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Hypercyclicity of the Adjoint of Weighted Composition Operators on the Reproducing Kernel Hilbert Spaces

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Abstract

The aim of this paper is to study the hypercyclicity of the adjoint of weighted composition operators on the vector-valued analytic reproducing kernel Hilbert spaces.

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1 Introduction

Denote by $\mathbb{D}$ the open unit disk in the complex plane $\mathbb{C}$. Let $\mathcal{E}$ be a Hilbert space and $\mathfrak{L}(\mathcal{E})$ the set of bounded linear operators on $\mathcal{E}$. An operator-valued function $K : \mathbb{D} \times \mathbb{D} \to \mathfrak{L}(\mathcal{E})$ is called an analytic kernel (cf. [6]) if for any fixed $w \in \mathbb{D}$, the operator-valued function $K(\cdot, w) : \mathbb{D} \to \mathfrak{L}(\mathcal{E})$ is analytic, and

$$\sum_{i,j=1}^{n} \left\langle K(w_i, w_j)\eta_j, \eta_i \right\rangle_{\mathcal{E}} \geq 0,$$

for all $\{w_i\}_{i=1}^n \subset \mathbb{D}$, $\{\eta_i\}_{i=1}^n \subset \mathcal{E}$ and $n \in \mathbb{N}^+$. In this case, by the Moore's Theorem ([7]), there exists a Hilbert space $\mathcal{H}_{\mathcal{E}}(K)$ of $\mathcal{E}$-valued analytic functions on $\mathbb{D}$ such that $\{K(\cdot, w)\eta : w \in \mathbb{D}, \eta \in \mathcal{E}\}$ is a total set in $\mathcal{H}_{\mathcal{E}}(K)$, and we call $\mathcal{H}_{\mathcal{E}}(K)$ the vector-valued analytic reproducing kernel Hilbert spaces.

Let ψ be a multiplier of $\mathcal{H}_{\mathcal{E}}(K)$, i.e., ψ is a complex-valued function on $\mathbb{D}$ satisfying $\psi \cdot \mathcal{H}_{\mathcal{E}}(K) \subset \mathcal{H}_{\mathcal{E}}(K)$. Suppose that φ is an analytic self-map of $\mathbb{D}$ such that $\big(f \circ \varphi\big)(z) = f\big(\varphi(z)\big) \in \mathcal{H}_{\mathcal{E}}(K)$, then the weighted composition operator $C_{\varphi,\psi}$ is defined by

$$\big(C_{\varphi,\psi} f\big)(z) = \psi(z) f\big(\varphi(z)\big),$$

for $f \in \mathcal{H}_{\mathcal{E}}(K)$ and $z \in \mathbb{D}$. The closed graph theorem shows that $C_{\varphi,\psi} : \mathcal{H}_{\mathcal{E}}(K) \to \mathcal{H}_{\mathcal{E}}(K)$ is bounded, so the adjoint of weighted composition operator $C^*_{\varphi,\psi}$ is also bounded on $\mathcal{H}_{\mathcal{E}}(K)$.

For a Banach space X, an operator $T \in \mathfrak{L}(X)$ is said to be hypercyclic if there exists a vector $x \in X$ such that the orbit of x under T, $Orb(x, T) := \big\{x, Tx, T^2x, \cdots\big\}$ is dense in X, and x is called the hypercyclic vector for T. In [2], Bourdon and Shapiro studied thoroughly the hypercyclicity of the composition operator (see also [3]).

Recently, Mundayadan and Sarkar characterized completely the hypercyclicity, as well other dynamic properties of the adjoint of the multiplication operator by the coordinate function on $\mathcal{H}_{\mathcal{E}}(K)$ in [6]. In 2011, Kamali et al. [5] studied the hypercyclicity of $C^*_{\varphi,\psi}$ acting on the scalar-valued reproducing kernel Hilbert space. In this paper, we concentrate on the more general case, namely the vector-valued analytic reproducing kernel Hilbert spaces $\mathcal{H}_{\mathcal{E}}(K)$.

2 Main Results

In this section, we use the following Hypercyclicity Criterion to give the sufficient conditions for $C^*_{\varphi,\psi} : \mathcal{H}_{\mathcal{E}}(K) \to \mathcal{H}_{\mathcal{E}}(K)$ to be hypercyclic.

Theorem 2.1. [1] *Let X be a Banach space and $T \in \mathfrak{L}(X)$. Suppose that there are dense subsets $\mathcal{D}_1, \mathcal{D}_2 \subset X$, an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers, and maps $S_{n_k} : \mathcal{D}_2 \to X$ such that for any $x \in \mathcal{D}_1$, $y \in \mathcal{D}_2$,*

(i) $T^{n_k} x \to 0$, *as* $k \to \infty$,

(ii) $S_{n_k} y \to 0$, *as* $k \to \infty$,

(iii) $T^{n_k} S_{n_k} y \to y$, *as* $k \to \infty$.

Then T is hypercyclic.

