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Construction of Conformal Mappings Between Several Rotating Surfaces

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Abstract

Conformal mappings are an important object to be investigated in differential geometry. In the paper, the authors mainly construct conformal mappings concerning several rotating surfaces and conclude the proportion of the area element between two surfaces under conformal mapping.

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1. Introduction

Conformal mappings are extremely important in the study of the curved surface theory. They are widely used in computer graphics, computer vision, geometric modeling, wireless sensor network, medical image, and other fields.

Qiu and Gu [5] led their team combining basic mathematical disciplines such as differential geometry and algebraic topology with computer science and creating a cross-domain discipline, Computational Conformal Geometry, by using conformal mappings.

In the paper [4], Gu–Luo–Yau expounded the basic concepts and theories involved in the cross-domain discipline, as well as briefly introduced some relevant applications in engineering.

In the paper [11], Wu and three coauthors proposed a conformal mapping method for tongue image alignment. The mapping on the boundary is first established by the Fourier descriptors, and then the mapping is extended to the inner region by the Cauchy integrals and finite difference method. This method not only improves the efficiency of image alignment and feature extraction, but also improves the accuracy of disease detection.

In the paper [1], Dai and Ben Amar expressed the shape of fresh flat leaves as a holomorphic function and used conformal mappings to study the complex morphology of growing leaves. This method can also be used for more two-dimensional organisms to study their growth patterns. Thus, the comparison and research between surfaces are of fundamental importance in the medical field and many projects.

Regarding the examples of conformal mappings, the textbook [2] of differential geometry gives the most classical stereographic projection [10], the mapping between the spherical surface and the plane [9, pp. 121–139], and the mapping between the spherical surface and the cylindrical surface [2, pp. 115–120].

In the paper [8], Mo and Ji studied the internal relation between the catenoid and the helicoid, two famous minimal surfaces. Through the parameter transformation, they found that the relation between the catenoid and the helicoid with different coefficients can form conformal mapping.

In the paper [6], basing on existing research, Li and six coauthors discussed the mapping from the ruled surface to the plane, obtained a formula for the conformal mapping from a special ruled surface to a plane, and presented two formulas for the conformal mappings from the plane to the elliptical cylinder and to the hyperbolic cylinder.

In this paper, we mainly aim to construct conformal mappings concerning several rotating surfaces, to study by using conformal mappings the correspondence of between area elements of surfaces, and extend known results in related literature.

2. Elementary knowledge

If a mapping between two surfaces causes the intersection angles of the corresponding curves on the surface to be equal, then we call it a conformal mapping [9].

On any surface $r(u,v)$ in $\mathbb{R}^3$, the differential of the arc length at some point is considered to be

$$\mathrm{d}\,s = \sqrt{E\,\mathrm{d}\,u^2 + 2F\,\mathrm{d}\,u\,\mathrm{d}\,v + G\,\mathrm{d}\,v^2}\,,$$

where

$$\begin{aligned} E &= E(u,v) = r_u(u,v)\cdot r_u(u,v),\\ F &= F(u,v) = r_u(u,v)\cdot r_v(u,v),\\ G &= G(u,v) = r_v(u,v)\cdot r_v(u,v). \end{aligned}$$

We call the quantity

$$I(u,v) = \mathrm{d}\,s^2 = E\,\mathrm{d}\,u^2 + 2F\,\mathrm{d}\,u\,\mathrm{d}\,v + G\,\mathrm{d}\,v^2$$

the first fundamental form of a surface and call E, F, G the coefficients of the first fundamental form of a surface [9].

A mapping between two surfaces is conformal if and only if the first fundamental forms of these two surfaces are proportional [2, pp. 115–120]. If taking the same parameters, then the first fundamental forms of these two surfaces are

$$I(u,v) = E\,\mathrm{d}\,u^2 + 2F\,\mathrm{d}\,u\,\mathrm{d}\,v + G\,\mathrm{d}\,v^2$$

and

$$I^*(u,v) = E^*\,\mathrm{d}\,u^2 + 2F^*\,\mathrm{d}\,u\,\mathrm{d}\,v + G^*\,\mathrm{d}\,v^2$$

respectively. Thus, we acquire the relation $I = \lambda^2(u,v)I^*$, where $\lambda(u,v) \neq 0$. See [2, pp. 115–120]. When $\lambda^2(u,v) = 1$, the first fundamental forms of these two surfaces are equal, so an isometric mapping must be conformal.

