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Optimal Control of Defined Contribution Pension Plan under Uncertain Optimistic Value Criterion

Xiantao Wang and Yuanguo Zhu

School of Mathematics and Statistics
Nanjing University of Science and Technology
Nanjing 210094, Jiangsu, China

Abstract

This paper studies a dynamic optimal investment decision of defined contribution (DC) pension with inflation. Fund managers invest capital in different assets to minimize the quadratic loss function. Considering financial market complexity and incompleteness of information, we use optimal control under uncertain optimistic value criterion to build an optimal control model for DC pension plan. The equation of optimality for the optimal control problem is used to get the optimal pension investment strategy. Finally, a numerical experiment is given as an illustration.

Mathematics Subject Classification: 91G80

Keywords: DC pension plan; Uncertainty theory; Optimistic value; Uncertainty contribution; Optimal investment

1 Introduction

In DC pension plan, the contribution of participants is fixed, the treatment after retirement is determined by the total contribution and the return of the fund, while participants undertake the investment risks. Therefore, the investment strategy management of DC pension has received a significant attention. Under the framework of probability theory, many scholars have studied investment problems of DC pension plans in different situations. Vigna and

Haberman [1] and Gerrard [2] considered an investment problem of the DC pension plan in discrete time. Cairns[3] constructed an optimal investment strategy model of DC pension under continuous time. Based on stochastic optimal control, Gao [4] used Legendre transform and duality theory to transform optimal control model into a duality problem. Tang [5] considered an optimal investment strategy of DC pension based on quadratic utility function under the constraint of inflation risk and minimum guarantee.

However, many studies show that there are indecisive phenomena in financial market that can not be explained by probability theory or fuzzy theory. To solve such problems, Zhu [6] proposed a problem of uncertain optimal control and obtained the optimality equation of optimal control problem by using the principle of dynamic programming based on the uncertainty theory proposed by Liu [7]. Deng and Zhu [8] studied an uncertain optimal control problem with jump. They applied the model to optimal control of pension funds. Gao and Wu [9] built an uncertain optimal control model for the DC pension plan intending to minimize the quadratic loss function. Sheng and Zhu [10] used optimistic value model to establish a problem of uncertain optimal control.

Furthermore, due to the long accumulation period of DC pension, an inflation risk cannot be ignored during an investment period while being affected by financial market fluctuations. Some researches take inflation risk into account in optimal pension investment strategy. Wang [11] proposed an optimal asset allocation problem for DC pension with stochastic wages under inflation risk.

This paper will use uncertain differential equations to describe dynamic risk-free assets and risky assets. Then, we set the present value of the quadratic loss function of the optimal benefit ratio and asset allocation ratio as the objective function to construct an uncertain optimal control model. Finally, the optimal investment strategy is obtained under the optimistic value criterion by considering inflation.

2 Preliminary Notes

To begin with, some concepts on uncertainty theory [7] are recalled. A Liu process C_t was defined by Liu [12], which satisfies: (i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous; (ii) C_t has stationary and independent increments; (iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2.

Suppose C_t is a Liu process, f and g are measurable functions. Then $dX_t = f(X_t, t)\,dt + g(X_t, t)\,dC_t$ is called an uncertain differential equation. And it is equivalent to the uncertain integral equation $X_s = X_0 + \int_0^t f(X_t, t)\,dt + \int_0^t g(X_t, t)\,dC_t$.

Let ξ be an uncertain variable, and $\alpha \in (0,1]$. Then $\xi_{\sup}(\alpha) = \sup\{h \mid \mathcal{M}\{\xi \geq h\} \geq \alpha\}$ is called the α-optimistic value to ξ; and $\xi_{\inf}(\alpha) = \inf\{h \mid \mathcal{M}\{\xi \leq h\} \geq \alpha\}$ is is called the α-pessimistic value to ξ. Let ξ and η be independent uncertain variables and $\alpha \in (0,1]$. Then we have

