s \in U$ and $T \in H$, with $(Tg)(s) < \varepsilon$;

iii. for every non-empty open set U, $g \in X'$ and every $\varepsilon > 0$, there exists a non-empty open set $V \subseteq U$ and $T \in H$ such that $(Tg)(s) < \varepsilon$ for all $s \in V$.

Proof. (i) $\Rightarrow$ (ii) Consider that $\inf H = 0$, but (ii) fails, that is, there exists an open non-empty U, $g \in X'$ and $\varepsilon > 0$ such that for every $T \in H$, $(Tg)(s) \geq \varepsilon$ on U for all s. Since X is completely regular, we find a nonzero $f \in C(X)_+$ and an operator $S \in B(X')_+$ such that $Sf \leq \varepsilon\mathbb{1}$ and Sf vanishes outside of U. Then $S \leq H$, which contradicts $\inf H = 0$.

(ii) $\Rightarrow$ (i) Suppose that (ii) holds and $\inf H \neq 0$. Then there exists $S \in B(X')_+$ such that $0 < S \leq H$. Next, we can find an open non-empty set U, $f \in C(X)$ and $\varepsilon > 0$ such that $Sf > \varepsilon$ on U. It follows that every $T \in H, g \in X'$ we obtain that Tg is greater than ε on U, which contradicts (ii).

Other implications can be clearly seen. $\square$

Now, we will define uo-convergence and characterize it on $B(X')_+$ by using operator nets.

Definition 3.2 *Let T_λ be a net in $B(X')_+$. $T_\lambda \xrightarrow{uo} T$ if and only if $|T_\lambda - T| \wedge f \xrightarrow{o} 0$ for all $f \in B(X')_+$. Since $T_\lambda \xrightarrow{uo} T$ if and only if $|T_\lambda - T| \xrightarrow{uo} 0$, it is enough to characterize $T_\lambda \xrightarrow{uo} 0$.*

Theorem 3.3 *Let X' be the dual of a Hausdorff locally solid vector lattice (and Tychonoff) X and (T_λ) be a net in $B(X')_+$. Then $T_\lambda \xrightarrow{uo} 0$ if and only if for every non-empty open set U, $f \in X'_+$ and every $\varepsilon > 0$, there exists an open non-empty $V \subseteq U$ and an index λ_0 such that $T_\lambda f$ is less than ε on V whenever $\lambda \geq \lambda_0$.*

Proof. Let $T_\lambda \xrightarrow{uo} 0$. Then $T_\lambda \wedge \mathbb{1} \xrightarrow{o} 0$. By the Lemma 2.4, we can find a set $H \subseteq B(X')_+$ such that $\inf H = 0$ and every element of H dominates a tail of $(T_\lambda \wedge \mathbb{1})$. Fix a non-empty open U and $\varepsilon \in (0,1)$. Choose such V and T that Lemma 3.1 (iii) holds. Since $T \in H$ then T dominates a tail of $(T_\lambda \wedge \mathbb{1})$, followingly, there exists λ_0 such that $T_\lambda \wedge \mathbb{1} \leq T$ for all $\lambda \geq \lambda_0$. In particular,

$$(T_\lambda f)(s) \wedge \mathbb{1}(s) = (T_\lambda f)(s) \wedge 1 \leq (Tf)(s) < \varepsilon$$

hence $(T_\lambda f)(s) < \varepsilon$ for all $s \in V$. This completes the proof.

