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Semi-Geodetic Coordinate Nets Through Isometries

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Abstract

Every coordinate net on a rotating surface is a semi-geodesic coordinate net composed of a family of curves of constant geodesic curvature. In this paper, using semi-geodesic coordinate nets on special rotating surfaces such as a conical surface, a catenoid, and a rotational hyperboloid surface, we give families of curves of constant geodesic curvature on some surfaces through isometric mappings. Also, with the aid of the software Mathematica, we draw images of the semi-geodesic coordinate nets and the family of curves obtained through isometric mappings.

Keywords: semi-geodesic coordinate net; isometry; constant geodesic curvature curve

1 Introduction

A semi-geodesic coordinate net is a special kind of coordinate net on a surface, which is not only a promotion of the polar coordinate net on a plane to a surface but also a necessary coordinate tool for studying the properties of the surface. Coordinate systems in which one family of coordinate lines represents geodesics on generic surfaces play an important role in differential geometry [1]. A semi-geodesic coordinate net is closely related to geodesics. It is needless to say that geodesic curvatures of regular curves are important invariants of curves. Since geodesic curvatures of curves are invariant, they play a very important role in

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studying surfaces from curve theoretic point of view. In the reference [2], by using isometries of surfaces in a Euclidean 3-space, we obtained the expressions of geodesics, and drew their images. Since geodesics are curves of constant null geodesic curvature, our next interest lies on curves of non-null constant geodesic curvature.

In this paper we first show that coordinate nets on rotating surfaces are semi-geodesic coordinate nets which are composed of geodesics and curves of constant geodesic curvature. In order to apply this result, we consider semi-geodesic coordinate nets on special rotating surfaces such as a conical surface, a catenoid, and a rotational hyperboloid surface. By using isometries of these surfaces [3], we obtain semi-geodesic coordinate nets on surfaces which are not rotating surfaces. We also draw figures of these semi-geodesic coordinate nets by using the software Mathematica.

2 Semi-geodesic coordinate nets on rotating surfaces

A coordinate network on a surface is composed of two families u, v of smooth regular curves satisfying the following condition: At each point $P_0(u_0, v_0)$ on this surface, there is a unique pair $u = u_0$, $v = v_0$ of curves passing through P_0 whose tangent vectors at this point are linearly independent.

Definition 1[4]. A coordinate network on a surface is said to be a semi-geodesic coordinate net, if one family is formed by geodesics and the other formed by curves which are orthogonal to geodesics in the previous family at their crossing points.

We denote by E, F, G the quantities of the first fundamental form of a surface $\boldsymbol{r} = \boldsymbol{r}(u,v)$. A coordinate net on a surface $\boldsymbol{r} = \boldsymbol{r}(u,v)$ is a semi-geodesic coordinate net if and only if $E_v = F = 0$ [5]. Therefore, a semi-geodesic coordinate net is a orthogonal coordinate grid ($F = 0$).

The method of calculating geodesic curvatures of curves under an orthogonal coordinate grid is given as follows.

Proposition 1[4]. (Liouville formula) When the coordinate net on the surface $\boldsymbol{r} = \boldsymbol{r}(u,v)$ is an orthogonal coordinate grid, the formula of geodesic curvature k_g of the curve c at a point is

$$k_g = \frac{d\theta}{ds} - \frac{1}{2\sqrt{G}} \frac{\partial \ln E}{\partial v} \cos\theta + \frac{1}{2\sqrt{E}} \frac{\partial \ln G}{\partial u} \sin\theta. \tag{1}$$

where s is the parameter of arclength of the curve c; θ represents the angle between the tangent vector $c(s)$ and r_u.

If the geodesic curvature of a curve on a surface is constant along this curve, we call it a curve of constant geodesic curvature. Geodesics are curves of constant null geodesic curvature.

We here study coordinate nets on rotating surfaces.

