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Free Final Time Stackelberg Differential Games

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Abstract

In this paper, we analyse a new formulation of Stackelberg differential games. We assume that the Leader can control not only the dynamics of the game, but also the length of the programming interval. This formulation of a free final time Stackelberg differential game is not explicitly considered in the literature and presents some interesting issues. After a formal definition of this kind of differential game, we show, using a practical example, the main difficulties associated with this new definition. We close the article by presenting two open questions related to this issue.

Mathematics Subject Classifications: 49N70, 91A23, 91A65

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1 Introduction

Differential game theory is a widely used tool to study economic and marketing problems that involve interactions between some decision makers. Game theory is essential for the formalisation of many problems [3], while optimal control theory is fundamental for their analysis [6]. Among the main applications of this theory, we find advertising models [5] and supply chain problems [2]. In both of these applications, hierarchical situations become increasingly relevant [4], making the corresponding concept of Stackelberg equilibrium a decisive matter. A differential game played la Stackelberg allows one to study situations where players have asymmetrical roles. For this reason, they are

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called Leader and Follower, and the game occurs as follows [3, Ch.5, p.113]: the Leader first declares his strategy, and then the Follower chooses his own best response to Leader's announcement. At this point, knowing Follower's response, the Leader picks his best strategy choice. Free final time games are useful to describe economic problems with time as a decision variable; although, to the best of our knowledge, there are no papers analysing free final time Stackelberg differential games [3]. When the final time is free, it becomes a decision variable for the Leader. In a free final time Stackelberg differential game, the sequential decision making is similar to what described above, except that, at first, the Leader declares both his strategy and the final time, and finally, computes the final time as well. We contribute to the literature by introducing the definition of a free final time Stackelberg differential game and by describing the analytical procedure to characterise such an equilibrium. The paper is organised as follows. In Section 2, we present the definition of a free final time Stackelberg differential game. In Section 3, we show how to find such an equilibrium in an example. In Section 4, we describe two open questions associated with this new definition.

2 Free final time Stackelberg differential game

Consider the following two-player differential game, where F denotes the Follower and L the Leader. The players' profits are

$$J_j = \int_0^T g_j(x(t), u_L(t), u_F(t), t)dt$$

where $j \in \{L, F\}$, and the motion equation is

$$\dot{x}(t) = f(x(t), u_L(t), u_F(t), t)$$

subject to the initial condition

$$x(0) = x_0 \in \mathbb{R}$$

and the final constraint

$$x(T) \geq \bar{x} \in \mathbb{R}\,.$$

We assume that functions g_F, g_L are continuously differentiable in all their variables and the controls $u_F(\cdot), u_L(\cdot)$ are in $L^1([0, +\infty), U_j)$, where $j \in \{L, F\}$ and $U_j \subset \mathbb{R}$, so that both objective functionals are well defined. Moreover, we assume that f is continuously differentiable in all its variables and Lipschitz continuous in the state variable x uniformly with respect to the control variables u_L, u_F so that the link between state and controls is well defined [6, Ch.2, p.73]. For the sake of simplicity, we deal with a one-dimensional instance of the problem; the multidimensional extension is straightforward.

Instead of a formal definition of free final time Stackelberg differential game, we prefer to illustrate a procedure to characterize an open-loop equilibrium. This approach is useful because it directly refers to necessary conditions, hence it is more practical and effective.

2.1 Follower's optimal control problem

First of all, in a free final time Stackelberg differential game the Leader announces the control path $\hat{u}_L(\cdot)$ and the final time $\hat{T} \in [0, \overline{T}]$, where $\overline{T}$ is the maximum feasible final time. We notice that in the literature the final time is fixed, whereas here it is part of Leader's strategy.

