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A Formula for Mertens' Function and Its Applications

R. L. Lewis, Jr.

Excelsior Labs, Elkton, Maryland 21921 USA

Abstract

In this article, we prove the limit formula

$$\lim_{x\to\infty} \frac{|M(x)|}{\pi(x)} = \lim_{x\to\infty} \frac{h}{log(x)} = 0,\ h = a\ constant$$

for Mertens' function $M(x)$ using arithmetic and analytic arguments based on theorems for the prime counting function $\pi(x)$ and the series $\sum \frac{\mu(k)}{k}$. The formula is evaluated using limit theorems to give: an alternative proof of $\lim_{x\to\infty} \frac{|M(x)|}{x} = 0$, a new disproof of Mertens' conjecture, proof of the Odlyzko–te Riele conjecture and a disproof of the Riemann hypothesis based on Littlewood's equivalence theorem.

Mathematics Subject Classification: 11N37; 40A05; 11F66; 11A25

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1. Introduction

Mertens' function, $M(x)$, is an arithmetic function defined by the sum

$$M(x) = \mu(1) + \mu(2) + \mu(3) + \cdots + \mu(n) = \sum_{n=1}^{n\leq x} \mu(n)$$

where n is an integer $\leq x$ and $\mu(n)$ is the Mobius function defined by

$\mu(n) = 0$ if n contains a square of a prime.
$= +1$ if $n = 1$ or is square-free and has an even number of primes.
$= -1$ if n is square-free and has an odd number of primes.

The Mertens function is of great interest because of its relation to the zeta function,

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s} \ , \ s = \sigma + it$$

and Euler's product formula (Edwards [2001], p. 22, p. 260):

$$\zeta(s)^{-1} = \prod \left(1 - \frac{1}{p^s}\right) = \sum_{1}^{\infty} \frac{\mu(n)}{n^s} = s \int_0^{\infty} M(x) x^{-s-1} dx$$

And these equalities relate $M(x)$ to the zeros of $\zeta(s)$, the distribution of primes and Riemann's hypothesis (RH) which has resisted resolution since 1859. We refer the reader to Odlyzko and te Riele [1985, Sect. 1 and 2] for an in-depth discussion of these relationships and some of the history. Here we state RH and briefly discuss its relevance to prime number theory.

Riemann's hypothesis can be stated as follows: *All the "non-trivial" zeros $s = \sigma + it$ of the zeta function, $\zeta(s)$, have real part σ equal to $\frac{1}{2}$.* The other "trivial" or simple zeros of $\zeta(s)$ are derived from its functional equation

$$\zeta(s) = 2^s \pi^{s-1} sin(s\pi/2) \prod (-s)\zeta(1-s)]$$

proved by Riemann. These zeros occur at negative even integers due to the zeros of the sine function at integer multiples of π.

Riemann did not explain his reasoning for making the conjecture; so his rationale for it is unknown. However, it was found that if RH is true, improved estimates of the error in the prime number theorem and of the difference between primes could be obtained: If RH is true, Von Koch [1901; Edwards 2001, p. 90] proved the relative error in $\pi(x) \sim Li(x)$, where $Li(x) = \int \frac{dt}{log(t)}$, is less than a constant times $(log\ x)^2 x^{-1/2}$ for sufficiently large x. On the other hand if RH is not true, then the relative error does not grow less rapidly than $x^{1/2+\epsilon}$.

Dudek [2014] proved that if RH is true there exists a prime in the interval $(x - \frac{4}{\pi}\sqrt{x} log(x), x)$ which is the latest improvement on results obtained by Cramér [1920] and von Koch [1901]. Letting $x = p_n$, where p_n is the n^{th} prime implies there exists a prime in the interval $(p_n - \frac{4}{\pi}\sqrt{p}_n log(p_n), p_n)$ if n is sufficiently large.

The RH is believed by many to be true and there are deep theoretical results and a large body of computational data regarding the zeros that suggested it

could be true [Edwards (2001) and Borwein et al. (2006)]. There also has been a substantial amount of theoretical and computational work which has called RH into question. Ivic's paper in Borwein, et al. (ch. 11) cites results to make a case against RH and includes a tentative proof RH is false. Edwards (p. 164) has speculated on Riemann's reasoning behind his hypothesis and he concludes:

> Even today ... one cannot really give any solid reasons for saying that the truth of the hypothesis is probable. The theorem of Bohr and Landau ... stating that for any $\delta > 0$ all but an infinitesimal portion of the roots ρ lie within δ of $Re\ s = \frac{1}{2}$ is the only positive result which lends real credence to the hypothesis. Also verification of the hypothesis for the first three and a half million roots ... perhaps make it more "probable." [The first 10 trillion zeros have been confirmed to lie on the "critical line" $x = \frac{1}{2}$.] However any real reason, any plausibility argument or heuristic basis for the statement seems entirely lacking.
>
> ... unless some basic cause is operating which has eluded mathematicians for 110 years, occasional roots ρ off the line are altogether possible.