Theorem 1. The coordinate net on a rotating surface $\boldsymbol{r}=\{\varphi(u)\cos\theta,\varphi(u)\sin\theta,\psi(u)\}$ is a semi-geodesic coordinate net composed of curves of constant geodesic curvature. Coordinate curves parameterized by u are geodesics, and the geodesic curvature of each coordinate curve parameterized by θ is given by

$$k_g=\frac{\varphi'(u)}{\varphi(u)\sqrt{\varphi'(u)^2+\psi'(u)^2}}.$$

Proof. Through the parametric equation of the surface, we find that the quantities of the first fundamental form for this rotating surface are $E=\varphi'(u)^2+\psi'(u)^2$, $F=0, G=\varphi(u)^2$. We hence obtain

$$E_\theta=F=0.$$

That is, the coordinate net on this rotating surface is a semi-geodesic coordinate net.

For each coordinate curve parameterized by θ (and u is a given constant) on this rotating surface, the angle between the tangent vector of this curve and r_u is $\alpha=\frac{\pi}{2}$. Using the Liouville formula (1), we find that the geodesic curvature of this coordinate curve is

$$k_g=\frac{\varphi'(u)}{\varphi(u)\sqrt{\varphi'(u)^2+\psi'(u)^2}}.$$

Thus, all coordinate curves parameterized by θ are of constant geodesic curvature.

For each coordinate curve parameterized by u (and v is a given constant) on this rotating surface, we have $\alpha=0$. Using the Liouville formula (1), we find that the geodesic curvature of this coordinate curve parameterized by u is $k_g=0$. Therefore, all coordinate curves parameterized by u are geodesics on this rotating surface.

We hence obtain that the coordinate net on the rotating surface is a semi-geodesic coordinate net composed of constant geodesic curvature curves.

3 Isometries of rotating surfaces and semi-geodesic coordinate nets

In this section, we apply Theorem 1 to some rotating surfaces to get semi-geodesic coordinate nets, and study their relationships between isometries.

Example 1. We take a punctured plane $R^2-\{0\}$, which can be regarded as a rotating surface. By Theorem 1, its coordinate $\boldsymbol{r}=\{\rho\cos\theta,\rho\sin\theta,0\}$ ($\rho>0$) is a semi-geodesic coordinate net composed by curves of constant geodesic curvature. Clearly, each coordinate cure parameterized by θ has a constant geodesic curvature $1/\rho$, and coordinate curves parameterized by ρ are geodesics.

Example 2. We take a conical surface $\boldsymbol{r}=\{u\cos v, u\sin v, u\}\,(u>0)$ and omit its vertex. By Theorem 1, its coordinate net is a semi-geodesic coordinate net of constant geodesic curvature. Each coordinate curve parameterized by v is a curve of constant geodesic curvature $k_g=\dfrac{\sqrt{2}}{2u}$, and coordinate curves parameterized by u are geodesics.

We can study Example 2 also by use of an isometry.

Lemma 1[6]**.** We consider a bijection of a plane $\boldsymbol{r}=\{\rho\cos\theta, \rho\sin\theta, 0\}$ to a conical surface $\boldsymbol{r}=\{u\cos v, u\sin v, u\}$ defined by $u=\dfrac{\sqrt{2}}{2}\rho, v=\theta$. If we omit their origins, its restriction on to the punctured plane is an isometry.

Through the isometry given in Lemma 1, the curve $\theta\to\{\rho_0\cos\theta, \rho_0\sin\theta, 0\}$ is mapped to the curve $\theta\to\left\{\dfrac{\sqrt{2}}{2}\rho_0\cos\theta, \dfrac{\sqrt{2}}{2}\rho_0\sin\theta, \dfrac{\sqrt{2}}{2}\rho_0\right\}$. Hence, we find that the coordinate $\boldsymbol{r}=\{u\cos v, u\sin v, u\}\,(u>0)$ is a semi-geodesic coordinate net of constant geodesic curvature. Each coordinate curve parameterized by v has constant geodesic curvature $1/\rho=\dfrac{\sqrt{2}}{2u}$.

The following shows the isometric transformation of the semi-geodetic coordinate net on a punctured plane to a conical surface. Here, red curves are of curves of non-zero constant geodesic curvature and blue curves are geodesics.

