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Personal and Historical Magic Squares

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Abstract

Parametrizations of 4×4 squares which allow to generate individual examples, using birthdays or other personally preferred numbers are developed. This will be done for magic squares that are delightful, perfect, skew symmetric, most perfect and pandiagonal (also called diabolic). Furthermore, the parametrizations explain the construction of famous historical magic squares. Also an idea for an artwork containing mathematics is given, called *MathArt*.

1 Historical Introduction

Magic squares are always of interest to people irrespective of age and their acquaintance with mathematics. Earlier, magic squares appeared often on temples, in paintings and on mythological objects.
Magic squares first appeared in ancient China, before they became an active subject westwards. They played a remarkable role in India, later in the Arabic world, in medieval Islam and finally in Europe and America.
Legend has it that the first magic square is over 4000 years old. It is said that

the mystical Emperor Yu in China discovered small black and white circles on the shell of a turtle that had emerged from the Lo river. The arrangement of the circles representing the numbers 1 to 9 were structured in a special 3×3 square (see [2]).

Here is the modern design of the so-called *Lo Shu magic square*

4	9	2
3	5	7
8	1	6

with the sum for the rows, columns and diagonals being 15. It was considered mystical with the *magic number 15*. The odd numbers forming a cross were interpreted as Yin, the even ones in the corners as Yang. Yu as a young man used the Pythagorean triangle for a successful water-control project to prevent floodings by the Yellow river. This might have played a vital role in Yu later becoming an emperor (see [6]).

The first indisputable reference to a magic square in China occurs in the first century AD (see[8]). It assigns numbers to each of the 9 chambers of the so-called Mingtang Hall. Later in the 10th century, the numbers were represented in a pseudo archaic form with the small black and white circles in the Lo Shu magic square mentioned above.

The *Sator-Square*

S	A	T	O	R
A	R	E	P	O
T	E	N	E	T
O	P	E	R	A
R	O	T	A	S

describes a palindrome sentence. It may be read horizontally, vertically forward and backward. It has the symmetry of the dihedral group D_2: Invariance under the two reflections in the diagonals and a rotation by 180°, but not under the symmetry of the square. It appeared in early Christian and mythological contexts on stones, wood and in handwritten books in monasteries. The oldest datable representation was found in the ruins of Pompeji, covered by the eruption of the Vesuvius vulcano in the year
AD 79. Since the medieval age, Sator-squares have also been present in America.

At a young age, *Ramanujan* was already very interested in magic squares. Here is a statement in his first famous notebook: In a 3×3 magic square, the elements in the middle row, middle column and diagonals are in arithmetic progression(see [3]).

2 Delightful Magic Squares DMS

We deal with the most popular size of 4×4 magic squares and start with the following four fundamental cases:

$$v_1 = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ \hline \end{array}$$

$v_2 = V \cdot v_1$, $v_3 = H \cdot v_1$, $v_4 = R_{180} \cdot v_1$, where V, H, R_{180} denote the vertical reflection, horizontal reflection and the rotation by 180° about the center, respectively. Note that $\{I, V, H, R_{180}\}$ with I being the identity is the *dihedral group* D_2, which is isomorphic to the *Klein Four Group*.

The linear combination

$$av_3 + bv_2 + cv_1 + dv_4 = \begin{array}{|c|c|c|c|} \hline a & b & c & d \\ \hline d & c & b & a \\ \hline b & a & d & c \\ \hline c & d & a & b \\ \hline \end{array}$$

contains permutations of the four elements in each row and each column and is therefore an example of a *Latin square*. The sum $M = a + b + c + d$, called the *Magic Number*, is the same for the following cases:

1. each row and each column,
2. the two diagonals,
3. the four corners,
4. the four corners of *each* 3×3 subsquare,
5. the entries of each 2x2 subsquare of the the 4x4 square, considered as a torus, with the following four exceptions: The 2x2-subsquare formed by the top left and the top right columns of lengths 2; the 2x2-subsquare formed by the bottom left and the bottom right columns of lengths 2; the middle 2x2-subsquares at the top and at the bottom. These exceptions have the sums 2a+2d or 2b+2c, resp.,
6. the two pairs of parallel subdiagonals of length 2.

Remark: Case 3 is included in case 5, because the four corners also form a 2×2 subsquare on the torus.