$$(c\xi)_{\sup}(\alpha) = c\xi_{\inf}(\alpha), \quad if\ c < 0. \tag{1}$$

Assume that $\boldsymbol{C}_t = (C_{t1}, C_{t2}, \ldots, C_{tk})^{\tau}$, where $C_{t1}, C_{t2}, \ldots, C_{tk}$ are independent Liu processes. For any $0 < t < T$, and confidence level $\alpha \in (0,1)$, an uncertain optimistic value optimal control problem for multidimensional case is as follows [6]

$$\begin{cases} J(t,\boldsymbol{x}) \equiv \sup\limits_{\boldsymbol{u}_t \in U} F_{\sup}(\alpha) \\ \text{subject to} \\ \quad d\boldsymbol{X}_s = \boldsymbol{\mu}(s,\boldsymbol{u}_s,\boldsymbol{X}_s)\,ds + \boldsymbol{\sigma}(s,\boldsymbol{u}_s,\boldsymbol{X}_s)\,d\boldsymbol{C}_s \text{ and } \boldsymbol{X}_t = \boldsymbol{x} \end{cases} \tag{2}$$

where $F = \int_t^T f(s,\boldsymbol{u}_s,\boldsymbol{X}_s)\,\mathrm{d}s + G(T,\boldsymbol{X}_T)$, and $F_{\sup}(\alpha) = \sup\{\bar{F} \mid \mathcal{M}\{F \geq \bar{F}\} \geq \alpha\}$ which denotes the α-optimistic value to F. The vector $\boldsymbol{X}_s$ is a state vector of dimension n, $\boldsymbol{u}_s$ is a control vector of dimension r subject to a constraint set U.

The function $f : [0,T] \times R^r \times R^n \longrightarrow R$ is an objective function, and $G : [0,T] \times R^n \longrightarrow R$ is a function of terminal reward. In addition, $\boldsymbol{u} : [0,T] \times R^r \times R^n \longrightarrow R^n$ is a vector-value function, and $\boldsymbol{\sigma} : [0,T] \times R^r \times R^n \longrightarrow R^n \times R^k$ is a matrix-value function. All functions mentioned are continuous.

Theorem 2.1 (Equation of Optimality)**.** *[10] Let $J(t,\boldsymbol{x})$ be twice differentiable on $[0,T] \times R^n$. Then we have*

$$\begin{aligned} -J_t(t,\boldsymbol{x}) = \sup_{\boldsymbol{u}_t \in U} \{ & f(t,\boldsymbol{u}_t,\boldsymbol{x}) + \nabla_{\boldsymbol{x}} J(t,\boldsymbol{x})^{\tau} \boldsymbol{\mu}(t,\boldsymbol{u}_t,\boldsymbol{x}) \\ & + \frac{\sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha} \|\nabla_x J(t,\boldsymbol{x})^{\tau} \boldsymbol{\sigma}(t,\boldsymbol{x},\boldsymbol{u})\|_1 \} \end{aligned} \tag{3}$$

where $J_t(t,\boldsymbol{x})$ is the partial derivative of the function $J(t,\boldsymbol{x})$ in t, $\nabla_{\boldsymbol{x}} J(t,\boldsymbol{x})$ is the gradient of $J(t,\boldsymbol{x})$ in $\boldsymbol{x}$, and $\|\cdot\|_1$ is the 1-norm for vectors, that is, $\|p\|_1 = \sum\limits_{i=1}^{n} |p_i|$ for $p = (p_1, p_2, \ldots, p_n)$.

3 Optimal DC plan model

Suppose that there are two assets in a financial market that can be traded continuously. One is a bond with a return rate of r. $B(t)$ is the price of the bond at time t satisfies the differential equation $dB_t = rB_t dt$. The second asset is a stock. S_t is the price of the stock at time t satisfies the differential equation $dS_t = S_t\left[(\lambda + r - \theta)dt + \sigma_1 dC_t\right]$, where λ is an excess return rate, θ is a cost rate and σ_1 is the volatility of stock.

