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Product Cordial Graph in the Context of Some Graph Operations on Crown, Helm, and Wheel Graph

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Abstract

A graph labeling is an assignment of integers to the vertices or edges or both subject to some conditions. The concept of cordial labeling was introduced by Ibrahim Cahit in 1987 as a weaker version of graceful and harmonious labeling. A product cordial labeling of a graph $G = (V(G), E(G))$ is a function $f: V(G) \to \{0,1\}$ with each edge uv assign label $f(u)f(v)$, such that the number of vertices with label 0 and the number of vertices with label 1 differ at most by 1, and the number of edges with label 0 and the number of edges with label 1 differ by at most 1. In this paper we investigate product cordial labeling of the graphs obtained by duplication of some graph elements in crown, helm and wheel graph.

Mathematics Subject Classification: 05C78

Keywords: crown graph, helm graph, wheel graph, product cordial graph

1 Introduction

Graph labeling is a captivating and emerging area of graph theory. In 1987, Ibrahim Cahit has introduced graceful and harmonious labeling as a weaker version which the concept of cordial labeling started. A binary vertex labeling $f: V(G) \to \{0,1\}$ of a graph G is called cordial labeling if for each $xy \in G$, the induced edge function $f^*: E(G) \to \{0,1\}$ is defined as $f^*(xy) = |f(x) - f(y)|$ and it satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. There are different types of cordial labeling that have been introduced like prime cordial labeling, A – cordial labeling, E – cordial labeling, H – cordial labeling, product cordial labeling and total product cordial labeling. In this paper, we will be investigating a new result on the product cordial labeling of the graphs obtained by duplication of some graph elements of crown, helm and wheel graph, which are the furthermost attraction of this study.

2 Preliminaries

We begin a simple, finite and undirected graph $G = (V(G), E(G))$ where $V(G)$ and $E(G)$ are the vertex set and edge set respectively.

Definition 2.1. [11] If the vertices or edges or both graphs are assigned values subject to certain conditions then it is known as vertex, edge and total labeling respectively.
Definition 2.2. [9] A binary vertex labeling of a graph G with induced edge labeling $f^* : E(G) \to \{0,1\}$ defined by $f^*(e = uv) = f(u)f(v)$ is called a *product cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph is called *product cordial* if it admits product cordial labeling.
Definition 2.3. [11] Duplication of a vertex of the graph G is the graph G' obtained from the graph G by adding a new vertex v' to G such that $N(v') = N(v)$.
Definition 2.4. [11] Duplication of a vertex v_k by a new edge $e = v'_k v''_k$ in a graph G produces a new graph G' such that $N(v'_k) = \{v_k, v''_k\}$ and $N(v''_k) = \{v_k, v'_k\}$.
Definition 2.5. [11] Duplication of an edge $e = uv$ by a new vertex w in a graph G produces a new graph G' such that $N(w) = \{u, v\}$.
Definition 2.6. [4] The crown graph C_n^* is obtained from a cycle C_n by attaching a pendant edge at each vertex of the $n - cycle$.
Definition 2.7. [11] The graph $W_n = C_n + K_1$ is called the wheel graph, and the vertex corresponding to K_1 is called an apex vertex, and C_n are called rim vertices.
Definition 2.8. [1] A *helm* H_n, $n \geq 3$ is the graph obtained from wheel W_n by adding a pendant edge at each vertex on the rim of the wheel W_n.

3 Results

Theorem 3.1. The graph obtained by duplicating each vertex in crown graph C_n^* is product cordial graph.

Proof. Let $v_1, v_2, \dots, v_n$ be the vertex of degree three and $u_1, u_2, \dots, u_n$ be the pendant vertex of crown graph C_n^*. Let G be the graph obtained from C_n^* by duplicating each pendant vertex by $u'_1, u'_2, \dots, u'_n$ and each vertex of degree three by $v'_1, v'_2, \dots, v'_n$ respectively for all $i = 1,2,\dots,n$. Then $|V(G)| = 4n$ and $|E(G)| = 6n$. Define a function $f: V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0, & if\ x = u_i, \quad i = 1,2,\dots,n, \quad or \\ & \quad x = u'_i, \quad i = 1,2,\dots,n \\ \\ 1, & if\ x = v_i, \quad i = 1,2,\dots,n, \quad or \\ & \quad x = v'_i, \quad i = 1,2,\dots,n \end{cases}$$

In view of the above labeling pattern, we have $v_f(0) = v_f(1) = 2n$ and $e_f(0) = e_f(1) = 3n$. Clearly, $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. Hence, G admits product cordial labeling. Therefore, the graph obtained by duplicating each vertex in crown graph C_n^* is product cordial graph.■

Illustration 3.1. Product cordial labeling of the graph obtained by duplication of each vertex in crown graph C_n^* for $n = 6$ is shown in **Figure 1**.