Lemma 1 ([6]). *Between the hyperbolic cylinder $r(u,v) = \{\sec u, \tan u, v\}$ and the plane $r^*(x,y) = \{x,y,0\}$, there exists a conformal mapping.*

Lemma 2 ([8]). *Between the catenoid*

$$r(u,v) = \left\{a\cosh\frac{v}{a}\cos u, a\cosh\frac{v}{a}\sin u, v\right\}$$

and the helicoid $r^*(\tilde{u},\tilde{v}) = \{\tilde{u}\cos\tilde{v}, \tilde{u}\sin\tilde{v}, b\tilde{v}\}$, if $a \neq b$, there exists a conformal mapping such that $\lambda(\tilde{u},\tilde{v}) = \frac{1}{b}\cos\big(\arctan\frac{\tilde{u}}{b}\big)\cosh\frac{\tilde{v}}{a}$.

Every point on any surface has a neighbourhood where one can establish a conformal mapping with a region of the Euclidean plane [2, pp. 115–120], that is, a conformal mapping between a surface and a plane can be established locally. Moreover, it can also be established between the whole surface. In addition to the stereographic projection, the mapping between the spherical surface and the plane or the mapping between the spherical surface and the cylindrical surface given in the textbook, a conformal mapping can also be established between a rotating surface and a cylinder [7, p. 26]. The proof of establishing conformal mappings between a rotating surface and a cylinder and between rotating surfaces are given as follows.

3. Conformal mappings between surfaces

We now start out to establish conformal mappings between surfaces.

Theorem 1. *There exists a conformal mapping between the catenoid*

$$r(u,v) = \left\{a\cosh\frac{u}{a}\cos v, a\cosh\frac{u}{a}\sin v, u\right\}, \quad 0 < v < 2\pi, u \in \mathbb{R}$$

and the cylinder

$$r^*(u,v) = \{a\cos v, a\sin v, u\}, \quad u, v \in \mathbb{R},$$

where $a > 0$ is a constant.

Proof. For the catenoid $r(u,v)$, a straightforward computation gives

$$E_1(u,v) = \sinh^2\frac{u}{a} + 1 = \cosh^2\frac{u}{a}, \quad F_1(u,v) = 0, \quad G_1(u,v) = a^2\cosh^2\frac{u}{a}.$$

The first fundamental form of the catenoid $r(u,v)$ is

$$I_1(u,v) = \cos h^2\frac{u}{a}\,\mathrm{d}\,u^2 + a^2\cos h^2\frac{u}{a}\,\mathrm{d}\,v^2 = \cos h^2\frac{u}{a}\big(\mathrm{d}\,u^2 + a^2\,\mathrm{d}\,v^2\big).$$

For the cylinder $r^*(u,v)$, a direct calculation gives

$$E_2(u,v) = 1, \quad F_2(u,v) = 0, \quad G_2(u,v) = a^2.$$

The first fundamental form of the cylinder $r^*(u,v)$ is

$$I_2(u,v) = \mathrm{d}\,u^2 + a^2\,\mathrm{d}\,v^2.$$

Therefore, we derive the relation $I_1(u,v) = \cosh^2\frac{u}{a}I_2(u,v)$. Hence, we can obtain $\lambda(u,v) = \cosh\frac{u}{a}$, which indicates that a conformal mapping can be established between the catenoid $r(u,v)$ and the cylinder $r^*(u,v)$. The proof of Theorem 1 is complete. $\square$

Theorem 2. *There exists a conformal mapping between the uniparted hyperboloid*

$$r(u,v) = \{\cosh u\cos v, \cosh u\sin v, \sinh u\}, \quad 0 < v < 2\pi, u \in \mathbb{R}$$

and the cylinder

$$r^*(\tilde{u},\tilde{v}) = \{\cos\tilde{v}, \sin\tilde{v}, \tilde{u}\}, \quad \tilde{u}, \tilde{v} \in \mathbb{R}.$$

