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Solutions of a Countable Set of Non-Elementary Integrals by Means of Simple Algebra

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Abstract

Many integrals can be solved in an elementary way. However, the overwhelming majority of functions cannot be integrated via one of the elementary integration techniques. Moreover, their antiderivative is not expressible in terms of elementary functions. In this note an alternative approach for the solution of a countable set of non-elementary integrals is provided. This approach involves the use of algebraic manipulation and an integral form of the Zeta function.

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1 Introduction

Integration and differentiation are the most important operations in calculus. Many integrals can be solved in an elementary way, or by applying a combination of elementary techniques such as integration by parts, substitution or the reverse chain rule. Also performing partial fraction expansion to express the integrand as a sum of integrands can be a viable tool to simplify computations. Similarly, if convergent, the integral can be described by the integral over its Taylor series expansion. The overwhelming majority of functions however cannot be integrated via one of the elementary integration techniques. Their antiderivative is not expressible in terms of elementary functions.

Nevertheless, at least in a few cases, it is possible to express definite solutions in terms of known and well defined functions. In spite of todays possibility to perform calculations by computer, interest and curiosity for formulae expressing the solution of non-elementary integrals are still well alive. In support of this statement, we recall here e.g. the recent publication of the books [1] and [2]. In the preface of both of them the authors write that following the publication of previous editions, they have been receiving letters and mails and comments from a large number of readers.

2 Preliminary Notes

A simple example of a function whose antiderivative is not expressible in terms of elementary functions is

$$f(x) = \frac{e^x}{x}. \tag{1}$$

This is quite common for expressions containing both exponential and rational functions. Another example is the solution to the infinite (uncountable) set of definite integrals

$$\int_0^\infty \frac{x^{\nu-1}}{e^{\mu x}+1}\,\mathrm{d}x = \frac{1}{\mu^\nu}(1-2^{1-\nu})\Gamma(\nu)\zeta(\nu), \qquad [Re(\mu)>0, Re(\nu)>0] \tag{2}$$

where $\Gamma(x)$ and $\zeta(x)$ are the Gamma- and Riemann Zeta functions respectively. Notice that there is no step by step procedure or technique to derive such equivalences by integration. And it is worth mentioning that integrals of this form are often found in quantum statistics, see e.g. [3]. It has been further shown (see e.g. [4]) that, by letting $\nu = 2n$, where $n \in N$, the solution in (2) simplifies to

$$\int_0^\infty \frac{x^{\nu-1}}{e^{px}+1}\,\mathrm{d}x = \left(\frac{2\pi}{p}\right)^\nu (1-2^{1-\nu})\frac{|B_\nu|}{2\nu}, \tag{3}$$

where $|B_\nu|$ is the absolute value of the ν-th Bernoulli number.

Bernoulli numbers are in fact known since the time of Ada Lovelace, who in 1842 developed what is considered to be the first algorithm in informatics to calculate them. We point out in this connection that a fascinating essay on the life of Ada, whose true surname was Byron (she was the daughter of the poet Lord Byron), and on the birth and development of mechanical calculators and the computational approach in a historical perspective can be found in [5].

The goal of this paper is to provide an alternative approach to calculate explicit solutions for the set of integrals in (3) when $p = \pi$. The derivation is based solely on simple algebra and on a well known result related to the Riemann Zeta function.

3 Main Results

All solutions of integrals in (3) when $p = \pi$ lie in the set of rational numbers. Consider the Riemann Zeta function given in the integral form (see e.g. [6])

$$\zeta(s) = \frac{2^{s-1}}{s-1} - 2^s \int_0^\infty \frac{\sin(s\arctan(x))}{(1+x^2)^{\frac{s}{2}}(e^{\pi x}+1)}\,\mathrm{d}x. \tag{4}$$

Note that applying the Euler formula on the numerator of the integrand in (4) we have

$$\begin{aligned}
\sin(s\arctan(x)) &= \frac{i}{2}\left(e^{-is\arctan(x)} - e^{is\arctan(x)}\right) \\
&= \frac{i}{2}\left[e^{-is\left(\frac{i}{2}ln\left(\frac{1-ix}{1+ix}\right)\right)} - e^{is\left(\frac{i}{2}ln\left(\frac{1-ix}{1+ix}\right)\right)}\right] \\
&= \frac{i}{2}\left[\left(\frac{1-ix}{1+ix}\right)^{\frac{s}{2}} - \left(\frac{1-ix}{1+ix}\right)^{-\frac{s}{2}}\right].
\end{aligned} \tag{5}$$