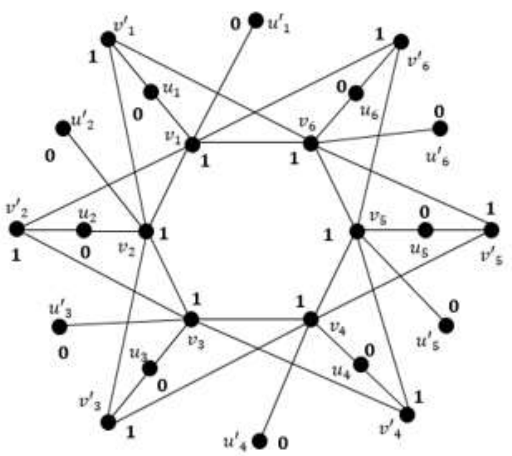

Figure 1. Duplication of each vertex in C_6^*

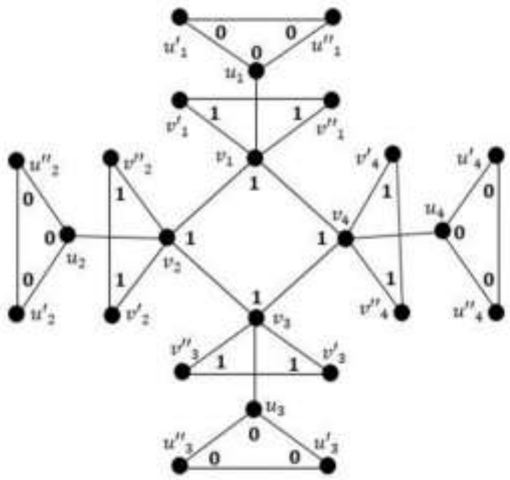

Figure 2. Duplication of each vertex by an edge in C_4^*

Theorem 3.2. The graph obtained by duplicating of each vertex by an edge in crown graph $\boldsymbol{C_n^*}$ is product cordial graph.

Proof. Let $v_1, v_2, \dots, v_n$ be the vertex of degree three and $u_1, u_2, \dots, u_n$ be the pendant vertex of crown graph C_n^*. Let G be the graph obtained from C_n^* by duplicating each pendant vertex by an edge u'_i, u''_i and each vertex of degree three by an edge v_i', v_i'' respectively for all $i = 1,2,\dots,n$. Then $|V(G)| = 6n$ and $|E(G)| = 8n$. Define a function $f\colon V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0, \ \ if \ \ x = u_i, & i = 1,2,\dots,n, \quad or \\ \qquad x = u'_i, & i = 1,2,\dots,n, \quad or \\ \qquad x = u''_i, & i = 1,2,\dots,n, \\ & \\ 1, \ \ if \ \ x = v_i, & i = 1,2,\dots,n, \quad or \\ \qquad x = v'_i, & i = 1,2,\dots,n, \quad or \\ \qquad x = v''_i, & i = 1,2,\dots,n, \end{cases}$$

In view of the above labeling pattern, we have $v_f(0) = v_f(1) = 3n$ and $e_f(0) = e_f(1) = 4n$. Clearly, $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. Hence, G admits product cordial labeling. Therefore, the graph obtained by duplicating each vertex by an edge in crown graph C_n^* is product cordial graph. ■

Illustration 3.2. Product cordial labeling of the graph obtained by duplication of each vertex by an edge in crown graph C_n^* for $n = 4$ is shown in **Figure 2**.

Theorem 3.3. The graph obtained by duplicating an arbitrary vertex by a new edge in crown graph C_n^* is product cordial graph.

Proof. Let $v_1, v_2, \dots, v_n$ be the vertex of degree three and $u_1, u_2, \dots, u_n$ be the pendant vertex of crown graph C_n^*. Let G be the graph obtained from C_n^* by duplicating an arbitrary vertex by a new edge with end vertices as u'_1 and u'_2. Then $|V(G)| = 2n + 2$ and $|E(G)| = 2n + 3$. Define a function $f\colon V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0, \ \ if \ \ x = u_i, & i > 1, \quad or \\ \qquad x = u'_i, & i = 1,2 \\ & \\ 1, \ \ if \ \ x = u_i, & i = 1, \quad or \\ \qquad x = v_i, & i = 1, 2, \dots, n \end{cases}$$

In view of the above labeling pattern, we have $v_f(0) = v_f(1) = n + 1$ and $e_f(0) = n + 2 = e_f(1) + 1$. Clearly, $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. Hence, G admits product cordial labeling. Therefore, the graph obtained by duplicating an arbitrary vertex by a new edge in crown graph C_n^* is product cordial graph. ■

Illustration 3.3. Product cordial labeling of the graph obtained by duplication of an arbitrary vertex by a new edge in crown graph C_n^* for $n = 4$ is shown in **Figure 3.**