Proof. It is easy to see that,

(1) for the cylinder $r^*(\tilde{u},\tilde{v})$, we have

$$E_3(\tilde{u},\tilde{v}) = 1, \quad F_3(\tilde{u},\tilde{v}) = 0, \quad G_3(\tilde{u},\tilde{v}) = 1.$$

Then the first fundamental form of the cylinder $r^*(\tilde{u},\tilde{v})$ is

$$I_3(\tilde{u},\tilde{v}) = \mathrm{d}\,\tilde{u}^2 + \mathrm{d}\,\tilde{v}^2.$$

(2) for the uniparted hyperboloid $r(u,v)$ considered in [3], we have

$$\begin{aligned} r_u(u,v) &= (\sinh u \cos v, \sinh u \sin v, \cosh u),\\ r_v(u,v) &= (-\cosh u \sin v, \cosh u \cos v, 0), \end{aligned}$$

and

$$E_4(u,v) = \sinh^2 u + \cosh^2 u, \quad F_4(u,v) = 0, \quad G_4(u,v) = \cosh^2 u.$$

Then the first fundamental form of the uniparted hyperboloid $r(u,v)$ is

$$I_4(u,v) = \big(\sinh^2 u + \cosh^2 u\big)\,\mathrm{d}\,u^2 + \cosh^2 u\,\mathrm{d}\,v^2.$$

Performing the parameter transformation $u = \operatorname{arcsinh}\alpha$ and $v = \delta$ leads to

$$\frac{\partial(u,v)}{\partial(\alpha,\delta)} = \begin{vmatrix} \frac{1}{\sqrt{\alpha^2+1}} & 0\\ 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{\alpha^2+1}} \neq 0,$$

where $\alpha \in \mathbb{R}$. Then we have

$$\begin{aligned} I_4(\alpha,\delta) &= (\alpha^2+1)\left[\frac{2\alpha^2+1}{\alpha^2+1}\,\mathrm{d}(\operatorname{arcsinh}\alpha)^2 + \mathrm{d}\,\delta^2\right]\\ &= (\alpha^2+1)\left[\frac{2\alpha^2+1}{(\alpha^2+1)^2}\,\mathrm{d}\,\alpha^2 + \mathrm{d}\,\delta^2\right]. \end{aligned}$$

Taking the parameter transformation $\alpha = \frac{\sqrt{2}}{2}\tan\beta$ and $\delta = \eta$ results in

$$\frac{\partial(\alpha,\delta)}{\partial(\beta,\eta)} = \begin{vmatrix} \frac{\sqrt{2}}{2\cos^2\beta} & 0\\ 0 & 1 \end{vmatrix} = \frac{\sqrt{2}}{2\cos^2\beta} \neq 0,$$

where $\beta \neq \frac{\pi}{2} + k\pi$. Then we acquire

$$\begin{aligned} I_4(\beta,\eta) &= \left(\frac{1}{2}\tan^2\beta + 1\right)\left[\frac{\tan^2\beta+1}{\big(\frac{1}{2}\tan^2\beta+1\big)^2}\,\mathrm{d}\left(\frac{\sqrt{2}}{2}\tan\beta\right)^2 + \mathrm{d}\,\eta^2\right]\\ &= \left(\frac{1}{2}\tan^2\beta + 1\right)\left[\frac{\sec^4\beta(\tan^2\beta+1)}{2\big(\frac{1}{2}\tan^2\beta+1\big)^2}\,\mathrm{d}\,\beta^2 + \mathrm{d}\,\eta^2\right]\\ &= \left(\frac{1}{2}\tan^2\beta + 1\right)\left[\frac{2\cos^2\beta}{(\sin^2\beta-2)^2(\sin^2\beta-1)^2}\,\mathrm{d}\,\beta^2 + \mathrm{d}\,\eta^2\right]. \end{aligned}$$