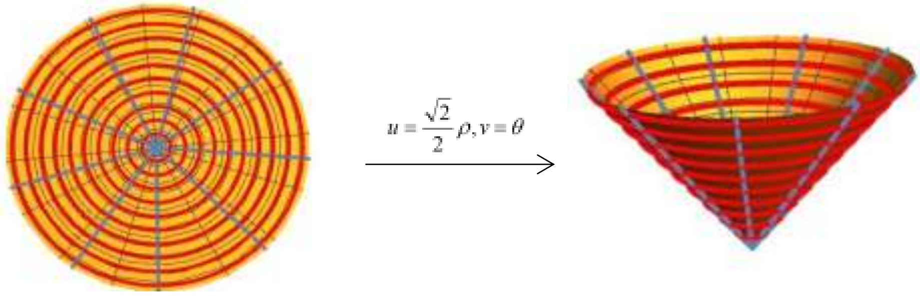

Fig. 1 Isometric transformation of a semi-geodetic coordinate net on a plane to that on a conical surface

Example 3. By Theorem 1, we find that the coordinate on a catenoid $\boldsymbol{r}=\{\cosh t\cos\theta, \cosh t\sin\theta, t\}$ is a semi-geodesic coordinate net. Each coordinate curve parameterized by θ has constant geodesic curvature $k_g=\dfrac{\tanh t}{\cosh t}$, hence $k_g=0$ if and only if $t=0$. For this curve, the angle α between its tangent vector and r_t is $\frac{\pi}{2}$. On the other hand, each coordinate curve parameterized by t is a geodesic and has $\alpha = 0$.

By using the above semi-geodesic coordinate net on a catenoid, we give a semi-geodesic coordinate net on a positive spiral surface.

Lemma 2[4]**.** The map of a catenoid $\boldsymbol{r} = \{\cosh t \cos\theta, \cosh t \sin\theta, t\}$ to a positive spiral surface $\boldsymbol{r} = \{u\cos v, u\sin v, v\}$ defined by $u = \sinh t, v = \theta$ is an isometry.

Theorem 2. On a positive spiral surface $\boldsymbol{r} = \{u\cos v, u\sin v, v\}$, the coordinate is a semi-geodesic coordinate net. Every coordinate curve parameterized by u is a geodesic, and each coordinate curve parameterized by v has constant geodesic curvature $k_g = \dfrac{u}{u^2+1}$.

Proof. Through the isometry defined in Lemma 2, the curve $\theta \to \{\cosh t \cos\theta, \cosh t \sin\theta, t\}$ is mapped to a curve $v \to \{\sinh t \cos v, \sinh t \sin v, v\}$. Hence, its geodesic curvature is $\sinh t / \cosh^2 t = u/(u^2+1)$.

The following is a figure of the isometric transformation of the semi-geodetic coordinate net on a catenoid to that on a positive spiral surface. Red curves have non-zero constant geodesic curvature, and blue curves are geodesics.

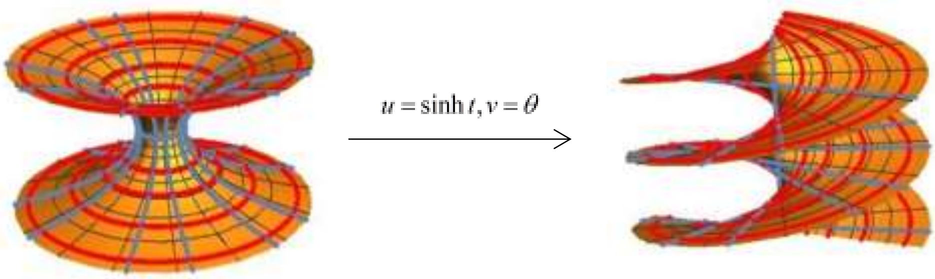

Fig. 2 Isometric transformation of a semi-geodetic coordinate net on a catenoid to that on a positive spiral surface

Example 4. We take a rotational hyperbolic surface $S = \{(x, y, z) | x^2 + y^2 - z^2 = 1, z > 0\}$. Its coordinate $\boldsymbol{r} = \{\cosh t \cos\theta, \cosh t \sin\theta, \sinh t\}$ is a semi-geodesic coordinate net by Theorem 1. All coordinate curves parameterized by t are geodesics, and each coordinate curve parameterized by θ has constant geodesic curvature $\sinh t / \left(\cosh t \sqrt{2\sinh^2 t + 1}\right)$.

By using the above semi-geodesic coordinate net on a rotational hyperbolic surface, we give a semi-geodesic coordinate net on a spiral surface.