At this point, the Follower has to find a best response function [3, Ch.2, p.17] to such a Leader's strategy. To compute which, the Follower solves the optimal control problem

$$\begin{array}{ll} \max\limits_{u_F(\cdot)} & \displaystyle\int_0^{\hat{T}} g_F(x(t), \hat{u}_L(t), u_F(t), t)dt \\ \text{s.t.} & \\ & \dot{x}(t) = f(x(t), \hat{u}_L(t), u_F(t), t) \\ & x(0) = x_0 \,. \end{array}$$

We observe that the Follower does not have to consider the final state constraint (otherwise the open-loop equilibrium would be time inconsistent, [1]). Follower's Hamiltonian function [6, Ch.2, p.85] is

$$H_F\left(x, \lambda_F, u_F, t |\, \hat{u}_L(t), \hat{T}\right) = g_F(x, \hat{u}_L(t), u_F, t) + \lambda_F f(x, \hat{u}_L(t), u_F, t)\,.$$

We assume that the necessary conditions for Follower's optimal control problem [6, Ch.2, p.85] are also sufficient. Hence, it is well defined the function

$$u_F^{\#}(x, \lambda_F, t|\, \hat{u}_L(t), \hat{T}) := \arg\max_{u_F} \left\{ H_F\left(x, \lambda_F, u_F, t |\, \hat{u}_L(t), \hat{T}\right)\right\}\,.$$

Moreover, we suppose that the two-point boundary value problem

$$\begin{cases} \dot{x}(t) = f(x(t), \hat{u}_L(t), u_F^{\#}(x(t), \lambda_F(t), t|\, \hat{u}_L(t), \hat{T}), t) \\ x(0) = x_0 \\ \dot{\lambda}_F(t) = -\partial_x H_F\left(x(t), \lambda_F(t), u_F^{\#}(x(t), \lambda_F(t), t|\, \hat{u}_L(t), \hat{T}), t \,\middle|\, \hat{u}_L(t), \hat{T}\right) \\ \lambda_F(\hat{T}) = 0 \end{cases}$$

has a unique solution $(x^{\#}(t), \lambda_F^{\#}(t))$, for all $t \in [0, \hat{T}]$ and for all $\hat{u}_L(t) \in L^1([0, \hat{T}], U_L)$.

Finally, we assume that U_F is a convex subset of $\mathbb{R}$ and the function

$$(x, u_F) \mapsto H_F\left(x, \lambda_F^{\#}(t), u_F, t \middle| \hat{u}_L(t), \hat{T}\right)$$

is concave for all $\hat{T} \in [0, \bar{T}]$, for all $t \in [0, \hat{T}]$ and for all $u_F \in U_F$. Under these hypotheses, the function

$$u_F^{\#}(x, \lambda_F, t|\, \hat{u}_L(t), \hat{T})$$

is the best response function of the Follower to any Leader's strategy [3, Ch.2, p.17].

2.2 Leader's optimal control problem

Now we can focus on the optimal control problem of the Leader.

$$\max_{T \in [0,\bar{T}], u_L(\cdot)} \int_0^T g_F(x(t), u_L(t), u_F^{\#}(x, \lambda_F, t|\, u_L(t), T), t)dt$$

$$\begin{aligned}
&\dot{x}(t) = f(x(t), u_L(t), u_F^{\#}(x(t), \lambda_F(t), t|\, u_L(t), T), t)\\
&x(0) = x_0\\
&x(T) \geq \bar{x}\\
&\dot{\lambda}_F(t) = -\partial_x H_F(x(t), \lambda_F(t), u_F^{\#}(x(t), \lambda_F(t), tu_L(t), T), t|\, u_L(t), T)\\
&\lambda_F(T) = 0\,.
\end{aligned}$$

This is a free final time optimal control problem; therefore, if we characterise the optimal time T^* and the optimal control $u_L^*(\cdot)$, we find an open-loop Stackelberg equilibrium for a free final time Stackelberg differential game.
This approach has two critical issues:

- The previous free final time optimal control problem is not standard since the differential equation for the adjoint function of the Follower is backward, so the function $\lambda_F(\cdot)$ becomes a new state function for the Leader, and, as a consequence, the standard necessary conditions for a free final time optimal control problem do not hold.

- Time consistency is crucial for an open-loop Nash equilibrium in a Stackelberg differential game; however, in this new framework, the standard condition about the controllability of the adjoint function of the Leader is not straightforward because the Leader can control the final time.