Making the parameter transformation $\beta = \arcsin\gamma$ and $\eta = \varepsilon$ arrives at

$$\frac{\partial(\beta,\eta)}{\partial(\gamma,\varepsilon)} = \begin{vmatrix} \frac{1}{\sqrt{1-\gamma^2}} & 0\\ 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{1-\gamma^2}} \neq 0,$$

where $-1 < \gamma < 1$. Then we gain

$$\begin{aligned} I_4(\gamma,\varepsilon) &= \frac{2-\gamma^2}{2(1-\gamma^2)}\left[\frac{2}{(2-\gamma^2)^2(1-\gamma^2)^2}\,\mathrm{d}\,\gamma^2 + \mathrm{d}\,\varepsilon^2\right] \\ &= \frac{2-\gamma^2}{2(1-\gamma^2)}\left[\mathrm{d}\left(\frac{1}{2}\ln\left|\frac{\gamma-\sqrt{2}}{\gamma+\sqrt{2}}\right| + \frac{\sqrt{2}}{2}\ln\left|\frac{\gamma+1}{\gamma-1}\right| + C\right)^2 + \mathrm{d}\,\varepsilon^2\right], \end{aligned}$$

where C is a constant. Finally, carrying out the substitution

$$\begin{cases} \tilde{u} = \dfrac{1}{2}\ln\left|\dfrac{\gamma-\sqrt{2}}{\gamma+\sqrt{2}}\right| + \dfrac{\sqrt{2}}{2}\ln\left|\dfrac{\gamma+1}{\gamma-1}\right| + C \\ \tilde{v} = \varepsilon \end{cases}$$

results in the relation

$$I_4 = \frac{2-\gamma^2}{2(1-\gamma^2)}\left(\mathrm{d}\,\tilde{u}^2 + \mathrm{d}\,\tilde{v}^2\right) = \frac{2-\gamma^2}{2(1-\gamma^2)} I_3,$$

where $\gamma = \frac{\sqrt{2}\sinh u}{\sqrt{2\sinh^2 u+1}}$, which can be written as $I_4 = \left(\cosh^2 u\right) I_3$. Accordingly, we obtain the equation $\lambda(u,v) = \cosh u$, which indicates that a conformal mapping is established between the uniparted hyperboloid $r(u,v)$ and the cylinder $r^*(\tilde{u},\tilde{v})$. The proof of Theorem 2 is complete. $\square$

Theorem 3. *There exists a conformal mapping between the catenoid*

$$r(u,v) = \{\cosh v\cos u, \cosh v\sin u, v\}, \quad 0 < u < 2\pi, v \in \mathbb{R}$$

and the uniparted hyperboloid

$$r^*(\tilde{u},\tilde{v}) = \{\sec\tilde{u}\cos\tilde{v}, \sec\tilde{u}\sin\tilde{v}, \tan\tilde{u}\}, \quad 0 < \tilde{v} < 2\pi, -\frac{\pi}{2} < \tilde{u} < \frac{\pi}{2}.$$

Proof. For the catenoid $r(u,v)$, by simple differentiation, it follows that

$$E_5(u,v) = \cosh^2 v, \quad F_5(u,v) = 0, \quad G_5(u,v) = \sinh^2 v + 1 = \cosh^2 v.$$

Thus, the first fundamental form of the catenoid $r(u,v)$ is

$$I_5(u,v) = \cosh^2 v\,\mathrm{d}\,u^2 + \cosh^2 v^2\,\mathrm{d}\,v^2 = \cosh^2 v\left(\mathrm{d}\,u^2 + \mathrm{d}\,v^2\right).$$

The first fundamental form of the plane $r_p(x,y) = \{x,y,0\}$ is

$$I_p(x,y) = \mathrm{d}\,x^2 + \mathrm{d}\,y^2.$$

Carrying out the parametric transform $x = u$ and $y = v$ reveals

$$I_5(u,v) = \cosh^2 v\left(\mathrm{d}\,x^2 + \mathrm{d}\,y^2\right) = \cosh^2 v I_p(x,y).$$

