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Mathematical Magnitudes

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Abstract

It is shown how the strictly determined countable infinite cardinality aleph-null $\aleph_0$ of the countable infinite set of the natural numbers **N** is used as unit of measurement for determined the cardinality of the different countable infinite sets. This enable to be proved that the cardinality $\aleph_q$ of the set of the rational numbers **Q** is bigger than the cardinality $\aleph_0$ of the set of the natural numbers **N**.

Mathematics Subject Classification: 03Exx, 03E47

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1. Introduction

There are three basic kinds number sets: a) the enumerable finite subsets of the countable infinite set of the natural numbers **N**; b) the countable infinite set of the natural numbers **N** and its countable infinite proper subsets, as well as the countable infinite set of the rational numbers **Q** and its countable infinite proper subsets; c) the one-dimensional uncountable infinite set of the real numbers **R** and its uncountable infinite proper subsets, as well as the next two-dimensional, three-dimensional and in general with a finite number of dimensions uncountable infinite sets and their respective uncountable infinite proper subsets. Similar to the physical magnitudes the number sets are mathematical magnitudes, which have size or theirs measure, called cardinality. Bernard Bolzano measured the cardinality of the uncountable infinite sets – see "Paradoxes of the Infinite" [1]. However and up to now there are not well established preliminary chosen units of measurements for determining the cardinality of the three basic kinds mathematical magnitudes.

2. Units of measurements

A set may be defined as a mentally created unity which contains well determined and distinct one from other things, called elements of the set. Main property of such standardly defined set is the containing of well determined and distinct one from other elements, which belong to the set according to a chosen by thinker their property. Since every kind of infinity represents a concrete determinateness which is idealized as unlimited, and the set **N** is based on all of the kinds of the simplest idealized quantitative regularities, therefore the properties of the standardly defined sets and the properties of their elements are revealed most easily when the elements of the set are countal (cardinal) natural numbers, such are the elements of the set **N**. (The set of ordinal natural numbers $\mathbf{N}_{\boldsymbol{\alpha}}$ is other kind number set.) Utmost general characteristic of countal number set X is the quantity of elements belonging to it, called cardinality of the set and marked with $|X|$. The set X is countable and infinite when at its juxtaposition with the countable infinite set **N** each element from the quantity of elements of the set X can be paired with exactly one element from the determined by a regularity infinite quantity of elements of the set **N**. The set X is enumerable and finite when at pairing of its quantity of elements through the countal unit 1 with the determined by a regularity infinite quantity of elements of the set **N**, the quantity of elements of the set X is exhausted up to a finite countal number n from the infinite quantity of elements of the set **N**. We will call the cardinality of such enumerable finite set enumerable finite cardinality n.

For creating of enumerable finite number sets it is not necessary to be used regular connection among the elements of this kind of sets. At juxtaposition between two such a sets with equal cardinalities, the harmonious combination of their elements, called bijection, can equivalently be realised unambigasly and reversibly one-to-one (1–1) by the every possible different ways in the case (usually many more than two).

Whereas the creating of countable infinite number set is impossible without the using of strictly determined regular connection among its elements, which regular connection is idealized as unlimited. Therefore, at the juxtaposition between two countable infinite sets the harmonious combination of their elements can be realized only through two essentially different ways: a) by an incompatible with their regularities unambiguous and reversible (incongruous mutually countable) correspondence between their elements – see Fig. 5, and b) by a conforming with their regularities unambiguous and reversible (congruous mutually countable) correspondence between their elements – see Fig. 1, Fig. 2 and Fig. 3.

On the one hand the infinite set **N** of the natural numbers is in incongruous mutually countable correspondence with every one of its infinite proper subsets, as they also are among each other on account of their infinity. That's why the incongruous mutually countable correspondence between their elements cannot be a criterion for the quantity of their elements. On the other hand their quantities of elements most often are essentially different due to the differences among the kinds of regularities, which determine the different kinds of countable infinite sets. For example, without doubt the infinite quantity of the even numbers represents exactly half from the entire infinite quantity of the elements $\aleph_0$ of the set **N**, whereas the other half of its infinite quantity of elements is exactly represented by the infinite quantity of its odd numbers. Note that at the so done estimating for the relation between the cardinalities of these countable infinite sets, one uses not the countal unit 1 for enumerable finite cardinality *n*, which unit of measurement is inapplicable in the case, but the countable infinite cardinality $\aleph_0$ as a unit of measurement for cardinality of countable infinite set. Therefore, at the juxtaposition between two such sets the relation between their countable infinite cardinalities must be determined not by the indistinguishing them incongruous mutually countable correspondence between their elements, but by the distinguishing them congruous mutually countable correspondence. Because of only uniform quantities may be reasonably compared and their uniformity at this juncture is determined from the sameness of the kind of regularity. In the general case, the greater of the two sets does not take part in the correspondence with all of its elements.