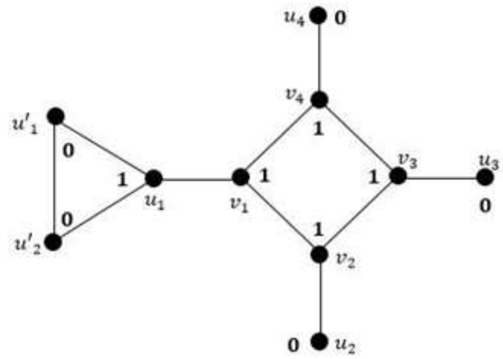

Figure 3. Duplication of an arbitrary vertex by a new edge in C_4^*

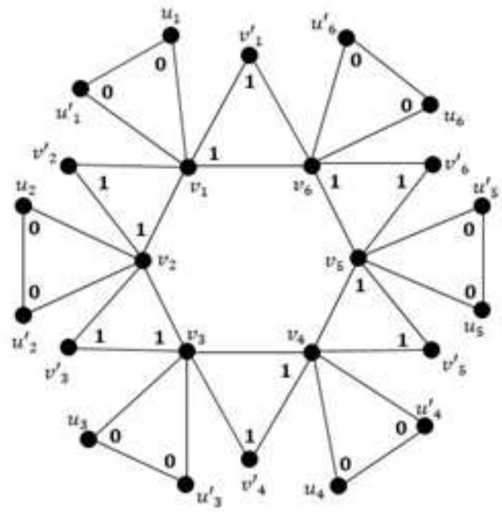

Figure 4. Duplication of an edge by a vertex in C_6^*

Theorem 3.4. The graph obtained by duplicating each edge by a vertex in crown graph C_n^* is product cordial graph.

Proof. Let $v_1, v_2, \dots, v_n$ be the vertex of degree three and $u_1, u_2, \dots, u_n$ be the pendant vertex of crown graph C_n^*. Let G be the graph obtained from C_n^* by duplicating each pendant edge by a vertex u_i' and each edge of the $n - cycle$ by a vertex v_i' for all $i = 1,2, \dots, n$ respectively. Then $|V(G)| = 4n$ and $|E(G)| = 6n$. Define a function $f: V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0, & if\ \ x = u_i, \quad i = 1,2, \dots, n, \quad or \\ & \quad\ x = u'_i, \quad i = 1,2, \dots, n \\ & \\ 1, & if\ \ x = v_i, \quad i = 1,2, \dots, n, \quad or \\ & \quad\ x = v'_i, \quad i = 1,2, \dots, n \end{cases}$$

In view of the above labeling pattern, we have $v_f(0) = v_f(1) = 2n$ and $e_f(0) = e_f(1) = 3n$. Clearly, $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Hence, G admits product cordial labeling. Therefore, the graph obtained by duplicating each edge by a vertex in crown graph C_n^* is product cordial graph.■

Illustration 3.4. Product cordial labeling of the graph obtained by duplication of each edge by a vertex in crown graph C_n^* for $n = 4$ is shown in **Figure 4.**

Theorem 3.5. The graph obtained by duplicating an arbitrary edge by a new vertex in crown graph C_n^* is product cordial graph.

Proof. Let $v_1, v_2, \dots, v_n$ be the vertex of degree three and $u_1, u_2, \dots, u_n$ be the pendant vertex of crown graph C_n^*. Let G be the graph obtained from C_n^* by duplicating an arbitrary edge by a new vertex v'. Then $|V(G)| = 2n+1$ and $|E(G)| = 2n+2$. Define a function $f: V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0, & if \ \ x = u_i, \qquad\qquad 2 \le i \le n, \quad or \\ & \ \ x = v' \\ & \\ 1, & if \ \ x = u_i, \qquad\qquad\quad i = 1, \quad or \\ & \ \ x = v_i, \qquad\qquad i = 1, 2, \dots, n \end{cases}$$

In view of the above labeling pattern, we have $v_f(0) = n = v_f(1) - 1$ and $e_f(0) = e_f(1) = n + 1$. Clearly, $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. Hence, G admits product cordial labeling. Therefore, the graph obtained by duplicating an arbitrary edge by a new vertex in crown graph C_n^* is product cordial graph.■

Illustration 3.5. Product cordial labeling of the graph obtained by duplication of an arbitrary edge by a new vertex in crown graph C_n^* for $n = 5$ is shown in **Figure 5.**

