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Circulant Hadamard Matrices and Hermitian Circulant Complex Hadamard Matrices

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Abstract

We prove that there exists a circulant Hadamard matrix of order n if and only if there exists a Hermitian circulant complex Hadamard matrix of order n and that there does not exist a Butson-type Hermitian circulant complex Hadamard matrix of order $n > 4$.

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1 Introduction

A Hadamard matrix H_n of order n is an $n \times n$ matrix with entries ± 1 such that $H_n {H_n}^T = nI_n$, where ${H_n}^T$ is the transpose of H_n and I_n is the identity matrix of order n. A complex Hadamard matrix K_n of order n is an $n \times n$ matrix whose entries are on the complex unit circle and which satisfies $K_n {K_n}^* = nI_n$, where ${K_n}^*$ is the conjugate transpose of K_n. In particular, a complex Hadamard matrix whose entries are q-th roots of unity is called a q-Butson Hadamard matrix.

In [5] Ryser conjectured that there is no circulant Hadamard matrix of order $n > 4$. Turyn [6] proved that, if there exists a circulant Hadamard matrix of order n, then n must be the form $n = 4m^2$ for some odd integer m. Brualdi [1] proved that there is no symmetric circulant Hadamard matrix of order $n > 4$. Craigen and Kharaghani [2] generalized Brualdi's theorem to Hermitian circulant 4-Butson Hadamard matrices.

In this paper we prove that there exists a circulant Hadamard matrix of order n if and only if there exists a Hermitian circulant complex Hadamard matrix of order n and generalize Craigen and Kharaghani's theorem to Hermitian circulant q-Butson Hadamard matrices for $q \geq 2$.

2 Equivalency

Denote by $\mathrm{circ}(a_0, \ldots, a_{n-1})$ the circulant matrix whose first row is $(a_0 \ldots a_{n-1})$, by $\mathrm{diag}(d_0, \ldots, d_{n-1})$ the diagonal matrix whose $i+1$-th diagonal element is d_i, and by F_n the $n \times n$ Fourier matrix whose $(i+1, j+1)$-th entry is ${\omega_n}^{ij}$, where $\omega_n = e^{2\pi\sqrt{-1}/n}$. Note that $\mathrm{circ}(a_0, a_1, \ldots, a_{n-1})^T = \mathrm{circ}(a_0, a_{n-1}, \ldots, a_1)$. It is well-known (e.g. [3]) that a circulant matrix $C_n = \mathrm{circ}(a_0, \ldots, a_{n-1})$ can be expressed as

$$\begin{aligned} C_n &= F_n \mathrm{diag}(d_0, \ldots, d_{n-1}) {F_n}^{-1} \\ &= \frac{1}{n} \mathrm{circ} \left(\sum_{k=0}^{n-1} d_k {\omega_n}^{-0k}, \sum_{k=0}^{n-1} d_k {\omega_n}^{-1k}, \ldots, \sum_{k=0}^{n-1} d_k {\omega_n}^{-(n-1)k} \right), \end{aligned}$$

where $d_i = \sum_{k=0}^{n-1} a_k {\omega_n}^{ik}$ for $0 \leq i \leq n-1$. We shall require the following lemma.

Lemma 2.1 ([4, Lemma 2.2])**.** *A circulant matrix C_n is a real orthogonal matrix if and only if $d_0, \ldots, d_{n-1}$ satisfy the following conditions:*

1. ${d_0}^2 = 1$ *and* $d_i d_{n-i} = 1$ *for* $1 \leq i \leq n-1$.
2. $d_i = \overline{d_{n-i}}$ *for* $1 \leq i \leq n-1$.

There exist the following relationships between circulant Hadamard matrices and Hermitian circulant complex Hadamard matrices.

Lemma 2.2. *Let H_n be a circulant Hadamard matrix of order n and let $n^{-1/2} H_n = F_n \mathrm{diag}(d_0, \ldots, d_{n-1}) {F_n}^{-1}$. Then $\mathrm{circ}(d_0, d_{n-1}, \ldots, d_1)$ is a Hermitian circulant complex Hadamard matrix.*

Proof. By Lemma 2.1, it holds that $d_i \overline{d_i} = 1$ for $0 \leq i \leq n-1$ and $d_i = \overline{d_{n-i}}$ for $1 \leq i \leq n-1$. Write $K_n = \mathrm{circ}(d_0, d_{n-1}, \ldots, d_1)$ and $f(x) = d_0 + d_{n-1} x + \cdots + d_1 x^{n-1}$. Then we have ${K_n}^* = \mathrm{circ}(\overline{d_0}, \overline{d_1} \ldots, \overline{d_{n-1}}) = K_n$. Since the entries of $n^{-1/2} H_n$ are $\pm n^{-1/2}$, namely,

$$\frac{1}{n} \sum_{k=0}^{n-1} d_k {\omega_n}^{-ik} = \frac{f({\omega_n}^i)}{n} = \pm \frac{1}{n^{1/2}}$$

for $0 \le i \le n-1$, we have

$$\begin{aligned} {F_n}^{-1} K_n {K_n}^* F_n &= {F_n}^{-1} K_n K_n F_n = {F_n}^{-1} \mathrm{circ}(d_0, d_{n-1}, \ldots, d_1)^2 F_n \\ &= \mathrm{diag}\left(f({\omega_n}^0), \ldots, f({\omega_n}^{n-1})\right)^2 = nI_n. \end{aligned}$$