Littlewood (whose theorem motivated the research in this article) later in life expressed sentiments about RH similar to those of Edwards:

> I believe this to be false. There is no evidence whatever for it (unless one counts that it is always nice when any function has only real roots). ... there is no imaginable reason why it should be true. [Good (1963), p. 390]

Earlier in his career, Littlewood [1912] outlined a proof of a theorem that states criteria for the Mertens function $M(x)$ that are equivalent to RH. (Edwards [2001, p. 261] later filled in details omitted from the proof.)

Theorem 1 (**Littlewood**). *Riemann's hypothesis is equivalent to the statement: for every $\epsilon > 0$ the function $M(x)/x^{1/2+\epsilon}$ approaches zero as $x \to \infty$.*

This statement can be expressed by the following limit:

$$\lim_{x\to\infty} \frac{M(x)}{x^{1/2+\epsilon}} = \ 0 \ for\ every\ \epsilon > 0$$

Littlewood's theorem reduces the proof or disproof of RH to an equivalent problem of finding the limit of an infinite series for $M(x)$. This forms the basis of the approach we take to resolve RH by writing $M(x)$ as a new infinite series and proving a formula for the series that allows evaluation of the above

limit. We show the formula is sufficiently general to have applications to other relations and conjectures involving $M(x)$.

It has been known for over 100 years that $M(x)$ does not oscillate within finite limits (Fatou [1906]) and $O(x)$ estimates of the function are well known (Landau [1909, Vol. II p. 570 and p. 594]; see also Wei [2010]); however no asymptotic or limit formula giving the rate of growth of $M(x)$ has been reported so far.

In this article a limit formula for $M(x)$ is proved using arithmetic arguments based on properties of $M(x)$, the prime counting function $\pi(x)$, prime number theorem (and Sierpinski's lemma) and theorems of Hall and Vallée Poussin for Euler's series $\sum \frac{\mu(k)}{k}$ which is equivalent to the prime number theorem (Landau [1911]). We evaluate the formula using standard limit theorems to obtain an alternative proof of

$$\lim_{x\to\infty} \frac{|M(x)|}{x} = 0,$$

a new disproof of Mertens' conjecture

$$\lim_{x\to\infty} \frac{|M(x)|}{x^{1/2}} < 1,$$

a proof of the Odlyzko–te Riele conjecture

$$\lim_{x\to\infty} sup \frac{|M(x)|}{x^{1/2}} = \infty$$

and a disproof of the Riemann hypothesis using the limit criteria in Littlewood's theorem.

Theorem 2.

$$\lim_{x\to\infty} \frac{|M(x)|}{\pi(x)} = \lim_{x\to\infty} \frac{h}{log(x)} = 0,\ h = a\ constant \tag{1}$$

PROOF.

Theorem 3. *When x tends to infinity*

$$M(x) = -\left[\pi(x) - \pi\left(\frac{x}{2}\right)\right] + \cdots - \mu(p_i \cdots)\left[\pi\left(\frac{x}{p_i \cdots}\right) - \pi\left(\frac{x}{2p_i \cdots}\right)\right] \cdots \tag{2}$$

where $\pi(x)$, $\pi(\frac{x}{2})$, $\pi(\frac{x}{p_i \cdots})$ and $\pi(\frac{x}{2p_i \cdots})$ is the prime counting function, $p_i \cdots$ represents a square-free integer containing either one odd prime (3, 5, 7, ...) or the product of two or more distinct odd primes, $\mu(p_i \cdots)$ is the Mobius function and terms in the series are arranged in the order of square-free odd integer divisors.

Corollary 1. *When x tends to infinity*

$$M(x) = \sum_{k=1}^{k<x} -\mu(k)\left[\pi\left(\frac{(-1)^{k-1}+1}{2k}x\right) - \pi\left(\frac{(-1)^{k-1}+1}{4k}x\right)\right] \tag{3}$$

where $k = 1, 2, 3, \cdots < x$ tends to ∞ and terms in the series are non-zero and square-free when k is odd and $\mu(k) \neq 0$ (i.e., $k = 1$ or $p_i \cdots$).