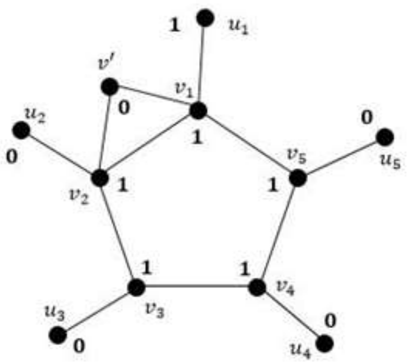

Figure 5. Duplication of arbitrary edge by a new vertex in C_5^*

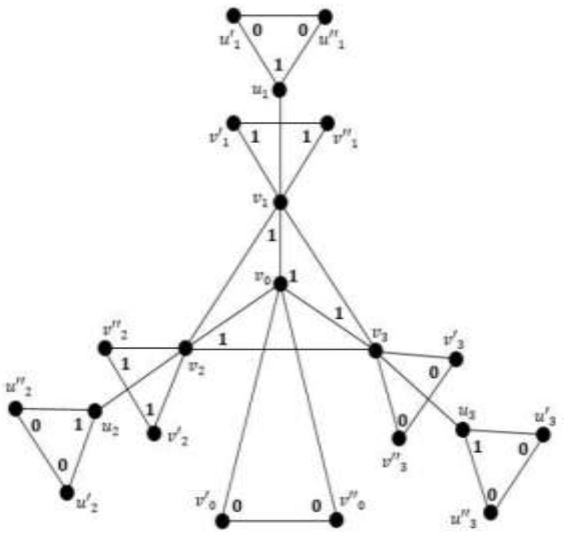

Figure 6. Duplication of each vertex by an edge in H_3

Meanwhile, we investigated 10 results for product cordial labeling of the graphs obtained by duplication of some graph elements in crown graph, 5 (five) Theorems and 5 (five) Illustrations.

Theorem 3.6. The graph obtained by duplicating each vertex by an edge in helm graph H_n is product cordial graph.

Proof. Let v_0 be the apex vertex, $v_1, v_2, \dots, v_n$ be the vertex of degree four and $u_1, u_2, \dots, u_n$ be the pendant vertex of helm graph H_n. Let G be the graph obtained from H_n by duplicating each pendant vertex by an edge u_i', u_i'' and each vertex of degree four by an edge v_i', v_i'' and an apex vertex v_0', v_0'' by an edge for all $i = 1,2, \dots, n$ respectively. Then $|V(G)| = 6n + 3$ and $|E(G)| = 9n + 3$. To define a function $f: V(G) \to \{0,1\}$ we consider following two cases.

<u>**Case 1:** When n is odd.</u>

$$f(x) = \begin{cases} 0, \quad if \;\; x = v'_0, & or \\ \quad x = v''_0, & or \\ \quad x = u'_i, \quad\quad i = 1, 2, \dots, n, & or \\ \quad x = u''_i, \quad\quad i = 1, 2, \dots, n, & or \\ \quad x = v'_i, \quad\quad \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, & or \\ \quad x = v''_i, \quad\quad \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n & \\ & \\ 1, \quad if \;\; x = v_0, & or \\ \quad x = v_i, \quad\quad i = 1, 2, \dots, n, & or \\ \quad x = u_i, \quad\quad i = 1, 2, \dots, n, & or \\ \quad x = v'_i, \quad\quad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, & or \\ \quad x = v''_i, \quad\quad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil & \end{cases}$$

In view of the above labeling pattern, we have $v_f(0) = 3n + 1 = v_f(1) - 1$ and $e_f(0) = e_f(1) = \frac{9n+3}{2}$. Clearly, $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. Thus, G admits product cordial labeling.

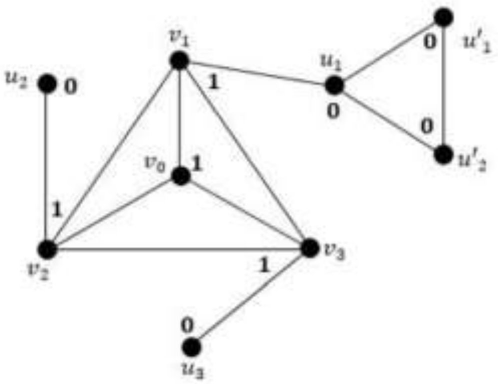

Figure 7. Duplication of arbitrary vertex by a new edge in H_3

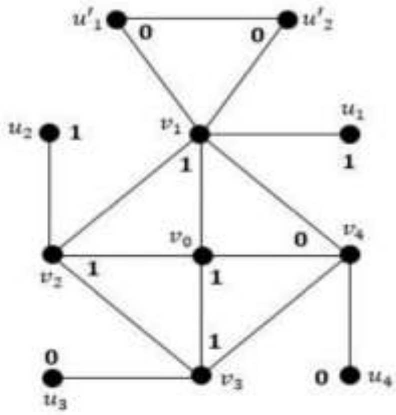

Figure 8. Duplication of arbitrary vertex by a new edge in H_4

Case 2: When n is even.

$$f(x)=\begin{cases} 0, & if\ \ x=v''_0, & & or\\ & x=u'_i, & i=1,2,\dots,n, & or\\ & x=u''_i, & i=1,2,\dots,n, & or\\ & x=v'_i, & \frac{n}{2}+1\le i\le n, & or\\ & x=v''_i, & \frac{n}{2}+1\le i\le n & \\ & & & \\ 1, & if\ \ x=v_0, & & or\\ & x=v'_0, & & or\\ & x=v_i, & i=1,2,\dots,n, & or\\ & x=u_i, & i=1,2,\dots,n, & or\\ & x=v'_i, & 1\le i\le \frac{n}{2}, & or\\ & x=v''_i, & 1\le i\le \frac{n}{2} & \end{cases}$$