In view of (5) the integrand in (4) simplifies to

$$\begin{aligned}
\frac{\sin(s\arctan(x))}{(1+x^2)^{\frac{s}{2}}(e^{\pi x}+1)} &= \frac{\frac{i}{2}\left[\left(\frac{1-ix}{1+ix}\right)^{\frac{s}{2}} - \left(\frac{1-ix}{1+ix}\right)^{-\frac{s}{2}}\right]}{((1+ix)(1-ix))^{\frac{s}{2}}(e^{\pi x}+1)} \\
&= \frac{\frac{i}{2}\left[\left(1+ix\right)^{-s} - \left(1-ix\right)^{-s}\right]}{e^{\pi x}+1}.
\end{aligned} \tag{6}$$

Thus, an alternative expression of the Riemann zeta function in (4) is

$$\zeta(s)=\frac{2^{s-1}}{s-1}-2^{s-1}i\int_0^\infty\frac{\big(1+ix\big)^{-s}-\big(1-ix\big)^{-s}}{e^{\pi x}+1}\,\mathrm{d}x. \tag{7}$$

It is time now to make use of the fact that the trivial zeros of the Riemann Zeta function lie at the even negative integers. Albeit not trivial to be verified when the Zeta function is expressed in one of the forms (4) or (7) above, this fact becomes immediately evident when the Zeta function is written in a form containing factor $\sin(\pi s/2)$.

Evaluating (7) at $s=-2$ yields

$$\begin{aligned}\zeta(-2)&=-\frac{1}{24}-\frac{i}{8}\int_0^\infty\frac{(1+2ix-x^2)-(1-2ix-x^2)}{e^{\pi x}+1}\,\mathrm{d}x\\&=-\frac{1}{24}+\frac{1}{2}\int_0^\infty\frac{x}{e^{\pi x}+1}\,\mathrm{d}x=0,\end{aligned} \tag{8}$$

which implies

$$\int_0^\infty\frac{x}{e^{\pi x}+1}\,\mathrm{d}x=\frac{1}{12}. \tag{9}$$

Similarly, substituting $s=-4$ in ((7)) yields

$$\zeta(-4)=-\frac{1}{160}+\frac{1}{4}\int_0^\infty\frac{x-x^3}{e^{\pi x}+1}\,\mathrm{d}x=0, \tag{10}$$

from which, by using (9) one also gets

$$\int_0^\infty\frac{x^3}{e^{\pi x}+1}\,\mathrm{d}x=-\frac{1}{40}+\int_0^\infty\frac{x}{e^{\pi x}+1}\,\mathrm{d}x=\frac{7}{120}. \tag{11}$$

Solutions of subsequent integrals can be obtained recursively by continuing to substitute the negative even integers into (7) and solving for the integral whose integrand is of highest order in the numerator. By making use of Pascal's triangle, this process can be generalized as follows:

$$\begin{aligned}&\frac{\zeta(-2n)}{2^{-2n-1}}=-\frac{1}{2n+1}\\&+2\int_0^\infty\frac{p_{2n,2}x-p_{2n,4}x^3+p_{2n,6}x^5-...-(-1)^n p_{2n,2n}x^{2n-1}}{e^{\pi x}+1}\,\mathrm{d}x=0,\end{aligned} \tag{12}$$

where $p_{2n,j}$ is the j-th entry from the left in the $(2n+1)$-th row of Pascal's triangle. Note that index j takes on only even values in the integrand, as the odd values cancel out. The imaginary terms appear twice with equal signs, yielding the factor of two in front of the definite integral. The solution to the integral in (12) with the highest order integrand is then expressed as

$$\int_0^\infty \frac{x^{2n-1}}{e^{\pi x}+1}\,\mathrm{d}x = \frac{\frac{1}{4n+2} + \sum_{k=1}^{n-1}\left[(-1)^k p_{2n,2k}\int_0^\infty \frac{x^{2k-1}}{e^{\pi x}+1}\,\mathrm{d}x\right]}{(-1)^{n+1}p_{2n,2n}}, \quad n \in N. \tag{13}$$

Thus, it is a simple scaled linear combination of all lower order integrals added to some offset.

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