Proof of Theorem 3.

Lemma 1. *For a number $x \geq 4$, $M(x)$ equals the sum of $\mu(n)$ of the square-free odd integers $n > \frac{x}{2}$ and $\leq x$.*

Proof. Let x be ≥ 4, then by definition $M(x)$ is the sum of $\mu(n)$ for all integers $n \leq x$; however, every square-free even integer $\leq x$ is a multiple of 2 of a square-free odd integer in the interval $[1, \frac{x}{2}]$; therefore, they have opposite signs for $\mu(n)$ since the even integer has one more prime divisor and cancel in pairs to 0. Also for even or odd integers that contain a square of a prime $\mu(n) = 0$. So the only integers for which $\mu(n)$ has not been canceled to 0 are the square-free odd integers $> \frac{x}{2}$ and $\leq x$.

Corollary 2. *$M(x)$ equals the sum of the number of square-free odd integers $> \frac{x}{2}$ and $\leq x$ containing $1, 3, 5, \ldots$ prime factors multiplied by $\mu(n) = -1$ and the number containing $2, 4, 6, \ldots$ prime factors multiplied by $\mu(n) = +1$.*

In what follows $\pi(x)$, $\pi(\frac{x}{2})$, $\pi(\frac{x}{p_i\cdots})$ and $\pi(\frac{x/2}{p_i\cdots}) = \pi(\frac{x}{2p_i\cdots})$ is the prime counting function which equals the number of primes $\leq x$, $\frac{x}{2}$, $\frac{x}{p_i\cdots}$ and $\frac{x/2}{p_i\cdots}$ respectively; where $p_i \cdots$ represents one or more odd primes $< \sqrt{x}$ of a square-free divisor ordered from smallest to largest. Using this function and Corollary 2 we are able to deduce the following conclusions and relations:

(1) For $x \geq 4$, square-free odd integers $> \frac{x}{2}$ and $\leq x$ containing one prime factor are the odd primes, p_k, in the interval $(\frac{x}{2}, x]$ which equals the difference between $\pi(x)$, the total number of primes $\leq x$, and $\pi(\frac{x}{2})$, the number of primes $\leq \frac{x}{2}$. So $\mu(p_k)$ times the number of primes $> \frac{x}{2}$ and $\leq x$ equals $(-1)[\pi(x) - \pi(\frac{x}{2})]$, the first term of the series for computing $M(x)$.

(2*a*) The integer part of $\frac{x}{p_i\cdots}$ and $\frac{x}{2p_i\cdots}$ equals the number of multiples of $p_i \cdots \leq x$ and $\frac{x}{2}$; therefore, $\pi(\frac{x}{p_i\cdots})$ and $\pi(\frac{x}{2p_i\cdots})$ equal the number of *prime multiples* of $p_i \cdots \leq x$ and $\frac{x}{2}$.

(2*b*) Square-free odd integers $\leq x$ that contain two or more prime factors must contain one or more odd primes $< \sqrt{x}$; otherwise, the product of two prime divisors of a square-free integer would be $> x$. Therefore, for an odd prime $p_i < \sqrt{x}$ the number of odd square-free integers $> \frac{x}{2}$ and $\leq x$ having two prime factors of the form p_ip_j, where $p_i < p_j$, equals $[\pi(\frac{x}{p_i}) - \pi(\frac{x}{2p_i})]$ *if x is sufficiently large* (i.e., $\frac{x}{p_i}$ and $\frac{x}{2p_i} \geq p_i$). This condition on the size of x is

necessary to exclude counting p_i^2 (which is a multiple of $p_i < x$) and to prevent double counting other square-free integers that contain p_i. When x tends to infinity, clearly this condition will be satisfied for any p_i.

If x is finite and $\frac{x}{p_i} \geq p_i$ but $\frac{x}{2p_i}$ is not then there will be $[\pi(\frac{x}{p_i}) - \pi(p_i)]$ integers $p_i p_j > \frac{x}{2}$ and $\leq x$. So $\mu(p_i p_j) = \mu(p_i)\mu(p_j) = -\mu(p_i)$ times the number of square-free integers $p_i p_j > \frac{x}{2}$ and $\leq x$ equals $-\mu(p_i)[\pi(\frac{x}{p_i}) - \pi(\frac{x}{2p_i})]$ or $-\mu(p_i)[\pi(\frac{x}{p_i}) - \pi(p_i)]$; otherwise it is zero. This calculation is made for all odd primes, $p_i < \sqrt{x}$ when x is finite. Similar considerations apply to computations for square-free odd integers having three or more prime factors.