In view of the above labeling pattern, we have $v_f(0)=3n+1=v_f(1)-1$ and $e_f(0)=\frac{9n+4}{2}=e_f(1)+1$. Clearly, $\left|v_f(0)-v_f(1)\right|\le 1$ and $\left|e_f(0)-e_f(1)\right|\le 1$. Thus, G admits product cordial labeling. Hence, all conditions admit product cordial labeling. Therefore, the graph obtained by duplicating each vertex by an edge in helm graph H_n is product cordial graph.■

Illustration 3.6. Product cordial labeling of the graph obtained by duplication of each vertex by an edge in helm graph H_n for $n=3$ is shown in **Figure 6.**

Theorem 3.7. The graph obtained by duplicating an arbitrary vertex by a new edge in helm graph H_n is product cordial graph.

Proof. Let v_0 be the apex vertex, $v_1,v_2,\dots,v_n$ be the vertex of degree four and $u_1,u_2,\dots,u_n$ be the pendant vertex of helm graph H_n. Let G be the graph obtained from H_n by duplicating an arbitrary vertex by a new edge with end vertices as u'_1 and u'_2. Then $|V(G)|=2n+3$ and $|E(G)|=3n+3$. To define a function $f\colon V(G)\rightarrow\{0,1\}$ we consider following three cases.

Case 1: When $n=3$.
The graph obtained by duplicating an arbitrary vertex by a new edge in helm graph H_3 and its product cordial labeling is shown in **Figure 7**.

Case 2: When n is odd $(n \neq 3)$.

$$f(x) = \begin{cases} 0, & if \quad x = u_i, & \left\lceil \frac{n}{2} \right\rceil \leq i \leq n, & or \\ & x = v_i, & \frac{n+5}{2} \leq i \leq n, & or \\ & x = u'_i, & i = 1,2 & \\ 1, & if \quad x = v_0, & & or \\ & x = u_i, & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, & or \\ & x = v_i, & 1 \leq i \leq \frac{n+3}{2} & \end{cases}$$

In view of the above labeling pattern, we have $v_f(0) = n + 1 = v_f(1) - 1$ and $e_f(0) = e_f(1) = \frac{3n+3}{2}$. Clearly, $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Thus, G admits product cordial labeling.

Case 3: When n is even.

$$f(x) = \begin{cases} 0, & if \quad x = u_i, & \frac{n}{2} + 1 \leq i \leq n, & or \\ & x = v_i, & \frac{n+4}{2} \leq i \leq n, & or \\ & x = u'_i, & i = 1,2 & \\ 1, & if \quad x = v_0, & & or \\ & x = u_i, & 1 \leq i \leq \frac{n}{2}, & or \\ & x = v_i, & 1 \leq i \leq \frac{n}{2} + 1 & \end{cases}$$

In view of the above labeling pattern, we have $v_f(0) = n + 1 = v_f(1) - 1$ and $e_f(0) = \frac{3n+4}{2} = e_f(1) + 1$. Clearly, $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Thus, G admits product cordial labeling. Hence, all conditions admit product cordial labeling. Therefore, the graph obtained by duplicating an arbitrary vertex by a new edge in helm graph H_n is product cordial graph.■

Illustration 3.7. Product cordial labeling of the graph obtained by duplication of an arbitrary vertex by a new edge in helm graph H_n for $n = 4$ is shown in **Figure 8.**

Theorem 3.8. The graph obtained by duplicating each edge by a vertex in helm graph H_n is product cordial graph.

Proof. Let v_0 be the apex vertex, $v_1, v_2, \dots, v_n$ be the vertex of degree four and $u_1, u_2, \dots, u_n$ be the pendant vertex of helm graph H_n. Let G be the graph obtained from H_n by duplicating each pendant edge by a vertex u_i' and an adjacent edge of the apex vertex by a vertex w_i' and each edge of the $n-cycle$ by a vertex v_i' for all $i = 1,2,3, \dots, n$ respectively. Then $|V(G)| = 5n+1$ and $|E(G)| = 9n$. To define a function $f: V(G) \to \{0,1\}$ we consider following two cases.

Case 1: When n is odd.

$$f(x) = \begin{cases} 0, & if\ x = u_i, & \left\lceil \frac{n}{2} \right\rceil \le i \le n, & or \\ & x = u'_i, & i = 1, 2, \dots, n, & or \\ & x = v'_i, & i = 1, 2, \dots, n & \\ & & & \\ 1, & if\ x = v_0, & & or \\ & x = v_i, & i = 1, 2, \dots, n, & or \\ & x = w'_i, & i = 1, 2, \dots, n, & or \\ & x = u_i, & 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor & \end{cases}$$

In view of the above labeling pattern, we have $v_f(0) = v_f(1) = \frac{5n+1}{2}$ and $e_f(0) - \frac{1}{2} = \frac{9n}{2} = e_f(1) + \frac{1}{2}$. Clearly, $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. Thus, G admits product cordial labeling.