For the uniparted hyperboloid $r^*(\tilde{u},\tilde{v})$, it is easy to obtain

$$\begin{aligned} r^*_{\tilde{u}} &= \left(\sec\tilde{u}\tan\tilde{u}\cos\tilde{v}, \sec\tilde{u}\tan\tilde{u}\sin\tilde{v}, \sec^2\tilde{u}\right), \\ r^*_{\tilde{v}} &= (-\sec\tilde{u}\sin\tilde{v}, \sec\tilde{u}\cos\tilde{v}, 0), \end{aligned}$$

and

$$E_6(\tilde{u},\tilde{v}) = \sec^2\tilde{u}\tan^2\tilde{u} + \sec^4\tilde{u}, \quad F_6(\tilde{u},\tilde{v}) = 0, \quad G_6(\tilde{u},\tilde{v}) = \sec^2\tilde{u}.$$

Hence, the first fundamental form of the uniparted hyperboloid $r^*(\tilde{u}, \tilde{v})$ is

$$I_6(\tilde{u}, \tilde{v}) = \left(\sec^2 \tilde{u} \tan^2 \tilde{u} + \sec^4 \tilde{u}\right) \mathrm{d}\,\tilde{u}^2 + \sec^2 \tilde{u}\, \mathrm{d}\,\tilde{v}^2.$$

Replacing the parameters $\tilde{u} = \arcsin \alpha$ and $\tilde{v} = \delta$ gives

$$\frac{\partial(\tilde{u}, \tilde{v})}{\partial(\alpha, \delta)} = \begin{vmatrix} \frac{1}{\sqrt{1-\alpha^2}} & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{1-\alpha^2}} \neq 0,$$

where $-1 < \alpha < 1$. So, we attain

$$\begin{aligned} I_6(\alpha, \delta) &= \frac{1}{1-\alpha^2}\left[\left(\frac{\alpha^2}{1-\alpha^2} + \frac{1}{1-\alpha^2}\right) \mathrm{d}(\arcsin \alpha)^2 + \mathrm{d}\,\delta^2\right] \\ &= \frac{1}{1-\alpha^2}\left[\left(\frac{\alpha^2+1}{1-\alpha^2}\frac{1}{1-\alpha^2}\right) \mathrm{d}\,\alpha^2 + \mathrm{d}\,\delta^2\right]. \end{aligned}$$

Making the substitution $\alpha = \tan \beta$ and $\delta = \eta$ results in

$$\frac{\partial(\alpha, \delta)}{\partial(\beta, \eta)} = \begin{vmatrix} \frac{1}{\cos^2 \beta} & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{\cos^2 \beta} \neq 0,$$

where $\beta \neq \frac{\pi}{2} + k\pi$. Consequently, we acquire

$$\begin{aligned} I_6(\beta, \eta) &= \frac{\cos^2 \beta}{\cos 2\beta}\left[\left(\frac{\cos^2 \beta}{\cos 2\beta} \sec^2 \beta \frac{\cos^2 \beta}{\cos 2\beta}\right) \mathrm{d}(\tan \beta)^2 + \mathrm{d}\,\eta^2\right] \\ &= \frac{\cos^2 \beta}{\cos 2\beta}\left[\left(\frac{\cos^2 \beta}{\cos 2\beta} \sec^2 \beta \frac{\cos^2 \beta}{\cos 2\beta} \sec^4 \beta\right) \mathrm{d}\,\beta^2 + \mathrm{d}\,\eta^2\right] \\ &= \frac{\cos^2 \beta}{\cos 2\beta}\left[\left(\frac{1}{\cos 2\beta}\frac{1}{\cos^2 \beta}\frac{1}{\cos 2\beta}\right) \mathrm{d}\,\beta^2 + \mathrm{d}\,\eta^2\right]. \end{aligned}$$