Therefore when $x \to \infty$ by summing all the terms generated by the above procedures (in the order of square-free odd integer divisors) one obtains the infinite sum (2) for $M(x)$ in Theorem 3.

Proof of Corollary 1.

In the series of Corollary 1 when k is even terms in parentheses $= 0$; so non-zero terms occur only if k is odd and their numerator $(= 2)$ and denominator cancel to become $\frac{1}{k}$ and $\frac{1}{2k}$. Multiplying the term within brackets by $-\mu(k)$ insures only odd divisors in which $k = 1$ or is square-free $(= p_i \cdots)$ occur in the series. As in (2) the $(-)$ sign adjusts for the fact that square-free integers generated by a divisor k contain one more prime than k.

From equation (3) for $M(x)$ we prove the following formula:

Theorem 4. *When x tends to infinity,*

$$\lim_{x\to\infty} \frac{M(x)}{\pi(x)} = -\sum_{k=1}^{k<x} \frac{\mu(k)}{k} \ = 0 \tag{4}$$

where $k = 1, 2, 3, \cdots < x$ tends to ∞.

Proof. For proof we use a lemma due to Sierpinski [1988].

Lemma 2 (Sierpinski)**.** *When x tends to infinity*

$$\lim_{x\to\infty} \frac{\pi(ax)}{\pi(bx)} = \frac{a}{b} \tag{5}$$

where a and b are positive real numbers, $0 < a < b$.

Let $x \to \infty$, then dividing (3) by $\pi(x)$, taking the limit and applying Sierpinski's lemma with $b = 1$ after combining terms we obtain

$$\lim_{x\to\infty} \frac{M(x)}{\pi(x)} = \sum_{k=1}^{k<x} -\mu(k)\left[\frac{(-1)^{k-1}+1}{4k}\right] \tag{6}$$

where $k = 1, 2, 3, \cdots < x$ tends to ∞.

Equation (6) can be written as

$$\lim_{x\to\infty} \frac{M(x)}{\pi(x)} = -\frac{1}{4}\left[\sum_{1}^{k<x} \frac{(-1)^{k-1}\mu(k)}{k} + \sum_{1}^{k<x} \frac{\mu(k)}{k}\right] \tag{7}$$

Hall [1972] and Apostol [1973] proved the following theorem:

Theorem (Hall).

$$\left(1 - \frac{1}{2^s}\right)\sum_{1}^{\infty} \frac{(-1)^{n-1}\mu(n)}{n^s} = \left(1 + \frac{1}{2^s}\right)\sum_{1}^{\infty} \frac{\mu(n)}{n^s} \tag{8}$$

where $s = \sigma + it$ and $\sigma \geq 1$

Using Hall's theorem with $s = 1$, (7) is written as

$$\lim_{x\to\infty} \frac{M(x)}{\pi(x)} = -\frac{1}{4}\left[\frac{1 + 2^{-1}}{1 - 2^{-1}}\sum_{1}^{k<x} \frac{\mu(k)}{k} + \sum_{1}^{k<x} \frac{\mu(k)}{k}\right] \tag{9}$$

then adding the two series within brackets and multiplying by $\frac{1}{4}$ gives

$$\lim_{x\to\infty} \frac{M(x)}{\pi(x)} = -\sum_{1}^{k<x} \frac{\mu(k)}{k} \tag{10}$$

where $k = 1, 2, 3, \cdots < x$ tends to ∞.
Von Mangoldt [1897] and Landau [1899] proved the sum (10) converges to 0, which proves Theorem 4.

Von Mangoldt's and Landau's proofs are indirect and only prove convergence. Vallée Poussin [1899, p. 63] improved on their result by proving a bound for the sum and a formula for its rate of convergence.

Theorem (Vallée Poussin). "The sum

$$\sum_{k<x} \frac{\mu(k)}{k}$$

tends to zero when x tends to infinity, and its absolute value remains less than an expression of the form

$$\frac{h}{lx}$$

where h is a fixed number, which can be assigned a precise value."

Vallée Poussin does not assign a precise value to h; therefore, below we only refer to h symbolically.