Case 2: When n is even.

$$f(x) = \begin{cases} 0, & if\ x = u_i, & \frac{n}{2} + 1 \le i \le n, & or \\ & x = u'_i, & i = 1, 2, \dots, n, & or \\ & x = v'_i, & i = 1, 2, \dots, n & \\ & & & \\ 1, & if\ x = v_0, & & or \\ & x = v_i, & i = 1, 2, \dots, n, & or \\ & x = w'_i, & i = 1, 2, \dots, n, & or \\ & x = u_i, & 1 \le i \le \frac{n}{2} & \end{cases}$$

In view of the above labeling pattern, we have $v_f(0) = \frac{5n}{2} = v_f(1) - 1$ and $e_f(0) = e_f(1) = \frac{9n}{2}$. Clearly, $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. Thus, G admits product cordial labeling. Hence, all conditions admit product cordial labeling. Therefore, the graph obtained by duplicating of each edge by a vertex in helm graph H_n is product cordial graph. ■

Illustration 3.8. Product cordial labeling of the graph obtained by duplication of each edge by a vertex in helm graph H_n for $n = 4$ and $n = 5$ are shown in **Figure 9** and **Figure 10** respectively.

Meanwhile, we investigated 8 results for product cordial labeling of the graphs obtained by duplication of some graph elements in helm graph, 3 (three) Theorems and 5 (five) Illustrations.

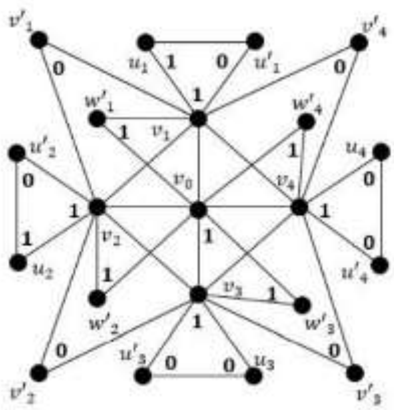

Figure 9. Duplication of each edge by a vertex in H_4

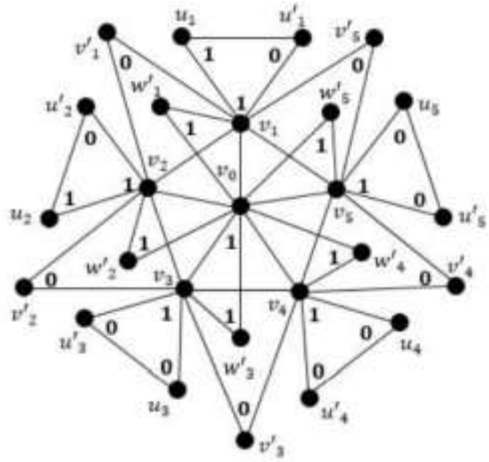

Figure 10. Duplication of each edge by a vertex in H_5

Theorem 3.9. The graph obtained by duplicating each vertex by an edge in wheel graph W_n is product cordial graph.

Proof. Let W_n be the wheel graph with the apex vertex v_0 and consecutive rim vertex $v_1, v_2, \dots, v_n$. Let G be the graph obtained from W_n by duplicating all vertices by an edge $v'_i v''_i$ respectively for all $i = 0,1,\dots,n$. Then $|V(G)| = 3n + 3$ and $|E(G)| = 5n + 3$. To define a function $f: V(G) \to \{0,1\}$ we consider following two cases.

<u>**Case 1:** When n is odd.</u>

$$f(x) = \begin{cases} 0, \ if \ \ x = v'_0, & & or \\ \quad x = v'_i, & \left\lceil \frac{n}{2} \right\rceil \le i \le n, & or \\ \quad x = v''_0, & & or \\ \quad x = v''_i, & 2 \le i \le n & \\ & & \\ 1, \ if \ \ x = v_0, & & or \\ \quad x = v_i, & i = 1, 2, \dots, n, & or \\ \quad x = v'_i, & 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor, & or \\ \quad x = v''_1 & & \end{cases}$$

In view of the above labeling pattern, we have $v_f(0) = v_f(1) = \frac{3n+3}{2}$ and $e_f(0) = e_f(1) = \frac{5n+3}{2}$. Clearly, $\left|v_f(0) - v_f(1)\right| \le 1$ and $\left|e_f(0) - e_f(1)\right| \le 1$. Thus, G admits product cordial labeling.