By [8, Proposition 3.2.], we know that if $C^*_{\varphi,\psi}$ is hypercyclic on $\mathcal{H}_{\mathcal{E}}(K)$, then φ must be an automorphism. Recall that a sequence $\{c_i\}_{i\in\mathbb{N}}$ of complex numbers is not a Blaschke sequence, if there exists $i_0 \in \mathbb{N}$ such that $|c_i| < 1$ for $i \geq i_0$ and $\sum_{i=1}^{\infty}(1-|c_i|) = \infty$. Moreover, for every $z \in \mathbb{D}$, the linear evaluation map $E_z : \mathcal{H}_{\mathcal{E}}(K) \to \mathcal{E}$ given by $E_z f = f(z)$ is bounded and so $E_z \circ E_w^* : \mathcal{E} \to \mathcal{E}$ is a bounded linear map. It is easy to verify that $K(z,w) = E_z \circ E_w^*$ and

$$\big\langle f, K(\cdot,w)\eta \big\rangle_{\mathcal{H}_{\mathcal{E}}(K)} = \big\langle E_w f, \eta \big\rangle_{\mathcal{E}}. \tag{1}$$

Theorem 2.2. *Let φ be a disk automorphism such that the sets*

$$P = \Big\{ w \in \mathbb{D} : \big\{ \psi\big(\varphi_n(w)\big) \big\}_{n=0}^{\infty} \ \textit{ is not a Blaschke sequence} \Big\}$$

and

$$Q = \Big\{ v \in \mathbb{D} : \Big\{ \big(\psi(\varphi_{-n}(v))\big)^{-1} \Big\}_{n=1}^{\infty} \ \textit{ is not a Blaschke sequence} \Big\}$$

have limit points in $\mathbb{D}$. If for each $w \in P$ and each $v \in Q$ the sequence $\big\{ K\big(\cdot, \varphi_n(w)\big)\eta \big\}_{n\ge 1}$ and $\big\{ K\big(\cdot, \varphi_{-n}(v)\big)\eta \big\}_{n\ge 1}$ are bounded for all $\eta \in \mathcal{E}_0$, where $\mathcal{E}_0$ is a dense subset of $\mathcal{E}$. Then $C^*_{\varphi,\psi}$ is hypercyclic on $\mathcal{H}_{\mathcal{E}}(K)$.

Proof. Claim: The sets

$$\mathcal{M}_P = \operatorname{span}\big\{ K(\cdot,w)\eta : w \in P, \eta \in \mathcal{E}_0 \big\}$$

and

$$\mathcal{M}_Q = \operatorname{span}\big\{ K(\cdot,v)\eta : v \in Q, \eta \in \mathcal{E}_0 \big\}$$

are dense in $\mathcal{H}_{\mathcal{E}}(K)$.

Indeed, suppose that $f \in \mathcal{H}_{\mathcal{E}}(K)$ and $f \perp K(\cdot,w)\eta$ for all $w \in P$ and $\eta \in \mathcal{E}_0$. Then

$$0 = \langle f, K(\cdot,w)\eta \rangle_{\mathcal{H}_{\mathcal{E}}(K)} = \langle f(w), \eta \rangle_{\mathcal{E}}.$$

We have $f(w) = 0$ as $\mathcal{E}_0$ is dense in $\mathcal{E}$. Since P have a limit point in $\mathbb{D}$, from the identity theorem it follows that $f \equiv 0$. Hence $\mathcal{M}_P$ is dense in $\mathcal{H}_{\mathcal{E}}(K)$. Similarly, $\mathcal{M}_Q$ is also dense in $\mathcal{H}_{\mathcal{E}}(K)$.

For any $f \in \mathcal{H}_{\mathcal{E}}(K)$, by (1),

$$\begin{aligned} \big\langle C^*_{\varphi,\psi}\big(K(\cdot,w)\eta\big), f \big\rangle_{\mathcal{H}_{\mathcal{E}}(K)} &= \big\langle K(\cdot,w)\eta, \psi \cdot (f \circ \varphi) \big\rangle_{\mathcal{H}_{\mathcal{E}}(K)} \\ &= \big\langle \eta, \psi(w) f\big(\varphi(w)\big) \big\rangle_{\mathcal{E}} \\ &= \big\langle \overline{\psi(w)} K\big(\cdot, \varphi(w)\big)\eta, f \big\rangle_{\mathcal{H}_{\mathcal{E}}(K)}. \end{aligned}$$

Thus we obtain

$$C^*_{\varphi,\psi}\big(K(\cdot,w)\eta\big) = \overline{\psi(w)} K\big(\cdot, \varphi(w)\big)\eta. \tag{2}$$

Inductively,

$$C^{*n}_{\varphi,\psi}\big(K(\cdot,w)\eta\big)=\prod_{j=0}^{n-1}\overline{\psi\big(\varphi_j(w)\big)}K\big(\cdot,\varphi_n(w)\big)\eta.$$

For $w\in P$, $\big\{\psi\big(\varphi_j(w)\big)\big\}_j$ is not a Blaschke sequence and we have

$$\sum_{j=1}^{\infty}\big(1-\big|\psi\big(\varphi_j(w)\big)\big|\big)=\infty,$$

which is equivalent to

$$\lim_{n\to\infty}\prod_{j=0}^{n-1}\overline{\psi\big(\varphi_j(w)\big)}=0.$$

On the other hand, $\big\{K(\cdot,\varphi_n\big(w)\big)\eta\big\}_{n\geq1}$ is bounded, then we have

$$\lim_{n\to\infty}C^{*n}_{\varphi,\psi}\big(K(\cdot,w)\eta\big)=0.$$

Therefore $\lim\limits_{n\to\infty}C^{*n}_{\varphi,\psi}f=0$ for any $f\in\mathcal{M}_P$.