Thus K_n is a Hermitian circulant complex Hadamard matrix. $\square$

Lemma 2.3. *Let K_n be a Hermitian circulant complex Hadamard matrix of order n and let $n^{-1/2} K_n = F_n \mathrm{diag}(d_0, \ldots, d_{n-1}) {F_n}^{-1}$. Then $\mathrm{circ}(d_0, d_{n-1}, \ldots, d_1)$ is a circulant Hadamard matrix.*

Proof. Since

$${F_n}^{-1} K_n {K_n}^* F_n = {F_n}^{-1} K_n K_n F_n = n\ \mathrm{diag}(d_0, \ldots, d_{n-1})^2 = nI_n,$$

it holds that ${d_i}^2 = 1$ for $0 \le i \le n-1$. Write $H_n = \mathrm{circ}(d_0, d_{n-1}, \ldots, d_1)$, $g(x) = d_0 + d_{n-1}x + \cdots + d_1 x^{n-1}$, and $h(x) = d_0 + d_1 x + \cdots + d_{n-1} x^{n-1}$. Since the absolute values of entries of $n^{-1/2} K_n$ are $n^{-1/2}$, namely,

$$\frac{1}{n^2} \left(\sum_{k=0}^{n-1} d_k {\omega_n}^{-ik} \right) \overline{\left(\sum_{k=0}^{n-1} d_k {\omega_n}^{-ik} \right)} = \frac{g({\omega_n}^i)\overline{g({\omega_n}^i)}}{n^2} = \frac{g({\omega_n}^i) h({\omega_n}^i)}{n^2} = \frac{1}{n}$$

for $0 \le i \le n-1$, we have

$$\begin{aligned} {F_n}^{-1} H_n {H_n}^T F_n &= \mathrm{diag}\left(g({\omega_n}^0), \ldots, g({\omega_n}^{n-1})\right) \mathrm{diag}\left(h({\omega_n}^0), \ldots, h({\omega_n}^{n-1})\right) \\ &= nI_n. \end{aligned}$$

Thus H_n is a circulant Hadamard matrix. $\square$

From Lemmas 2.2 and 2.3, we obtain immediately the following theorem.

Theorem 2.4. *There exists a circulant Hadamard matrix of order n if and only if there exists a Hermitian circulant complex Hadamard matrix of order n.*

3 Hermitian circulant q-Butson Hadamard matrices

Craigen and Kharaghani [2, Lemma 4] proved the following result.

Lemma 3.1 (Craigen-Kharaghani)**.** *Let H_n be a circulant Hadamard matrix of order n satisfying ${H_n}^m = n^{m/2} I_n$ for some $m > 0$. Then $n \le 4$.*

Using this result, we prove nonexistence of Hermitian circulant q-Butson Hadamard matrices.

Theorem 3.2. *Let $q \geq 2$ and $n > 4$. Then there is no Hermitian circulant q-Butson Hadamard matrix of order n.*

Proof. Suppose that K_n is a Hermitian circulant q-Butson Hadamard matrix matrix of order $n > 4$. Let $n^{-1/2}K_n = F_n \mathrm{diag}(d_0, \ldots, d_{n-1}){F_n}^{-1}$. By Lemma 2.3, $H_n = \mathrm{circ}(d_0, d_{n-1}, \ldots, d_1)$ is a circulant Hadamard matrix. Since the entries of K_n are q-th roots of unity, namely,

$$\frac{1}{n^{q/2}} \left(\sum_{k=0}^{n-1} d_k {\omega_n}^{-ik} \right)^q = 1$$

for $0 \leq i \leq n-1$, we have

$${F_n}^{-1} {H_n}^q F_n = \mathrm{diag}\left(g({\omega_n}^0), \ldots, g({\omega_n}^{n-1}) \right)^q = n^{q/2} I_n.$$

However, this contradicts Lemma 3.1. □

References

[1] R.A. Brualdi, A note on multipliers of difference sets, *J. Res. Nat. Bur. Standards Sect.B Math. and Math. Phys.,* **69** (1965), 87 - 89. https://doi.org/10.6028/jres.069b.008

[2] R. Craigen and H. Kharaghani, On the nonexistence of Hermitian circulant complex Hadamard matrices, *Australas. J. Combin.,* **7** (1993), 225 - 227.

[3] P.J. Davis, *Circulant matrices,* Chelsea, New York, 1994.

[4] N. Matsuki, A note on symmetric orthogonal circulant matrices, *Int. Math. Forum,* **14** (2019), 263 - 266. https://doi.org/10.12988/imf.2019.91141

[5] H.R. Ryser, *Combinatorial mathematics,* Wiley, New York, 1963. https://doi.org/10.5948/upo9781614440147

[6] R. Turyn, Character sums and difference sets, *Pacific J. Math.,* **15** (1965), 319 - 346. https://doi.org/10.2140/pjm.1965.15.319

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