Case 2: When n is even.

$$f(x)=\begin{cases} 0, \ \ if \ \ x = v'_0, & or \\ \quad x = v'_i, \quad \dfrac{n}{2}+1 \le i \le n, & or \\ \quad x = v''_0, & or \\ \quad x = v''_i, \quad 2 \le i \le n & \\ & \\ 1, \ \ if \ \ x = v_0, & or \\ \quad x = v_i, \quad i = 1, 2, \dots, n, & or \\ \quad x = v'_i, \quad 1 \le i \le \dfrac{n}{2}, & or \\ \quad x = v''_1 & \end{cases}$$

In view of the above labeling pattern, we have $v_f(0) = \frac{3n+2}{2} = v_f(1) - 1$ and $e_f(0) = \frac{5n+2}{2} = e_f(1) - 1$. Clearly, $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. Thus, G admits product cordial labeling. Hence, all conditions admit product cordial labeling. Therefore, the graph obtained by duplicating each vertex by an edge in wheel graph W_n is product cordial graph.■

Illustration 3.9. Product cordial labeling of the graph obtained by duplication of each vertex by an edge in wheel graph W_n for $n = 3$ and $n = 4$ are shown in **Figure 11** and **Figure 12** respectively**.**

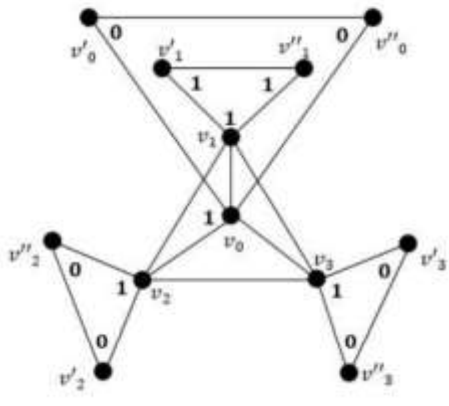

Figure 11. Duplication of each vertex by an edge in W_3

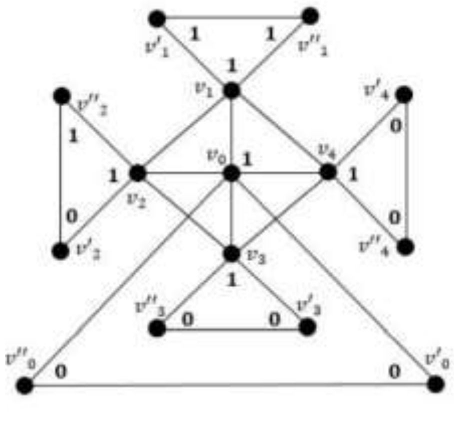

Figure 12. Duplication of each vertex by an edge in □4

Theorem 3.10. The graph obtained by duplicating an arbitrary vertex by a new edge in wheel graph W_n is product cordial graph when n is even.

Proof. Let W_n be the wheel graph with the apex vertex v_0 and consecutive rim vertex $v_1, v_2, \dots, v_n$. Let G be the graph obtained from W_n by duplicating an arbitrary vertex by a new edge with end vertices as v' and v''. Then $|V(G)| = n + 3$ and $|E(G)| = 2n + 3$. To define a function $f: V(G) \rightarrow \{0,1\}$ we consider following two cases.

Case 1: When n is odd.
In order to satisfy the vertex condition for product cordial graph it is essential to assign label 1 to $\left(\frac{n+3}{2}\right)$ vertices out of $(n+3)$ vertices. The vertices with label 0 will give rise at least $(n+3)$ edges with label 0 and at most n edge with label 1 out of total $(2n+3)$ edges. Clearly, $\left|e_f(0) - e_f(1)\right| = |\,(n+3) - (n)| = 3 > 1$. Thus, the edge condition for product cordial graph is violated. Hence, G is not a product cordial graph.

Case 2: When n is even.

$$f(x) = \begin{cases} 0, & if \;\; x = v', \quad or \\ & x = v'', \quad or \\ & x = v_i, \quad \frac{n+4}{2} \le i \le n \\ 1, & if \;\; x = v_0, \quad or \\ & x = v_i, \quad 1 \le i \le \frac{n+2}{2} \end{cases}$$

In view of the above labeling pattern, we have $v_f(0) = \frac{n+2}{2} = v_f(1) - 1$ and $e_f(0) = n + 2 = e_f(1) + 1$. Clearly, $\left|v_f(0) - v_f(1)\right| \le 1$ and $\left|e_f(0) - e_f(1)\right| \le 1$. Thus, G admits product cordial labeling. Therefore, the graph obtained by duplicating an arbitrary vertex by a new edge in wheel graph W_n is product cordial graph when n is even. ■

Illustration 3.10. Product cordial labeling of the graph obtained by duplication of an arbitrary vertex by a new edge in wheel graph W_n for $n = 6$ is shown in **Figure 13.**

Theorem 3.11. The graph obtained by duplicating each edge by a vertex in wheel graph W_n is product cordial graph when n is even.