Due to the changes in pension policies and the existence of inflation, it is not practical to simply set the contribution amount as a constant. Therefore, we assume that the contribution amount u_t obeys an uncertain growth and satisfies the uncertain differential equation $du_t = u_t\left(\mu dt + \sigma_2 dC_t\right)$, where μ is the expectation of inflation rate and σ_2 is the volatility.

In a DC pension plan, assume that the proportion of the fund invested in stocks is listed as w. P_t is the payment amount of pension at time t, which is a predictable process. X_t represents the wealth value of the fund at time t, so we can use an uncertain differential equation to describe X_t by

$$dX_t = \left[rX_t + wX_t(\lambda - \theta) + u_t\mu - P_t\right]dt + \left(wX_t\sigma_1 + \sigma_2 u_t\right)dC_t. \tag{4}$$

According to Cairns [3], we assume that the objective of the fund is to select the optimal payouts and asset allocation. Therefore, we take minimizing the present value of the quadratic loss function as the objective function. In this way, we consider the optimal control problem of pension fund under optimistic value criterion as follows

$$\begin{cases} J(t,x,u) = \min\limits_{w,P_t}\left\{\int_0^\infty e^{-\beta s}\left[\delta_1\left(P_t - p_m\right)^2 + \delta_2\left(wX_s - x_p\right)^2\right]ds\right\}_{\inf}(\alpha) \\ \text{subject to} \\ \quad dX_t = \left[rX_t + wX_t(\lambda - \theta) + u_t\mu - P_t\right]dt + \left(wX_t\sigma_1 + \sigma_2 u_t\right)dC_t \\ \quad du_t = u_t\left(\mu dt + \sigma_2 dC_t\right) \\ \quad X_t = x, u_t = u. \end{cases} \tag{5}$$

Theorem 3.1. *Suppose the optimal DC plan model with considering inflation under uncertain optimistic value criterion is model (5). The optimal payout is* $P_t = p_m - \frac{1}{2\delta_1}\frac{\beta-2r}{4\tilde{b}}(2x + 2\frac{\tilde{d}}{r-\tilde{e}}u + \frac{2\tilde{c}}{r})$ *and the optimal proportion of the investment in the stock is* $w = \frac{1}{x}\left[x_p + \frac{\tilde{a}}{2\delta_2}\frac{\beta-2r}{4\tilde{b}}(2x + 2\frac{\tilde{d}}{r-\tilde{e}}u + \frac{2\tilde{c}}{r})\right]$.

If $J_x \geq 0$, $J_u \geq 0$, *then,* $\tilde{a} = \left[(\lambda-\theta) + \frac{\sqrt{3}}{\pi}\sigma_1\ln\frac{1-\alpha}{\alpha}\right]$, $\tilde{b} = \frac{1}{4\delta_1} + \frac{\tilde{a}^2}{4\delta_2}$, $\tilde{c} = \tilde{a}x_p - p_m$, $\tilde{d} = \tilde{e} = \mu + \frac{\sqrt{3}}{\pi}\sigma_2\ln\frac{1-\alpha}{\alpha}$.

If $J_x \geq 0$, $J_u < 0$, *then,* $\tilde{a} = \left[(\lambda-\theta) + \frac{\sqrt{3}}{\pi}\sigma_1\ln\frac{1-\alpha}{\alpha}\right]$, $\tilde{b} = \frac{1}{4\delta_1} + \frac{\tilde{a}^2}{4\delta_2}$, $\tilde{c} = \tilde{a}x_p - p_m$, $\tilde{d} = \mu + \frac{\sqrt{3}}{\pi}\sigma_2\ln\frac{1-\alpha}{\alpha}$, $\tilde{e} = \mu - \frac{\sqrt{3}}{\pi}\sigma_2\ln\frac{1-\alpha}{\alpha}$.