In the following section, we introduce an example to better explain this procedure.

3 Numerical example

In this section, we propose a numerical example to show how to characterise an open-loop equilibrium for a free final time Stackelberg differential game. The objective functional of the Leader is

$$J_L = \int_0^T e^{-t}\left(u_L(t) - \frac{1}{2}u_F(t)u_L^2(t)\right) dt$$

while the objective functional of the Follower is

$$J_F = \int_0^T \left(u_F(t) - \frac{1}{2}u_F^2(t)\right) dt\,.$$

The motion equation is described by the Cauchy problem

$$\begin{cases} \dot{x}(t) = -u_L(t) - u_F(t) \\ x(0) = 4\,. \end{cases}$$

Moreover, we assume that the Leader has to satisfy the final constraint

$$x(T) \geq 0\,.$$

Both controls must be positive, i.e. $u_L(t), u_F(t) \geq 0$ for all $t \in [0, T]$, and the Leader can choose the final time in the interval $[0, 2.5]$.
Let us start our analysis by assuming that the Leader proposes a strategy to the Follower. We denote by $\hat{u}_L(\cdot)$ and $\hat{T}$ this strategy (with $\hat{T} > 0$). At this point, we find the best response function of the Follower.
Follower's Hamiltonian function is

$$H_F(x, u_F, \lambda_F, t) = u_F - \frac{1}{2}u_F^2 + \lambda_F\left(-\hat{u}_L(t) - u_F\right)\,.$$

We compute

$$\partial_{u_F} H_F(x, u_F, \lambda_F, t) = 1 - u_F - \lambda_F$$

and

$$\partial^2_{u_F u_F} H_F(x, u_F, \lambda_F, t) = -1\,;$$

hence, Follower's best response function is

$$u_F^{\#}(x, \lambda_F, t) = 1 - \lambda_F\,.$$

Moreover, from the adjoint equation and the transversality condition, we have $\dot{\lambda}_F(t) = 0$ and $\lambda_F(\hat{T}) = 0$; therefore, $\lambda_F(t) = 0$ for all $t \in [0, \hat{T}]$. Thus, the best response function becomes

$$u_F^{\#}(x, \lambda_F, t) = 1\,.$$

We notice that the Hamiltonian function of the Follower is concave in the state and in the control, hence the sufficient conditions [6, Ch.2, p.105] are satisfied. In this example:

- the adjoint equation is uncoupled from the motion equation, therefore we can explicitly solve it, and, as a consequence, Follower's adjoint equation does not become a motion equation for Leader's problem;
- Follower's best response function does not depend on $\hat{T}$, because of the simplicity of the model.

Now, we study Leader's optimal control problem. The Hamiltonian function is

$$H_L(x, u_L, \lambda_L, t) = \lambda_0 e^{-t}\left(u_L - \frac{1}{2}u_L^2\right) + \lambda_L(-u_L - 1)\,.$$

The necessary conditions [6, Ch.2, p.143] are

1. $(\lambda_0, \lambda_L(t)) \neq (0, 0)$ for all $t \in [0, T^*]$;
2. $u_L^*(t) \in \arg\max_w\{H_L(x^*(t), w, \lambda_L(t), t)\}$ for all $t \in [0, T^*]$;
3. $\lambda_0 \in \{0, 1\}$;
4. $\dot{\lambda}_L(t) = 0$ for a.e. $t \in [0, T^*]$;
5. $\lambda_L(T^*) \geq 0, \quad x^*(T^*) \geq 0, \quad \lambda_L(T^*)x^*(T^*) = 0$;
6. $H_L(x^*(T^*), u_L^*(T^*), \lambda_L(T^*), T^*) = 0$.