Since $M(x)$ and $\pi(x)$ are continuous functions, taking the absolute values of the terms in (10) gives

$$\left|\lim_{x\to\infty}\frac{M(x)}{\pi(x)}\right| = \lim_{x\to\infty}\frac{|M(x)|}{\pi(x)} \;=\; \left|\sum_{1}^{k<x}\frac{\mu(k)}{k}\right| \tag{11}$$

Substituting Vallée Poussin's formula for the sum in (11) we can write

$$\lim_{x\to\infty}\frac{|M(x)|}{\pi(x)} = \lim_{x\to\infty}\frac{h}{log(x)} = 0 \tag{12}$$

This completes the proof of Theorem 2.

2. Applications and Results

2.1 *Proof of* $\lim_{x\to\infty}\frac{|M(x)|}{x} = 0$ (13)

Von Mangoldt [1897] first proved this relation using a neat ϵ/δ type proof. (See Landau [1899] for details and [1909, Vol. II, p. 588] for his own proof. See also Kalecki [1967].) We prove it from Theorem 2 in two different ways by applying limit theorems. The simplest proof of (13) comes from noting that since $x > \pi(x)$ for $x > 0$, the function

$$\frac{|M(x)|}{x} \le \frac{|M(x)|}{\pi(x)} \tag{14}$$

so by the limit theorems if

$$\lim_{x\to\infty}\frac{|M(x)|}{\pi(x)} = 0,\ then\ \lim_{x\to\infty}\frac{|M(x)|}{x} \le 0 \tag{15}$$

However, the limit cannot be negative; so it must $= 0$.

For our second proof we multiply both sides of (12) by $\lim_{x\to\infty}\frac{\pi(x)}{x}$ to obtain

$$\lim_{x\to\infty}\frac{|M(x)|}{\pi(x)}\lim_{x\to\infty}\frac{\pi(x)}{(x)} = \lim_{x\to\infty}\frac{h}{log(x)}\lim_{x\to\infty}\frac{\pi(x)}{x} \tag{16}$$

Then using the limit theorems we can write (16) as

$$\lim_{x\to\infty}\frac{|M(x)|}{\pi(x)}\frac{\pi(x)}{x} = \lim_{x\to\infty}\frac{h}{log(x)}\lim_{x\to\infty}\frac{\pi(x)}{x}\frac{log(x)}{log(x)} \tag{17}$$

which implies

$$\lim_{x\to\infty}\frac{|M(x)|}{x} = \lim_{x\to\infty}\frac{h}{log(x)}\lim_{x\to\infty}\frac{\pi(x)log(x)}{x}\lim_{x\to\infty}\frac{1}{log(x)} \tag{18}$$

By the prime number theorem we have

$$\lim_{x\to\infty} \frac{\pi(x)}{x/log(x)} = \lim_{x\to\infty} \frac{\pi(x)log(x)}{x} = 1 \tag{19}$$

Then using the prime number theorem (18) is simplified to give

$$\lim_{x\to\infty} \frac{|M(x)|}{x} = \lim_{x\to\infty} \frac{h}{log(x)} \lim_{x\to\infty} \frac{1}{log(x)} = \lim_{x\to\infty} \frac{h}{log^2(x)} = 0 \tag{20}$$

2.2 *Disproof of* $\lim_{x\to\infty} \frac{|M(x)|}{x^{1/2}} < 1$ (21)

This relation, conjectured by Mertens, was disproved first by Odlyzko and te Riele [1985] using computational methods. Based on the techniques they utilized and the trends in their data, they conjectured that it seemed very probable that

$$\lim_{x\to\infty} sup \frac{|M(x)|}{x^{1/2}} = \infty \tag{22}$$

In what follows we apply Theorem 2 and limit theorems to give a new disproof of Mertens' conjecture and concurrently a proof of the Odlyzko–te Riele conjecture. We begin by multiplying both sides of (12) by $\lim_{x\to\infty} \frac{\pi(x)}{x^{1/2}}$ to obtain

$$\lim_{x\to\infty} \frac{|M(x)|}{\pi(x)} \lim_{x\to\infty} \frac{\pi(x)}{x^{1/2}} = \lim_{x\to\infty} \frac{h}{log(x)} \lim_{x\to\infty} \frac{\pi(x)}{x^{1/2}} \tag{23}$$