If $J_x < 0$, $J_u \geq 0$, *then,* $\tilde{a} = \left[(\lambda-\theta) - \frac{\sqrt{3}}{\pi}\sigma_1\ln\frac{1-\alpha}{\alpha}\right]$, $\tilde{b} = \frac{1}{4\delta_1} + \frac{\tilde{a}^2}{4\delta_2}$, $\tilde{c} = \tilde{a}x_p - p_m$, $\tilde{d} = \mu - \frac{\sqrt{3}}{\pi}\sigma_2\ln\frac{1-\alpha}{\alpha}$, $\tilde{e} = \mu + \frac{\sqrt{3}}{\pi}\sigma_2\ln\frac{1-\alpha}{\alpha}$.

If $J_x < 0$, $J_u < 0$, *then,* $\tilde{a} = \left[(\lambda-\theta) - \frac{\sqrt{3}}{\pi}\sigma_1\ln\frac{1-\alpha}{\alpha}\right]$, $\tilde{b} = \frac{1}{4\delta_1} + \frac{\tilde{a}^2}{4\delta_2}$, $\tilde{c} = \tilde{a}x_p - p_m$, $\tilde{d} = \tilde{e} = \mu - \frac{\sqrt{3}}{\pi}\sigma_2\ln\frac{1-\alpha}{\alpha}$.

Proof We apply the equation of optimality (3) to solve the model (5) by

$$\begin{aligned}
-J_t &= \max_{w,P_t}\Big\{-e^{-\beta s}[\delta_1(P_t-p_m)^2+\delta_2(wx-x_p)^2]+[rx+wx(\lambda-\theta)+u\mu-P_t]J_x \\
&\quad +\mu J_u+\frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}[|J_x(wx\sigma_1+\sigma_2 u)|+|J_u\sigma_2 u|]\Big\} \\
&= \max_{w,P_t} L(w,P_t) \qquad (6)
\end{aligned}$$

where $L(w,P_t)$ represents the term in the braces.

If $J_x<0$, $J_u\geq 0$, we differentiate $L(w,P_t)$ with respect to w and P_t:

$$\begin{cases}
\dfrac{\partial L}{\partial w}=-e^{-\beta t}2x\delta_2\left(wx-x_p\right)+x(\lambda-\theta)J_x-\dfrac{\sqrt{3}}{\pi}\sigma_1\ln\dfrac{1-\alpha}{\alpha}J_x x=0 \\
\dfrac{\partial L}{\partial P_t}=-e^{-\beta t}2\delta_1\left(P_t-p_m\right)-J_x=0.
\end{cases} \qquad (7)$$

Solving the equation (7), we get

$$w=\frac{1}{x}\left(x_p+\frac{1}{2\delta_2}\tilde{a}J_x e^{\beta t}\right),\quad P_t=p_m-\frac{1}{2\delta_1}J_x e^{\beta t} \qquad (8)$$

where $\tilde{a}=\left[(\lambda-\theta)-\frac{\sqrt{3}}{\pi}\sigma_1\ln\frac{1-\alpha}{\alpha}\right]$.

Substitute w and P_t into equation (6) to get

$$-J_t e^{\beta t}=\tilde{b}\left(J_x e^{\beta t}\right)^2+(rx+\tilde{c}+u\tilde{d})J_x e^{\beta t}+\tilde{e}uJ_u e^{\beta t} \qquad (9)$$

where $\tilde{b}=\frac{1}{4\delta_1}+\frac{\tilde{a}^2}{4\delta_2}$, $\tilde{c}=\tilde{a}x_p-p_m$, $\tilde{d}=\mu-\frac{\sqrt{3}}{\pi}\sigma_2\ln\frac{1-\alpha}{\alpha}$, $\tilde{e}=\mu+\frac{\sqrt{3}}{\pi}\sigma_2\ln\frac{1-\alpha}{\alpha}$.