Suppose, at first, that $\lambda_0 = 0$. Then $\lambda_L(t) = \bar{\lambda}$ for all $t \in [0, T^*]$ and $\bar{\lambda}$ must be strictly positive. However, by maximising the Hamiltonian function, we have $u_L^*(t) = 0$ for all $t \in [0, T^*]$, which is not feasible because

$$H_L(x^*(T^*), u_L^*(T^*), \lambda_L(T^*), T^*) = \lambda_L(T^*)(-u_L^*(T^*) - 1) = -\bar{\lambda} < 0\,.$$

Hence, let us assume that $\lambda_0 = 1$. We compute

$$\partial_{u_L} H_L(x, u_L, \lambda_L, t) = e^{-t}(1 - u_L) - \lambda_L$$

and

$$\partial^2_{u_L u_L} H_L(x, u_L, \lambda_L, t) = -e^{-t}\,;$$

thus,

$$u_L^{\#}(x, \lambda_L, t) = \left[1 - \bar{\lambda}e^t\right]^+\,.$$

If $\bar{\lambda} = 0$, then $u_L^*(t) = 1$ for all $t \in [0, T^*]$; therefore the solution is not feasible because

$$H_L(x^*(T^*), u_L^*(T^*), \lambda_L(T^*), T^*) = e^{-T^*}/2 > 0\,.$$

Thus, it must be $\bar{\lambda} > 0$. We notice that the map $t \mapsto 1 - \bar{\lambda}e^t$ is a strictly decreasing function, hence either $u_L^*(T^*) = 0$, or $u_L^*(T^*) > 0$. If $u_L^*(T^*) = 0$, then

$$H_L(x^*(T^*), u_L^*(T^*), \lambda_L(T^*), T^*) = -\bar{\lambda} < 0\,,$$

hence this solution is not feasible. Therefore, for all $t \in [0, T^*]$

$$u_L^{\#}(x, \lambda_F, t) = 1 - \bar{\lambda} e^t \,.$$

From the motion equation, we obtain $x^*(t) = 4 - 2t + \bar{\lambda}(e^t - 1)$ and, by the transversality condition, we get $x^*(T^*) = 4 - 2T^* + \bar{\lambda}(e^{T^*} - 1) = 0$, which gives us

$$\bar{\lambda} = \frac{2(T^* - 2)}{(e^{T^*} - 1)} \,,$$

that is feasible if and only if $T^* > 2$.
Finally, by the free final time condition, we have

$$H_L(x^*(T^*), u_L^*(T^*), \lambda_L(T^*), T^*) = e^{-T^*} u_L^*(T^*) \left(1 - \frac{1}{2} u_L^*(T^*) - \bar{\lambda} e^{T^*}\right) - \bar{\lambda} = 0 \,,$$

which becomes

$$4(T^* - 2)^2 e^{T^*} - 8(T^* - 2)\left(e^{T^*} - 1\right) + e^{-T^*}(e^{T^*} - 1)^2 = 0 \,,$$

whose unique solution is $T^* \approx 2.1179$.
Futhermore, we observe that $H_L(x^*(T), u_L^*(T), \lambda_L(T), T)$ is positive for $T < T^*$, while it becomes negative for $T > T^*$. Therefore, the sufficient conditions [6, Ch.2, p.145] are satisfied. Hence, we have completely characterized the open-loop equilibrium for the free final time Stackelberg differential game.

4 Conclusions and open problems

In this paper we have introduced the definition of open-loop equilibrium for a free final time Stackelberg differential game. Then, we have proposed a numerical example to prove that this equilibrium can be explicitly characterized. After our analysis, two issues remain open.

First of all, in the numerical example introduced in Section 2, the adjoint function of the Follower can be explicitly solved, hence the Leader has to solve a standard free final time optimal control problem. In the numerical example, the condition about the vanishing of the Hamiltonian function in the optimal final time is correct. However, it cannot be used in general because of the backward motion equation introduced by the adjoint function of the Follower.

Furthermore, time consistency is a key issue in Stackelberg differential games. In our numerical example, time consistency is trivially satisfied because the strategy of the Follower is uniform with respect to Leader's one. In general, this is not true and time consistency become more relevant to preserve the credibility of an equilibrium. It seems interesting to characterize some classes of problem that have a simple structure such that time consistency is automatically satisfied.

In conclusion, even though this new definition seems to be a straightforward extension of the original Stackelberg differential game, its analysis appears to be rich of new stimulant situations.

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