Now suppose that $\big\{K(\cdot,v)\eta : v\in Q,\eta\in\mathcal{E}_0\big\}$ is linearly independent. In this case, we can define a linear map

$$S:\mathcal{M}_Q\to\mathcal{M}_Q$$

by extending the definition

$$S\big(K(\cdot,v)\eta\big)=\overline{\psi\big(\varphi^{-1}(v)\big)}^{-1}K\big(\cdot,\varphi^{-1}(v)\big)\eta$$

linearly to $\mathcal{M}_Q$. Since $\varphi^{-1}(v)\in Q$ whenever $v\in Q$, $S\big(K(\cdot,v)\eta\big)\in\mathcal{M}_Q$, and we can define S^n for all $n\geq1$. It is easy to see that

$$S^n\big(K(\cdot,v)\eta\big)=\prod_{j=1}^{n}\overline{\psi\big(\varphi_{-j}(v)\big)}^{-1}K\big(\cdot,\varphi_{-n}(v)\big)\eta.$$

Since $\big\{\big(\psi(\varphi_{-j}(v))\big)^{-1}\big\}_j$ is not a Blaschke sequence, we can obtain

$$\lim_{n\to\infty}\prod_{j=1}^{n}\overline{\psi\big(\varphi_{-j}(v)\big)}^{-1}=0.$$

For $v\in Q$, $\big\{K(\cdot,\varphi_{-n}\big(v)\big)\eta\big\}_{n\geq1}$ is bounded. It follows that

$$\lim_{n\to\infty}S^n\big(K(\cdot,v)\eta\big)=0.$$

Hence $\lim\limits_{n\to\infty} S^n f = 0$ for any $f \in \mathcal{M}_Q$.

Moreover, if $v \in Q$, then

$$\begin{aligned} C^*_{\varphi,\psi} S\big(K(\cdot, v)\eta\big) &= C^*_{\varphi,\psi} \overline{\psi\big(\varphi^{-1}(v)\big)}^{-1} K\big(\cdot, \varphi^{-1}(v)\big)\eta \\ &= \overline{\psi\big(\varphi^{-1}(v)\big)}^{-1} \overline{\psi\big(\varphi^{-1}(v)\big)} K\big(\cdot, \varphi(\varphi^{-1}(v))\big)\eta \\ &= K(\cdot, v)\eta, \end{aligned}$$

i.e., $C^*_{\varphi,\psi} S = I$ on $\mathcal{M}_Q$. We can conclude that $C^*_{\varphi,\psi}$ is hypercyclic on $\mathcal{H}_{\mathcal{E}}(K)$ by the Hypercyclicity Criterion.

In the case $\big\{K(\cdot, v)\eta : v \in Q, \eta \in \mathcal{E}_0\big\}$ is linearly dependent. We adopt the same method in [4, Theorem 4.5] due to Godefroy and Shapiro. Enumerate a countable dense subset $Q_1 = \{v_n : n \geq 1\}$ of Q, and inductively choose a subsequence $\{g_n\}_n$ as follows. Take $g_1 = v_1$,

$$Q_2 = Q_1 \big\backslash \big\{v \in Q_1 : K(\cdot, v)\eta \in \operatorname{span}\{K(\cdot, g_1)\eta\}\big\}.$$

Denote the first element of Q_2 by g_2 and let

$$Q_3 = Q_2 \big\backslash \big\{\nu \in Q_2 : K(\cdot, v)\eta \in \operatorname{span}\{K(\cdot, g_1)\eta, K(\cdot, g_2)\eta\}\big\}.$$

Let g_3 be the first element of Q_3 and continue this process we get a subset $R = \{g_n : n \geq 1\}$ of Q such that the set

$$\{K(\cdot, g)\eta : g \in G, \eta \in \mathcal{E}_0\}$$

is linearly independent and

$$\mathcal{M}_G = \operatorname{span}\{K(\cdot, g)\eta : g \in G, \eta \in \mathcal{E}_0\} = \operatorname{span}\{K(\cdot, v)\eta : v \in Q_1, \eta \in \mathcal{E}_0\}$$

is dense in $\mathcal{H}_{\mathcal{E}}(K)$. Define $S : \mathcal{M}_G \to \mathcal{M}_G$ as above. Similarly, we also have $C^*_{\varphi,\psi} S = I$ on $\mathcal{M}_G$ and $S^n \to 0$ pointwise on $\mathcal{M}_G$. Therefore $C^*_{\varphi,\psi}$ is hypercyclic on $\mathcal{H}_{\mathcal{E}}(K)$ and the proof is completed. □

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