Assume that $J(t,u,x)=e^{-\beta t}K\left(x^2+2Pxu+Qu^2+Rx+Su+T\right)$, and we take $J(t,u,x)$ into the equation (9) and simplify the equation to get

$$\begin{aligned}
&(4K\tilde{b}+2r-\beta)x^2+(8PK\tilde{b}+2Pr+2\tilde{d}+2P\tilde{e}-2P\beta)xu \\
&+\left(4P^2K\tilde{b}+2P\tilde{d}+2Q\tilde{e}-Q\beta\right)u^2+(4RK\tilde{b}+Rr+2\tilde{c}-R\beta)x \\
&+\left(4PRK\tilde{b}+2P\tilde{c}+R\tilde{d}+S\tilde{e}-S\beta\right)u+\left(R^2K\tilde{b}+R\tilde{c}-\beta T\right)=0. \qquad (10)
\end{aligned}$$

Solve equation (10) to find $K=\frac{\beta-2r}{4\tilde{b}}$, $P=\frac{\tilde{d}}{r-\tilde{e}}$, $Q=\frac{\tilde{d}^2}{(r-\tilde{e})^2}$, $R=\frac{2\tilde{c}}{r}$, $S=\frac{2\tilde{c}\tilde{d}}{r(r-\tilde{e})}$, $T=\frac{\tilde{c}^2}{r^2}$. Substituting K, P, Q, R, S, T to $J(t,u,x)$, and differentiate $J(t,u,x)$ with respect to x and u, we get:

$$\begin{cases}
J_x=e^{-\beta t}K(2x+2Pu+R) \\
\quad =e^{-\beta t}\dfrac{\beta-2r}{4\tilde{b}}(2x+2\dfrac{\tilde{d}}{r-\tilde{e}}u+\dfrac{2\tilde{c}}{r}) \\
J_u=e^{-\beta t}K(2Qu+2Px+S) \\
\quad =e^{-\beta t}\dfrac{\beta-2r}{4\tilde{b}}\left(2\dfrac{\tilde{d}^2}{(r-\tilde{e})^2}u+2\dfrac{\tilde{d}}{r-\tilde{e}}x+\dfrac{2\tilde{c}\tilde{d}}{r(r-\tilde{e})}\right)
\end{cases} \qquad (11)$$

Due to $J_x < 0,\ J_u \geq 0$, so we get

$$\begin{cases} (\beta - 2r)\left(x + \dfrac{\mu - \frac{\sqrt{3}}{\pi}\sigma_2 \ln \frac{1-\alpha}{\alpha}}{r - \mu - \frac{\sqrt{3}}{\pi}\sigma_2 \ln \frac{1-\alpha}{\alpha}}u \right. \\ \qquad \left. + \dfrac{[(\lambda - \theta) - \frac{\sqrt{3}}{\pi}\sigma_1 \ln \frac{1-\alpha}{\alpha}]x_p - p_m}{r}\right) < 0, \\ \dfrac{\tilde{d}(\beta - 2r)}{r - \tilde{e}}\left(x + \dfrac{\mu - \frac{\sqrt{3}}{\pi}\sigma_2 \ln \frac{1-\alpha}{\alpha}}{r - \mu - \frac{\sqrt{3}}{\pi}\sigma_2 \ln \frac{1-\alpha}{\alpha}}u \right. \\ \qquad \left. + \dfrac{[(\lambda - \theta) - \frac{\sqrt{3}}{\pi}\sigma_1 \ln \frac{1-\alpha}{\alpha}]x_p - p_m}{r}\right) \geq 0 \end{cases} \tag{12}$$

And finally we get the optimal payout and the optimal proportion of the investment in the stock by

$$w = \frac{1}{x}\left[x_p + \frac{\tilde{a}}{2\delta_2}\frac{\beta - 2r}{4\tilde{b}}(2x + 2\frac{\tilde{d}}{r - \tilde{e}}u + \frac{2\tilde{c}}{r})\right],$$

$$P_t = p_m - \frac{1}{2\delta_1}\frac{\beta - 2r}{4\tilde{b}}(2x + 2\frac{\tilde{d}}{r - \tilde{e}}u + \frac{2\tilde{c}}{r}).$$

We apply the equation of optimality to solve the remaining three cases and get the analytical expression for w and P_t similarly and omit the process here. The theorem is proven.