If we add

$$k \cdot \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 1 & -1 & -3 & 3 \\ \hline -2 & 2 & 2 & -2 \\ \hline 1 & -1 & 1 & -1 \\ \hline \end{array} \qquad \text{with } k \in \mathbb{Z}$$

there is no change in the sum M for all cases 1 to 6 and we get the following type of so-called *Delightful Magic Squares (DMS)* with 5 parameters:

$$DMS = \begin{array}{|c|c|c|c|} \hline a & b & c & d \\ \hline d+k & c-k & b-3k & a+3k \\ \hline b-2k & a+2k & d+2k & c-2k \\ \hline c+k & d-k & a+k & b-k \\ \hline \end{array}$$

It contains the following four sequences of arithmetic progressions: $\{a, a+k, a+2k, a+3k\}$, $\{b-3k, b-2k, b-k, b\}$, $\{c-2k, c-k, c, c+k\}$, $\{d-k, d, d+k, d+2k\}$.

A similar type to DMS, but with 4 parameters, can be found in [1].

Let us construct an example with primes only. For $k = 6$, the smallest sequences of primes in consecutive arithmetic progressions of length ≥ 4 are

$$\{5, 11, 17, 23, 29\}, \quad \{41, 47, 53, 59\}, \quad \{61, 67, 73, 79\},$$
$$\{251, 257, 263, 269\}, \quad \{601, 607, 613, 619\}, \quad \{1091, 1097, 1103, 1109\}.$$

There exists no sequence of length larger than 4, except the first one. A calculation shows that there are 24 sequences with numbers $< 10^4$.

If we take any four different sequences, then the magic square consists of *four disjoint sets of consecutive primes in arithmetic progression.*

The choice $a = 5,\ b = 59,\ c = 73,\ d = 257$ results in the example with the smallest possible primes and the magic number $M = 394$:

$$P = \begin{array}{|c|c|c|c|} \hline 5 & 59 & 73 & 257 \\ \hline 263 & 67 & 41 & 23 \\ \hline 47 & 17 & 269 & 61 \\ \hline 79 & 251 & 11 & 53 \\ \hline \end{array}.$$

3 Perfect Magic Squares PMS

To increasee the number of parameters, we add the following fundamental cases under the rotation about an angle of 90°: $w_i = R_{90} \cdot v_i$, where $i = 1, 2, 3, 4$.

Together with the v_i, there are 8 vectors of dimension 16. Gaussian elimination shows, that they span a vector space of dimension 7. The linear dependence can be verified with $\sum v_i = \sum w_i =$ magic square with all 16 entries being 1.

We consider the linear combination $x_1v_1 + x_2w_3 + x_3w_2 + x_4v_2 + x_5v_4 + x_6w_1 + x_7w_4$:

x_7	$x_2 + x_4$	$x_1 + x_6$	$x_3 + x_5$
$x_5 + x_6$	$x_1 + x_3$	$x_4 + x_7$	x_2
$x_3 + x_4$	x_6	$x_2 + x_5$	$x_1 + x_7$
$x_1 + x_2$	$x_5 + x_7$	x_3	$x_4 + x_6$

It has the same sum $M = \sum x_i$ for all cases 1, 2, 3 ,4, and 6 with the following modified case 5: there are four additional 2×2 subsquares, which do not have the sum M in general, caused by the rotation R_{90} of the 4 described exceptions.

The geometrical pattern of all cases with sum M has the symmetry of the square (dihedral group D_4): It is invariant under its 8 transformations (4 rotations and 4 reflections). Let us change to the following 7 new parameters:

The top row shall be a, b, c, d as before and $e = x_2$, $f = x_6$, $g = x_3$. Therefore,
$x_7 = a$
$x_2 + x_4 = b \Rightarrow x_4 = b - e$
$x_1 + x_6 = c \Rightarrow x_1 = c - f$
$x_3 + x_5 = d \Rightarrow x_5 = d - g$

leads to the type of the so-called *Perfect Magic Squares (PMS)* with 7 parameters:

$PMS =$

a	b	c	d
$d + f - g$	$c + g - f$	$a + b - e$	e
$b + g - e$	f	$d + e - g$	$c + a - f$
$c + e - f$	$a + d - g$	g	$b + f - e$

The Indian number theorist from Kerala, *C.S.Venkatraman (1918-1994)*, honored *Ramanujan* in 1976 with the following personalized magic square of type PMS by using his birth date in the top row and choosing $e = 2$, $f = 16$, $g = 82$:

$$R_1 = \begin{array}{|c|c|c|c|}\hline 22 & 12 & 18 & 87 \\ \hline 21 & 84 & 32 & 2 \\ \hline 92 & 16 & 7 & 24 \\ \hline 4 & 27 & 82 & 26 \\ \hline \end{array}$$

In [1], a parametrization is given with 8 parameters describing all magic squares where in general only the rows, columns and the two diagonals have the same sum.