Further making a substitution $\beta = \arcsin \gamma$ and $\eta = \varepsilon$ leads to

$$\frac{\partial(\beta, \eta)}{\partial(\gamma, \varepsilon)} = \begin{vmatrix} \frac{1}{\sqrt{1-\gamma^2}} & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{1-\gamma^2}} \neq 0,$$

where $-\frac{\sqrt{2}}{2} < \gamma < \frac{\sqrt{2}}{2}$. Accordingly, we deduce

$$\begin{aligned} I_6(\gamma, \varepsilon) &= \frac{1-\gamma^2}{1-2\gamma^2}\left[\frac{1}{(1-2\gamma^2)^2}\frac{1}{(1-\gamma^2)^2} \mathrm{d}\,\gamma^2 + \mathrm{d}\,\varepsilon^2\right] \\ &= \frac{1-\gamma^2}{1-2\gamma^2}\left[\mathrm{d}\left(\frac{1}{2}\ln\left|\frac{\gamma-1}{\gamma+1}\right| - \frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}\,\gamma-1}{\sqrt{2}\gamma+1}\right| + C\right)^2 + \mathrm{d}\,\varepsilon^2\right], \end{aligned}$$

where C is constant. Taking the replacement

$$\begin{cases} x = \dfrac{1}{2}\ln\left|\dfrac{\gamma-1}{\gamma+1}\right| - \dfrac{\sqrt{2}}{2}\ln\left|\dfrac{\sqrt{2}\,\gamma-1}{\sqrt{2}\,\gamma+1}\right| + C \\ y = \varepsilon \end{cases}$$

for C being a constant yields

$$I_6 = \frac{1-\gamma^2}{1-2\gamma^2}\left(\mathrm{d}\,x^2 + \mathrm{d}\,y^2\right) = \frac{1-\gamma^2}{1-2\gamma^2} I_p,$$

which can be reformulated as

$$I_5 = \frac{1-2\gamma^2}{1-\gamma^2}\cosh^2 y I_6 = \cos^2\tilde{u}\cosh^2\tilde{v} I_6,$$

where $\gamma = \frac{\sin\tilde{u}}{\sqrt{\sin^2\tilde{u}+1}}$. This means that $\lambda(\tilde{u},\tilde{v}) = \cos\tilde{u}\cosh\tilde{v}$, which indicates that a conformal mapping between the catenoid $r(u,v)$ and the uniparted hyperboloid $r^*(\tilde{u},\tilde{v})$ has been established. The proof of Theorem 3 is complete. □

Remark 1. Theorems 2 and 3 tell us that conformal mappings can be established for the same surface with different parametric equations.

4. A relation between area elements of surfaces and examples

In this section, we pay our attention on finding a relation between area elements of surfaces and taking several examples.

Theorem 4. *The area elements of two surfaces under a conformal mapping are proportional.*

Proof. If a mapping between two surfaces satisfies that the first fundamental forms of the corresponding surfaces are proportional, then the mapping between them is a conformal mapping. Thus, we obtain

$$\begin{aligned}\mathrm{d}\,\sigma &= \sqrt{EG-F^2}\,\mathrm{d}\,u\,\mathrm{d}\,v = \sqrt{\lambda^2E^*\lambda^2G^*-(\lambda^2F^*)^2}\,\mathrm{d}\,u\,\mathrm{d}\,v\\ &= \sqrt{\lambda^4[E^*G^*-(F^*)^2]}\,\mathrm{d}\,u\,\mathrm{d}\,v = \lambda^2\,\mathrm{d}\,\sigma^*.\end{aligned}$$

The proof of Theorem 4 is complete. □

Example 1. The relation between area elements of the hyperbolic cylinder

$$r(u,v) = \{\sec u, \tan u, v\}$$

and the plane $r^*(x,y) = \{x,y,0\}$ is $\mathrm{d}\,\sigma = \mathrm{d}\,\sigma_p$.

Proof. According to Lemma 1, the first fundamental forms of the hyperbolic cylinder $r(u,v)$ and the plane $r^*(x,y)$ have the relation $I = I_p$. Therefore, the relation between area elements of these two surfaces is

$$\mathrm{d}\,\sigma = \sqrt{EG-F^2}\,\mathrm{d}\,u\,\mathrm{d}\,v = \sqrt{\lambda^2E_p\lambda^2G_p}\,\mathrm{d}\,u\,\mathrm{d}\,v = \sqrt{E_pG_p}\,\mathrm{d}\,u\,\mathrm{d}\,v = \mathrm{d}\,\sigma_p. \quad \square$$

Example 2. The relation between area elements of the catenoid

$$r(u,v) = \left\{a\cosh\frac{v}{a}\cos u, a\cosh\frac{v}{a}\sin u, v\right\}$$

and the plane $r^*(x,y) = \{x,y,0\}$ is $\mathrm{d}\,\sigma = \cosh^2\frac{v}{a}\,\mathrm{d}\,\sigma_p$.