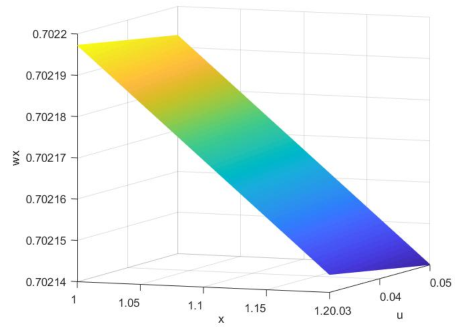
4 Numerical example

To better illustrate our proposed approach for the uncertain optimal control of DC pension plan model under optimistic value criterion and compare the effects of different parameters, we give a numerical example.

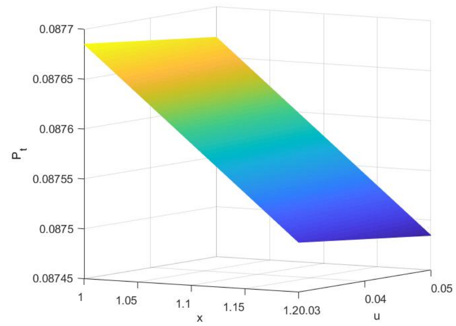
Suppose that excess return rate of stock $\lambda = 0.08$, volatility $\sigma_1 = 0.3$, cost rate $\theta = 0.0025$, inflation rate $\mu = 0.025$, volatility of inflation rate $\sigma_2 = 0.1$, target payoff $p_m = 0.08$, target funding level of stock $x_p = 0.7$, risk-free return rate $r = 0.03$, discount rate $\beta = 0.061$, $\alpha = 0.9$, $\delta_1 = \delta_2 = 0.05$.

For the parameters given above, we can known from the equation (12) that $J_x \geq 0$, $J_u < 0$. According to Theorem 3.1, we get the optimal w and P_t which are functions of x and u. Draw the plots of $w = w(x, u)$ and $P_t = P_t(x, u)$ for $x \in (1, 1.2)$ and $u \in (0.03, 0.05)$ by Figure 1.

We can see in Figure 1, as pension contribution u increases, both w and P_t increase, wx and P_t get closer to target funding level of stock $x_p = 0.7$ and

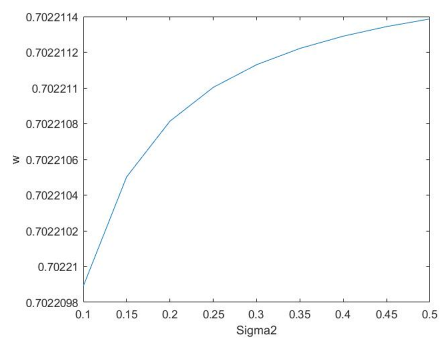


(a) The impact of x and u on wx

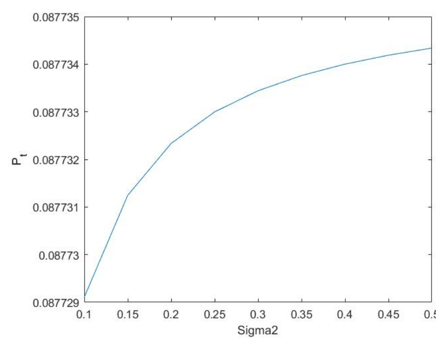


(b) The impact of x and u on P_t

Figure 1: The dynamic planning of the optimal investment strategy wx and payoff P_t



(a) The impact of σ_2 on w



(b) The impact of σ_2 on P_t

Figure 2: The impact of σ_2 on the optimal strategy w and payoff P_t

target payoff $p_m = 0.08$, respectively. As total pension amount x increases, w decreases and P_t increases, both wx and P_t get closer to their target values.

In addition, we discuss the impact of inflation rate volatility σ_2. As shown in Figure 2, we still assume total pension amount is equal to 1, so $wx = w$. With the increasing of σ_2, both w and P_t increases. An increase in σ_2 represents uncertainty in the inflation rate, which will have an impact on the objective function, and the higher the uncertainty, wx and P_t deviate even more from the target value x_p and P_m.

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