4 Skew Symmetric Magic Squares

To obtain the general parametrization of the skew symmetric square of type PMS, we consider the two corners with entries a and $b+f-e$

$$a + (b + f - e) = \frac{a+b+c+d}{2} = \frac{M}{2} \Longrightarrow f = \frac{-a-b+c+d}{2} + e.$$

Adding of the two values in the positions (2,1) and (3,4) yields

$$(d + f - g) + (c + a - f) = \frac{a+b+c+d}{2} = \frac{M}{2} \Longrightarrow g = \frac{a-b+c+d}{2}.$$

Substituting f and g in PMS leads to the type PMS_{sym} of *skew symmetric Perfect Magic Squares* with the 5 parameters a, b, c, d, e:

$$PMS_{sym} = \begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline d-a+e & a+c-e & a+b-e & e \\ \hline \frac{a+b+c+d}{2}-e & \frac{-a-b+c+d}{2}+e & \frac{-a+b-c+d}{2}+e & \frac{3a+b+c-d}{2}-e \\ \hline \frac{a+b+c-d}{2} & \frac{a+b-c+d}{2} & \frac{a-b+c+d}{2} & \frac{-a+b+c+d}{2} \\ \hline \end{array}$$

The magic square depicted in the famous painting by *Albrecht Drer* from 1514 entitled *'Melancholia'* can be obtained by choosing $a = 4,\ b = 15,\ c = 14,\ d = 1,\ e = 12$, followed by the reflection of the square about the horizontal symmetry axis:

$$D = \begin{array}{|c|c|c|c|}\hline 16 & 3 & 2 & 13 \\ \hline 5 & 10 & 11 & 8 \\ \hline 9 & 6 & 7 & 12 \\ \hline 4 & 15 & 14 & 1 \\ \hline \end{array}$$

It is indeed skew symmetric 6+11 = 4+13 = ... = 17 and it contains all numbers from
1 to 16.

Let us further specialize by requesting that *all* 2×2 subsquares of the torus have the same sum M. The condition for the 2×2 subsquare at the top in the middle must be

$$2a + 2b + 2c - 2e = a + b + c + d \Rightarrow e = (a + b + c - d)/2$$

Substituting e into PMS_{sym} leads to the type $MPMS_{sym}$ of *skew symmetric Most Perfect Magic Squares* with 4 parameters

$$MPMS_{sym} = \begin{array}{|c|c|c|c|} \hline a & b & c & d \\ \hline \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ \hline d & c & b & a \\ \hline \bar{d} & \bar{c} & \bar{b} & \bar{a} \\ \hline \end{array}$$

where $a + \bar{a} = b + \bar{b} = c + \bar{c} = d + \bar{d} = M/2$.

Here is an example featuring *Evariste Galois's* birth date with $M = 64$:

$$G = \begin{array}{|c|c|c|c|} \hline 25 & 10 & 18 & 11 \\ \hline 7 & 22 & 14 & 21 \\ \hline 11 & 18 & 10 & 25 \\ \hline 21 & 14 & 22 & 7 \\ \hline \end{array}$$

5 Most Perfect Pandiagonal Magic Squares

Now we give up skew symmetry, but realize the property that *all* 2×2 subsquares of the torus of the type PMS have the same sum $M = a + b + c + d$.