Proof. According to Lemma 2, the first fundamental forms of the catenoid $r(u,v)$ and the plane $r^*(x,y)$ satisfy the relation $I = \cosh^2\frac{v}{a}I_p$. Therefore, the relation between area elements of these two surfaces is

$$\mathrm{d}\,\sigma = \sqrt{EG-F^2}\,\mathrm{d}\,u\,\mathrm{d}\,v = \sqrt{\lambda^2 E_p \lambda^2 G_p}\,\mathrm{d}\,u\,\mathrm{d}\,v$$
$$= \lambda^2\sqrt{E_p G_p}\,\mathrm{d}\,u\,\mathrm{d}\,v = \cosh^2\frac{v}{a}\,\mathrm{d}\,\sigma_p. \quad \square$$

Example 3. The relation between area elements of the helicoid

$$r(\tilde u,\tilde v) = \{\tilde u\cos\tilde v, \tilde u\sin\tilde v, b\tilde v\}$$

and the plane $r^*(x,y) = \{x,y,0\}$ is

$$\mathrm{d}\,\sigma = \frac{b^2}{\cos^2\left(\arctan\frac{\tilde u}{b}\right)}\,\mathrm{d}\,\sigma_p.$$

Proof. According to Lemma 2, the relation between the first fundamental forms of the helicoid $r(\tilde u,\tilde v)$ and the plane $r^*(x,y)$ is $I = \frac{b^2}{\cos^2(\arctan\frac{\tilde u}{b})} I_p$. Therefore, the relation between area elements of these two surfaces is

$$\mathrm{d}\,\sigma = \sqrt{EG-F^2}\,\mathrm{d}\,\tilde u\,\mathrm{d}\,\tilde v = \sqrt{\lambda^2 E_p \lambda^2 G_p}\,\mathrm{d}\,\tilde u\,\mathrm{d}\,\tilde v$$
$$= \lambda^2\sqrt{E_p G_p}\,\mathrm{d}\,\tilde u\,\mathrm{d}\,\tilde v = \frac{b^2}{\cos^2\left(\arctan\frac{\tilde u}{b}\right)}\,\mathrm{d}\,\sigma_p. \quad \square$$

Example 4. For $a \neq b$, the relation between area elements of the catenoid

$$r(u,v) = \left\{a\cosh\frac{v}{a}\cos u, a\cosh\frac{v}{a}\sin u, v\right\}$$

and the helicoid $r^*(\tilde u,\tilde v) = \{\tilde u\cos\tilde v, \tilde u\sin\tilde v, b\tilde v\}$ is

$$\mathrm{d}\,\sigma = \frac{\cos^2\left(\arctan\frac{\tilde u}{b}\right)}{b^2}\cosh^2\frac{\tilde v}{a}\,\mathrm{d}\,\sigma^*.$$

Proof. From Lemma 2, it is immediate that

$$I = \frac{\cos^2\left(\arctan\frac{\tilde u}{b}\right)}{b^2}\cosh^2\frac{\tilde v}{a} I^*.$$

Therefore, we acquire

$$\mathrm{d}\,\sigma = \sqrt{EG-F^2}\,\mathrm{d}\,u\,\mathrm{d}\,v = \sqrt{\lambda^2 E^* \lambda^2 G^*}\,\mathrm{d}\,u\,\mathrm{d}\,v$$
$$= \lambda^2\sqrt{E^* G^*}\,\mathrm{d}\,u\,\mathrm{d}\,v = \frac{\cos^2\left(\arctan\frac{\tilde u}{b}\right)}{b^2}\cosh^2\frac{\tilde v}{a}\,\mathrm{d}\,\sigma^*. \quad \square$$

Example 5. The relation between area elements of the cylinder

$$r(u,v) = \{a\cos v, a\sin v, u\}$$

and the plane $r^*(x,y) = \{x,y,0\}$ is $\mathrm{d}\,\sigma_2 = \mathrm{d}\,\sigma_p$.