Proof. Let W_n be the wheel graph with the apex vertex v_0 and consecutive rim vertex $v_1, v_2, \dots, v_n$. Let G be the graph obtained from W_n by duplicating each edge of the $n - cycle$ by vertices v'_i and adjacent edge to apex vertex by vertices v''_i respectively for all $i = 1,2,\dots,n$. Then $|V(G)| = 3n + 1$ and $|E(G)| = 6n$. To define a function $f: V(G) \rightarrow \{0,1\}$ we consider following two cases.
Case 1: When n is odd.
In order to satisfy the vertex condition for product cordial graph it is essential to assign label 1 to $\left(\frac{3n+1}{2}\right)$ vertices out of $(3n+1)$ vertices. The vertices with label

0 will give rise at least $(3n+1)$ edges with label 0 and at most $(3n-1)$ edge with label 1 out of total $6n$ edges. Clearly, $\left|e_f(0)-e_f(1)\right| = |\,(3n+1)-(3n-1)| = 2 > 1$. Thus, the edge condition for product cordial graph is violated. Hence, G is not a product cordial graph.

Case 2: When n is even.

$$f(x)=\begin{cases} 0, & if\;\; x=v'_i, \qquad i=1,2,\dots,n, \quad or \\ & x=v''_i, \qquad \frac{n}{2}+1\le i\le n \\ \\ 1, & if\;\; x=v_0, \qquad or \\ & x=v_i, \qquad i=1,2,\dots,n, \quad or \\ & x=v''_i, \qquad 1\le i\le \frac{n}{2} \end{cases}$$

In view of the above labeling pattern, we have $v_f(0)=\frac{3n}{2}=v_f(1)-1$ and $e_f(0)=e_f(1)=3n$. Clearly, $\left|v_f(0)-v_f(1)\right|\le 1$ and $\left|e_f(0)-e_f(1)\right|\le 1$. Thus, G admits product cordial labeling. Therefore, the graph obtained by duplicating each edge by a vertex in wheel graph W_n is product cordial graph when n is even.■

Illustration 3.11. Product cordial labeling of the graph obtained by duplication of each edge by a vertex in wheel graph W_n for $n=6$ is shown in **Figure 14.**

Meanwhile, we investigated 7 results for product cordial labeling of the graphs obtained by duplication of some graph elements in wheel graph, 3 (three) Theorems and 4 (four) Illustrations. Hence, we have 26 results that we derived for product cordial labeling of the graphs obtained by duplication of some graph elements in crown, helm and wheel graph.

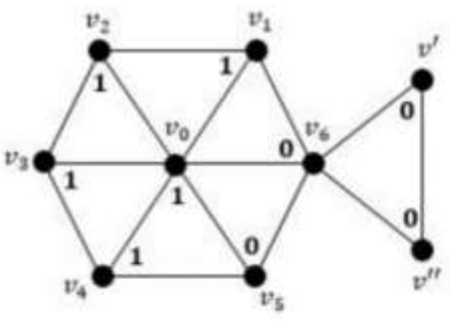

Figure 13. Duplication of an arbitrary vertex by a new edge in W_6

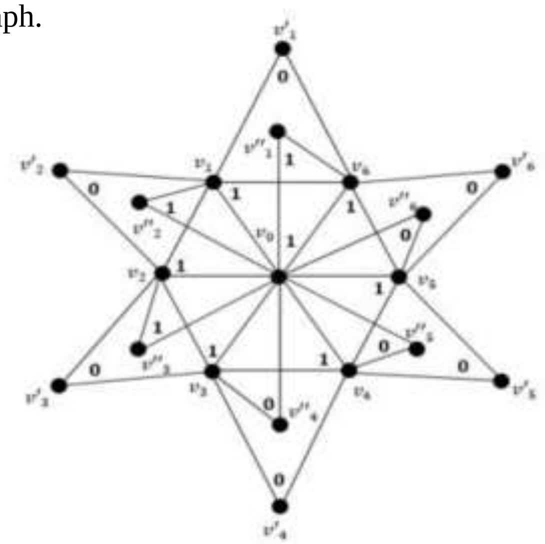

Figure 14. Duplication of each edge by a vertex in W_6

4 Conclusion

In this paper, we investigated that the graph obtained by duplication of some graph elements in crown, helm and wheel graph are product cordial graph. Thus, we obtained the following results:

1. For the crown graph C_n^*, it shows that the graph obtained by duplication of each vertex, duplication of each vertex by an edge, duplication of an arbitrary vertex by a new edge, duplication of each edge by a vertex and duplication of an arbitrary edge by a new vertex is product cordial graph.
2. For the helm graph H_n, it shows that the graph obtained by duplication of each vertex by an edge, duplication of an arbitrary vertex by a new edge and duplication of each edge by a vertex is product cordial graph.
3. For the wheel graph W_n, it shows that the graph obtained by duplication of each vertex by an edge, duplication of an arbitrary vertex by a new edge and duplication of each edge by a vertex is product cordial graph. On the other hand, we also discussed that the graph obtained by duplication of an arbitrary vertex by a new edge in wheel graph W_n is not product cordial when n is odd; and the graph obtained by duplication of each edge by a vertex in wheel graph W_n is not product cordial when n is odd.

These results may serve as a good reference for further studies on the product cordial labeling of graphs. Similar problems can be discussed in the context of these graph operations on closed helm, armed helm and flower graph.

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