For the middle low subsquare, we get the restriction

$$f + (d + e - g) + (d + a - g) + g = a + b + c + d \Longrightarrow f = b + c - d - e + g.$$

For the right middle subsquare, we get the restriction

$$(a + b - e) + e + (d + e - g) + (c + a - f) = a + b + c + d \Longrightarrow f = a + e - g.$$

Adding or subtracting the two expressions for f leads to

$$f = \frac{a + b + c - d}{2} \text{ or } g = \frac{a - b - c + d}{2} + e, \text{ respectively.}$$

$$MPPMS = \begin{array}{|c|c|c|c|} \hline a & b & c & d \\ \hline b+c-e & d-b+e & a+b-e & e \\ \hline \frac{a+b-c+d}{2} & \frac{a+b+c-d}{2} & \frac{-a+b+c+d}{2} & \frac{a-b+c+d}{2} \\ \hline \frac{-a-b+c+d}{2}+e & \frac{a+b+c+d}{2}-e & \frac{a-b-c+d}{2}+e & \frac{a+3b+c-d}{2}-e \\ \hline \end{array}$$

Therefore, the two parameters f and g in PMS have to be replaced and we get the following type of *Most Perfect Pandiagonal Magic Square* MPPMS, also called *Diabolic Square*, with 5 parameters and the magic number $M = a+b+c+d$:

The *Pandiagonality* can be described as follows: If we consider two identical MPPMS which are put side by side, then all the 8 diagonals (4 going up and 4 going down) sum up to M.

The following properties, also described in [10], can be verified with MPPMS:

P1 Rows, columns and all 8 diagonals have the sum M (pandiagonality).

P2 All 2×2 subsquares of the torus, including the 4 corners, have the sum M.

P3 The corners of all 3×3 subsquares have the sum M.

P4 The sum of two entries on any diagonal of length 3 or 4 with distance 2, so-called *corresponding cells*, is $\frac{M}{2}$.

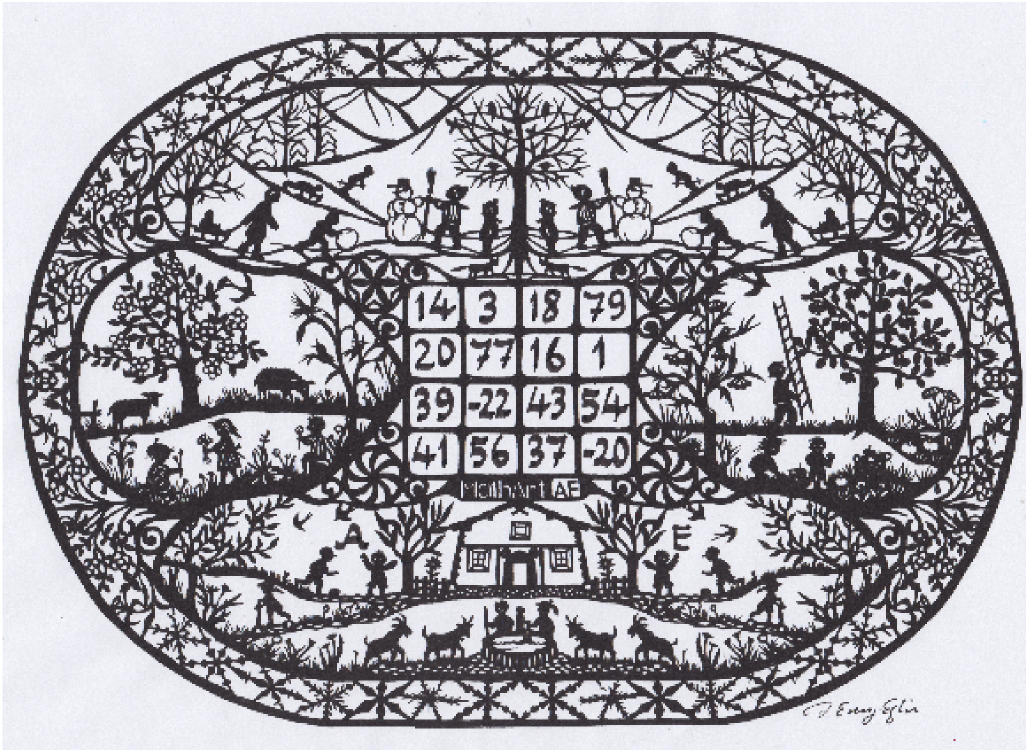
P5 If M is even, all entries are integers. If M is odd, the entries in the lower half square are half-integers.

P6 The following transformations and their compositions transform a diabolic square into a diabolic square: Cyclic permutations of rows and columns, exchanging rows 1 and 3, rows 2 and 4, columns 1 and 3, columns 2 and 4, the 4 reflections and 4 rotations (including the identity) that leave the square invariant (dihedral groupD_4).

P7 If four identical diabolic squares are fillerd into an 8×8 square, then each 4×4 subsquare is also a diabolic square. This is a consequence of the fact, that cyclic permutations of rows and columns transform a diabolic square into a diabolic square (see also [11]).