Then using the limit theorems (23) is written as

$$\lim_{x\to\infty} \frac{|M(x)|}{\pi(x)} \frac{\pi(x)}{x^{1/2}} = \lim_{x\to\infty} \frac{h}{log(x)} \lim_{x\to\infty} \pi(x) \frac{x^{1/2}}{x} \frac{log(x)}{log(x)} \tag{24}$$

This gives the same setup to apply the prime number theorem as in the previous proof. So we write (24) as

$$\lim_{x\to\infty} \frac{|M(x)|}{x^{1/2}} = \lim_{x\to\infty} \frac{h}{log(x)} \lim_{x\to\infty} \frac{\pi(x)log(x)}{x} \lim_{x\to\infty} \frac{x^{1/2}}{log(x)} \tag{25}$$

and apply the prime number theorem and limit theorems to get

$$\lim_{x\to\infty} \frac{|M(x)|}{x^{1/2}} = \lim_{x\to\infty} \frac{hx^{1/2}}{log^2(x)} \tag{26}$$

Finally, to evaluate the limit on the right we apply L'Hospital's rule by taking the derivative of the numerator and denominator two times. This gives

$$\lim_{x\to\infty} \frac{|M(x)|}{x^{1/2}} = \lim_{x\to\infty} \frac{h}{2^3} x^{1/2} = \infty \tag{27}$$

disproving Mertens' conjecture and proving the Odlyzko–te Riele conjecture.

2.3 *Disproof of* $\lim_{x\to\infty} \frac{|M(x)|}{x^{1/2+\epsilon}} = 0$, *for every* $\epsilon > 0$ *and RH* (28)

In the Introduction it was stated that the limit (28) expresses Littlewood's theorem which is equivalent to RH. The previous result $\lim_{x\to\infty} \frac{|M(x)|}{x} = 0$ proves the criteria for $\epsilon \geq \frac{1}{2}$, and since the theorem excludes $\epsilon = 0$, the disproof of Mertens' conjecture and proof of the Odlyzko–te Riele conjecture do not exclude the possibility for RH to be true. *[Note that if Mertens' conjecture was true then RH would also be true.]* So we are left with the case $0 < \epsilon < \frac{1}{2}$. To evaluate the limit for this case we proceed as we did above by multiplying both sides of (12) by $\lim_{x\to\infty} \frac{\pi(x)}{x^{1/2+\epsilon}}$ to obtain

$$\lim_{x\to\infty} \frac{|M(x)|}{\pi(x)} \lim_{x\to\infty} \frac{\pi(x)}{x^{1/2+\epsilon}} = \lim_{x\to\infty} \frac{h}{log(x)} \lim_{x\to\infty} \frac{\pi(x)}{x^{1/2+\epsilon}} \tag{29}$$

Then using the limit theorems we can write (29) as

$$\lim_{x\to\infty} \frac{|M(x)|}{\pi(x)} \frac{\pi(x)}{x^{1/2+\epsilon}} = \lim_{x\to\infty} \frac{h}{log(x)} \lim_{x\to\infty} \pi(x) \frac{x^{1/2-\epsilon}}{x} \frac{log(x)}{log(x)} \tag{30}$$

Applying limit theorems again, (30) is written as

$$\lim_{x\to\infty} \frac{|M(x)|}{x^{1/2+\epsilon}} = \lim_{x\to\infty} \frac{h}{log(x)} \lim_{x\to\infty} \frac{\pi(x)log(x)}{x} \lim_{x\to\infty} \frac{x^{1/2-\epsilon}}{log(x)} \tag{31}$$

By the prime number theorem and limit theorems (31) is simplified to give

$$\lim_{x\to\infty} \frac{|M(x)|}{x^{1/2+\epsilon}} = \lim_{x\to\infty} \frac{hx^{1/2-\epsilon}}{log^2(x)} \tag{32}$$

Then applying L'Hospital's rule to the limit on the right we obtain

$$\lim_{x\to\infty} \frac{|M(x)|}{x^{1/2+\epsilon}} = \lim_{x\to\infty} \frac{h(\frac{1}{2}-\epsilon)^2 x^{1/2-\epsilon}}{2} \tag{33}$$

The term $\frac{1}{2} - \epsilon$ is positive for $0 < \epsilon < \frac{1}{2}$; so the limit is infinite for this case since any positive power of x tends to infinity. This result proves an exception to Littlewood's criteria therefore disproving Riemann's hypothesis.

Before ending we note that the limit relations proved earlier can be proved using the formula in (33) by letting $\epsilon = \frac{1}{2}$ and 0; however the approach that we have taken may be clearer and more informative.

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