Proof. According to Theorem 1, the first fundamental form of the cylinder $r(u,v)$ is

$$I_2(u,v) = \mathrm{d}\,u^2 + a^2\,\mathrm{d}\,v^2 = \mathrm{d}\,u^2 + \mathrm{d}(av)^2.$$

Letting $x = u$ and $y = av$ leads to

$$I_2(u,v) = \mathrm{d}\,u^2 + \mathrm{d}(av)^2 = \mathrm{d}\,x^2 + \mathrm{d}\,y^2 = I_p(x,y).$$

This implies that $\lambda(u,v)=1$. Accordingly, the relation between area elements of these two surfaces is

$$\mathrm{d}\,\sigma_2=\sqrt{E_2G_2-F_2^2}\,\mathrm{d}\,u\,\mathrm{d}\,v=\sqrt{\lambda^2E_p\lambda^2G_p}\,\mathrm{d}\,u\,\mathrm{d}\,v=\sqrt{E_pG_p}\,\mathrm{d}\,u\,\mathrm{d}\,v=\mathrm{d}\,\sigma_p.$$
□

Example 6. The relation between area elements of the catenoid

$$r(u,v)=\left\{a\cosh\frac{u}{a}\cos v, a\cosh\frac{u}{a}\sin v, u\right\}$$

and the cylinder $r^*(u,v)=\{a\cos v, a\sin v, u\}$ is $\mathrm{d}\,\sigma_1=\cosh^2\frac{u}{a}\,\mathrm{d}\,\sigma_2$.

Proof. From Theorem 1, it is immediate that $I_1(u,v)=\cosh^2\frac{u}{a}I_2(u,v)$. Therefore, it follows that

$$\begin{aligned}\mathrm{d}\,\sigma_1&=\sqrt{E_1G_1-F_1^2}\,\mathrm{d}\,u\,\mathrm{d}\,v=\sqrt{\lambda^2E_2\lambda^2G_2}\,\mathrm{d}\,u\,\mathrm{d}\,v\\&=\lambda^2\sqrt{E_2G_2}\,\mathrm{d}\,u\,\mathrm{d}\,v=\cosh^2\frac{u}{a}\,\mathrm{d}\,\sigma_2.\end{aligned}$$
□

Similarly, using Theorems 2 and 3, we can obtain the following examples.

Example 7. The relation between area elements of the uniparted hyperboloid

$$r(u,v)=\{\cosh u\cos v,\cosh u\sin v,\sinh u\}$$

and the cylinder $r^*(\tilde{u},\tilde{v})=\{\cos\tilde{v},\sin\tilde{v},\tilde{u}\}$ is $\mathrm{d}\,\sigma_4=\cosh^2 u\,\mathrm{d}\,\sigma_3$.

Example 8. The relation between area elements of the uniparted hyperboloid

$$r(\tilde{u},\tilde{v})=\{\sec\tilde{u}\cos\tilde{v},\sec\tilde{u}\sin\tilde{v},\tan\tilde{u}\}$$

and the plane $r^*(x,y)=\{x,y,0\}$ is $\mathrm{d}\,\sigma_6=\frac{1}{\cos^2\tilde{u}}\,\mathrm{d}\,\sigma_p$.

Example 9. The relation between area elements of the catenoid

$$r(u,v)=\{\cosh v\cos u,\cosh v\sin u, v\}$$

and the uniparted hyperboloid

$$r^*(\tilde{u},\tilde{v})=\{\sec\tilde{u}\cos\tilde{v},\sec\tilde{u}\sin\tilde{v},\tan\tilde{u}\}$$

is $\mathrm{d}\,\sigma_5=\cos^2\tilde{u}\cosh^2\tilde{v}\,\mathrm{d}\,\sigma_6$.

5. Declarations

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