It follows from the pandiagonality P1 that the 4 diagonals of length 3 plus to corner opposite have the sum of M as well as the two parallel pairs of diagonals of length 2.

Here, Einstein is honored with a piece of *MathArt*, with his birth date in the top row of a MPPMS, integrated in a paper cutting.



It is not possible to avoid negative integers because $79 > 14 + 3 + 18$.

The silhouette with the four seasons was created by Dora Erny-Eglin, living in the Engadine, an area in the south-east of Switzerland. The geometric elements, called Sgraffittis, are found on the walls of traditional houses.

Winter and summer are symmetric, but spring and autumn are not. The frame has the symmetry of a rectangle (dihedral group D_2): it is invariant under vertical and horizontal reflection as well as under the rotation about $180°$.

The most perfect pandiagonal magic square for *Ramanujan's* birth date contains half integers in the lower half and two negative entries:

$$R_2 = \begin{array}{|c|c|c|c|} \hline 22 & 12 & 18 & 87 \\ \hline 29 & 76 & 33 & 1 \\ \hline 51.5 & -17.5 & 47.5 & 57.5 \\ \hline 36.5 & 68.5 & 40.5 & -6.5 \\ \hline \end{array}$$

Choosing $a = n - 3$, $b = 1$, $c = n - 6$, $d = 8$, $e = 2$, we get the famous historical work (from the second century AD) of the Buddhist philosopher

$$K = \begin{array}{|c|c|c|c|}\hline n-3 & 1 & n-6 & 8 \\ \hline n-7 & 9 & n-4 & 2 \\ \hline 6 & n-8 & 3 & n-1 \\ \hline 4 & n-2 & 7 & n-9 \\ \hline \end{array}$$

Nagarjuna, called *Kaksaputa*, with $M = 2n$ and corresponding cells with sum n:

Finally, let us investigate *skew symmetric MPPMS*. The two equations

$$M/2 - e + c = M/2 \Rightarrow e = c$$
$$a + b - e + \frac{a+b+c-d}{2} = M/2 \Rightarrow e = a + b - d$$

lead to the condition $a + b = c + d$ with 3 parameters and the magic number $M = 2(a + b) = 2(c + d)$.

The solution is the Latin Square

$$MPPMS_{sym} = \begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline b & a & d & c \\ \hline d & c & b & a \\ \hline c & d & a & b \\ \hline \end{array} \qquad \text{with } a + b = c + d.$$

6 Euler Squares for the Entries 1 to 16

At this stage one might wonder how to generate magic squares with the entries 1,2,3,...,16 and sum $M = 34$. The Swiss mathematician *Leonhard Euler (1707-1783)* worked on this problem and came up with the idea of merging two Latin squares into a so-called *Graeco-Latin square*, or *Euler square (ES):*

$$ES1 = \begin{array}{|c|c|c|c|}\hline a\alpha & b\delta & c\beta & d\gamma \\ \hline d\beta & c\gamma & b\alpha & a\delta \\ \hline b\gamma & a\beta & d\delta & c\alpha \\ \hline c\delta & d\alpha & a\gamma & b\beta \\ \hline \end{array}$$

It has the characteristics of the Delightful Magic Squares DMS which means, that any four entries of the cases 1 to 6 are composed of all 4 latin letters as well as all 4 greek letters. The square with the latin letters and the square with the greek letters are said to be *orthogonal,* because all 16 orderded pairs in the Euler square have latin letters in the first and greek letters in the second positions.

Let us consider the bottom left corner and the opposite diagonal of length 3. The sufficient and necessary conditions for most perfect pandiagonal Euler squares are
$2c + 2b = a + b + c + d$ and $2\delta + 2\alpha = \alpha + \beta + \gamma + \delta$, which can be directly verified in ES1 using the simplified conditions $b + c = a + d$ and $\beta + \gamma = \alpha + \delta$.