Lemma 3[7]**.** The map of an open part of a rotational hyperbolic surface $\boldsymbol{r}=\{\cosh t\cos\theta,\cosh t\sin\theta,\sinh t\}\left(t>0,0\le\theta<2\pi\right)$ to a spiral surface $\boldsymbol{r}=\{u\cos v,u\sin v,u+v\}$ defined by $u=\sinh t, v=\theta-\arctan(\sinh t)$ is an isometry.

Theorem 3. The coordinate net

$\boldsymbol{r}=\{u\cos(s-\arctan u),u\sin(s-\arctan u),s+u-\arctan u\}$

on a spiral surface is a semi-geodesic coordinate net. Every coordinate curve parameterized by u is a geodesic, and each coordinate curve parameterized by s has constant geodesic curvature $u\big/\sqrt{(u^2+1)(2u^2+1)}$.

Proof. Through the isometry defined in Lemma 3, the curve $\theta\to\{\cosh t\cos\theta,\cosh t\sin\theta,\sinh t\}$ is mapped to a curve

$s\to\{\sinh t\cos(s-\arctan(\sinh t)),\sinh t\sin(s-\arctan(\sinh t)),s+\sinh t-\arctan(\sinh t)\}.$

Its geodesic curvature is $\sinh t\big/\cosh t\sqrt{2\sinh^2 t+1}=u\big/\sqrt{(u^2+1)(2u^2+1)}$.

The following is a figure of the isometric transformation of the semi-geodetic coordinate net on a rotational hyperbolic surface to a spiral surface. Red curves have non-zero constant geodesic curvature, and blue curves are geodesics.

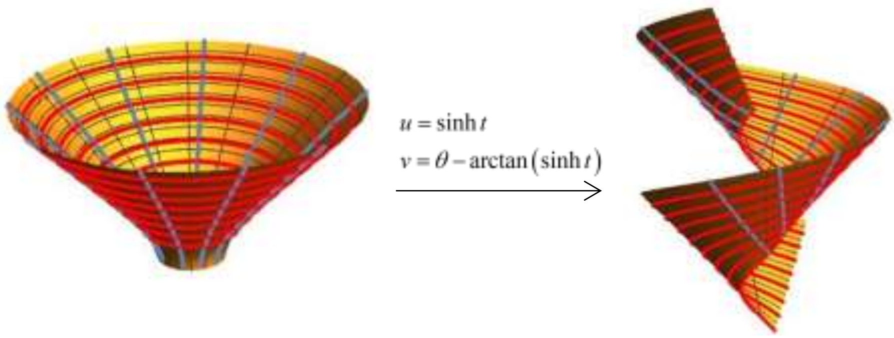

Fig. 3 Isometric transformation of a semi-geodetic coordinate net on a rotational hyperbolic surface to that on a spiral surface

Our technique guarantees that we can get many other semi-geodesic coordinate nets on some surfaces formed by geodesics and curves of non-zero constant geodesic curvature through isometries of rotating surfaces.

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References

[1] T. Hoffmann, W.K. Schief and J. Steinmeier, On Discrete Conjugate Semi-Geodesic Nets, *International Mathematics Research Notices*, **11** (2022), 8685-8739. https://doi.org/10.1093/imrn/rnaa394

[2] Yuying Wu, Tuya Bao and Xiaozhou Liu, Geodesic in the Perspective of Isometric Correspondence, to appear in *Journal of College Mathematics*.

[3] Tianchang Xu, On the Egualling Correspondence of Two Curved Surfaces, *Journal of Anqing Teachers College (Natural Science Edition)*, **4** (2005), 101-102.

[4] Xiangming Mei, Jingzhi Huang, *Differential Geometry*, 4th edition, Beijing Higher Education Press, 2008.

[5] Benlu Fu, Guomin Wang, The Semi-geodesic Coordinates Net and Semi-Chebyshev Net, *Journal of LiaoNing University (Natural Science Edition)*, **03** (1999), 216-218.

[6] Jinfang Rong, *Analysis and Extraction of Geodesic Curve in Curved Surface*, Northeastern University, 2010, 19-22.

[7] Weihuan Chen, *Differential Geometry*, Peking University Press, 2006, 111-241.

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