Conversely, suppose that the condition in the theorem holds. Since $\mathbb{1}$ is a weak unit, it is enough to prove that $T_\lambda \wedge \mathbb{1} \xrightarrow{o} 0$. We will use Lemma 2.4 to prove it. Fix a nonempty set U and $\varepsilon > 0$. We assume that $V \subseteq U$ is a non-empty set and λ_0 is an index such that $T_\lambda f$ is less than ε on V for all $f \in X'_+$ whenever $\lambda \geq \lambda_0$. Fix any $s \in V$. Since X is completely regular, we can find $g \in C(X)_+$ such that $g(s) = 0$ and g equals 1 outside of V. Put $k = g \vee \varepsilon\mathbb{1}$. Then

$$k(s) = g(s) \vee \varepsilon \cdot \mathbb{1}(s) = \varepsilon.$$

We claim that $(T_\lambda \wedge \mathbb{1})(f) \leq k$ for every $\lambda \geq \lambda_0$. Indeed, if $m \in V$ then

$$(T_\lambda f)(m) < \varepsilon \leq k(s)$$

and if $m \notin V$ then

$$[(T_\lambda \wedge \mathbb{1})(f)](m) \leq 1 = g(s) \leq k(s).$$

Repeat this process for every pair (U, ε) where U is open and non-empty and $\varepsilon > 0$. Let H be the set of the resulting functions k. Each such k dominates a tail of $(T_\lambda \wedge \mathbb{1})$ Lemma 3.1 implies $\inf H = 0$, completes the proof. $\square$

Corollary 3.4 *Let X be a Hausdorff locally solid vector lattice (and Tychonoff), X' be a topological dual of it and $(T_\lambda)_{\lambda \in A}$ be a net in $B(X')_+$. Then (T_λ) is uo$-$Cauchy if and only if for every non-empty set U, $f \in X'_+$ and every $\varepsilon > 0$, there exists an open non-empty $V \subseteq U$ and an index λ_0 such that*

$$|(T_\lambda f)(s) - (T_\beta f)(s)| < \varepsilon$$

for all $s \in V$ and $\lambda, \beta \geq \lambda_0$.

Proof. ($\Rightarrow$) Let (T_λ) be an $uo-Cauchy$ net. That is, for $\lambda, \beta \in A$, the net $(T_\lambda - T_\beta)_{A\times A}$ is $uo-$null. By the Theorem 3.3 we get the result directly.
($\Leftarrow$) It can be easily seen by using Theorem 3.3 again. □

Acknowledgements. The authors would like to thank the BAP (Scientific Research Projects) YTU and TUBITAK (The Scientific and Technological Research Council of Trkiye) for their scientific supports.

References

[1] E. Bilokopytov, V.G. Troitsky, Order and uo-convergence in spaces of continuous functions, *Topology and its Applications*, **308** (2022), Paper No. 107999, 9. https://doi.org/10.1016/j.topol.2022.107999

[2] A. M. Dabboorasad, E. Y. Emelyanov, Unbounded convergence in the convergence vector lattices: a survey, *Vladikavkaz Mathematical Journal*, **20** (2) (2018), 49-56. https://doi.org/10.23671/VNC.2018.2.14720

[3] R. DeMaar, Partially ordered linear spaces and locally convex linear topological spaces, *Illinois Journal of Mathematics*, **8** (1964), 601-606.

[4] N. Gao, F. Xanthos, Unbounded order convergence and application to martingales without probability, *Journal of Mathematical Analysis and Applications*, **415** (2) (2014), 931-947.
http://dx.doi.org/10.1016/j.jmaa.2014.01.078

[5] S. Kaplan, On unbounded order convergence, *Real Analysis Exchange*, **23**(1) (1998-1999), 175-184.

[6] P. Meyer-Nieberg, *Banach Lattices,* Springer-Verlag, Berlin, 1991.
https://doi.org/10.1007/978-3-642-76724-1

[7] H. Nakano, Ergodic theorems in semi-ordered linear spaces, *Annals of Mathematics (2)*, **49** (1948), 538-556.
https://doi.org/10.2307/1969044

[8] J. H. van der Walt, The universal completion of $C(X)$ and unbounded order convergence, *ournal of Mathematical Analysis and Applications*, **460** (2018), 76-97. https://doi.org/10.1016/j.jmaa.2017.11.011

[9] A. W. Wickstead, Weak and unbounded order convergence in Banach lattices, *Journal of the Australian Mathematical Society Series A*, **24** (1977), 312-319.

Received: May 12, 2023; Published: June 14, 2023