The use of the numbers 0, 1, 2, 3 for the latin and greek letters implies

$$b + c = a + d = \beta + \gamma = \alpha + \delta = 3$$

with the following $8^2 = 64$ possibilities for the choices of $(abcd)$ and $(\alpha\beta\gamma\delta)$:

$$\begin{array}{cc|cccccccc}
a & \alpha & 0 & 0 & 1 & 2 & 2 & 1 & 3 & 3 \\
b & \beta & 1 & 2 & 0 & 0 & 3 & 3 & 1 & 2 \\
c & \gamma & 2 & 1 & 3 & 3 & 0 & 0 & 2 & 1 \\
d & \delta & 3 & 3 & 2 & 1 & 1 & 2 & 0 & 0
\end{array}$$

To get a *Most Perfect Pandiagonal Euler square* with the properties P1 to P6 and entries 0,1,2,. . . ,15, we represent them to the base 4. Then, we increase all entries by 1 to get the entries 1,2,3,. . . , 16. Corresponding cells always have the sum of $M/2 = 34/2 = 17$.

To honor the universal genius *Nicolaus Cusanus (1401-1464)*, who worked in such different fields as theology, philosophy and mathematics, we create an example with the decimal numbers 14 and 01 in the top center. To get 13 and 0, it follows that $b = 3, \delta = 1, c = \beta = 0$. Choice: $(abcd) = (2301)$; $(\alpha\beta\gamma\delta) = (2031)$.

$$\text{Nicolaus Cusanus:}\quad \begin{array}{|c|c|c|c|}
\hline
22 & 31 & 00 & 13 \\ \hline
10 & 03 & 32 & 21 \\ \hline
33 & 20 & 11 & 02 \\ \hline
01 & 12 & 23 & 30 \\ \hline
\end{array}
\;\mapsto\;
\begin{array}{|c|c|c|c|}
\hline
11 & 14 & 01 & 8 \\ \hline
5 & 4 & 15 & 10 \\ \hline
16 & 9 & 6 & 3 \\ \hline
2 & 7 & 12 & 13 \\ \hline
\end{array}$$

To the left are the representations of the numbers to the base 4, to the right the corresponding decimal representations increased by 1.

Here is another Euler square with the same top row as $ES1$ (from [13]):

$$ES2 = \begin{array}{|c|c|c|c|} \hline a\alpha & b\delta & c\beta & d\gamma \\ \hline c\gamma & d\beta & a\delta & b\alpha \\ \hline d\delta & c\alpha & b\gamma & a\beta \\ \hline b\beta & a\gamma & d\alpha & c\delta \\ \hline \end{array}$$

There are hundreds of Euler squares: Two independently chosen permutations of the 4 latin and greek letters transform an Euler square into an Euler square. Because $ES1$ and $ES2$ have identical top rows, they cannot be transformed with such permutations. They form two different classes.

The permutations $(abcd) \mapsto (abcd)$ and $(\alpha\beta\gamma\delta) \mapsto (\alpha\gamma\delta\beta)$ of $ES2$ lead to a diabolic square with top row $(a\alpha\ \ b\beta\ \ c\gamma\ \ d\delta)$.

The construction and enumeration of all diabolic squares is given in [9]. It is proven in [5], that there exists exactly 3 mutually orthogonal latin squares (MOLS) with common first row.

A beautiful historical example is the *Jaina Square,* created in the 10th Century on the *Parshvanath temple in Khajuraho*. It can be generated, using the Euler square $ES2$ with $(abcd) = (1203)$ and $(\alpha\beta\gamma\delta) = (2013)$:

$$J = \begin{array}{|c|c|c|c|} \hline 7 & 12 & 1 & 14 \\ \hline 2 & 13 & 8 & 11 \\ \hline 16 & 3 & 10 & 5 \\ \hline 9 & 6 & 15 & 4 \\ \hline \end{array}$$

Let us calculate skew symmetric ES1 with entries 0 to 15 (therefore not pandiagonal):

The two conditions $c + d = 3$ and $\gamma + \delta = 3$ for skew symmetry have to be respected.

Here is an example obtained by choosing $(abcd) = (1230)$ and $(\alpha\beta\gamma\delta) = (3012)$:

$$\begin{array}{|c|c|c|c|} \hline 7 & 10 & 12 & 1 \\ \hline 0 & 13 & 11 & 6 \\ \hline 9 & 4 & 2 & 15 \\ \hline 14 & 3 & 5 & 8 \\ \hline \end{array}$$

Remark: The Dürer square D cannot be generated with an Euler square, because the neighbors 15 and 14, reduced by 1, are 14 and 13. But in the base 4, the first digit for both is 3. However, all the neighbors (horizontally or vertically) need to have different Latin and greek letters. Only a subset of diabolic squares with elements 1 to 16 can be generated with Euler squares.

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