When the cardinalities of two countable infinite number sets are essentially different, as for example the cardinality $\aleph_0$ of the set **N** and the cardinality a_s of the set of the even numbers, then the harmonious combination between their elements in the case can be done congruously in three different ways, as described below:

a) Injection first at $a_s < \aleph_0$ – see Fig. 1. When the determined by the regularity evenness smaller infinite quantity of elements of the set of the even numbers are put in congruous mutually countable one-to-one (1–1) correspondence with the determined by the same regularity infinite quantity of elements of the proper subset of the even numbers of the set **N**, at which in the case, the odd numbers from the set with bigger quantity of elements remain without correspondence to the elements of the set with the smaller quantity of elements.

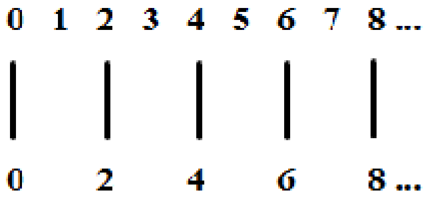

Fig. 1. Illustration of the congruous mutually countable injection first.

b) Injection second at $a_s < \aleph_0$ – see Fig. 2. When the determined by the regularity evenness smaller infinite quantity of elements of the set of the even numbers are put in congruous mutually countable one-to-one (1–1) correspondence with the determined by the analogical regularity infinite quantity of elements of the proper subset of the odd numbers of the set **N**, at which in the case, the even numbers from the set with bigger quantity of elements remain without correspondence to the elements of the set with the smaller quantity of elements.

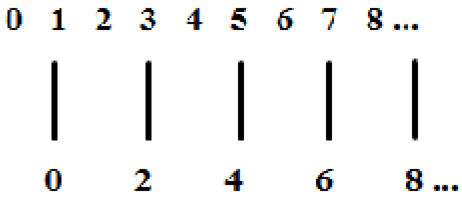

Fig. 2. Illustration of the congruous mutually countable injection second.

c) Surjection at $\aleph_0 > a_s$ – see Fig. 3. When more than one elements of the set with the bigger quantity of elements are put in a congruous, in this case two-to-one (2–1) (responding to the kind of regularities mutually countable) correspondence with every one of the elements of the set with the smaller quantity of elements, at which circumstance there are no remaining without correspondence elements of the set with the bigger quantity of them.

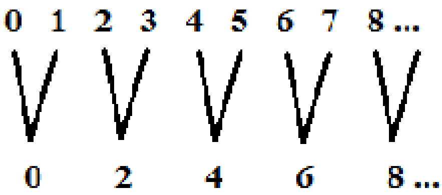

Fig. 3. Illustration of congruous mutually countable surjection.

When the cardinalities of the two countable infinite number sets are the same, as in the case of the cardinality a_{s1} of the set of the even numbers and the cardinality a_{s2} of the set of the odd numbers, then the harmonious combination between their elements can be done only in one way, to call it analogically:

d) Bijection at $a_{s1} = a_{s2}$ – see Fig. 4. When the infinite quantities of elements of the two sets are put in congruous mutually countable one-to-one (1–1) correspondence without remainder and without surjection, with which the equivalence of the cardinalities of two countable infinite sets is proved, similar to the bijection at the proving the equivalence of the cardinalities of two enumerable finite sets.

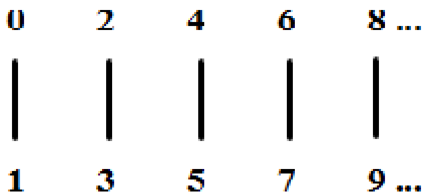

Fig. 4. Illustration of bijection at countable infinite sets.

For a more exact finding of the relation between the cardinality $\aleph_0$ of the set **N** and the cardinality of any of its infinite proper subsets, as well as the relation between the cardinalities of any two of its infinite proper subsets, we should use the foreseen by Bolzano relation between the respective sums of all of the terms, which are in the scope of a determined distance from the beginning of the sequence of the natural numbers. For example, the relation between the cardinality $\aleph_0$ of the set **N** and the cardinality of the set of squares of natural numbers is found correctly and more and more exactly by the relations of the successively determined respective sums: 1 to 1; 1 + 2 + 3 + 4 = 10 to 5 = 1 + 4; 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45 to 14 = 1 + 4 + 9; and so on. In the case these sums are determined for the distances of which are located several consecutive initial terms of the sequence of the squares of the natural numbers. Thus on the one hand we obtain the sum, presented only by the terms of the sequence of the squares of the natural numbers for a determined distance from the beginning of the sequence of the natural numbers. Whereas on the other hand to the same sum is added the sum, presented by the natural numbers, which are not exact squares of such numbers for the same distance. The infinite set of the numbers, which are not exact squares, increases in progressing arithmetic progression, which after the first step ($2^2 = 4$), consists of the numbers 2 and 3, and with every next step of the sequence of the squares of the natural numbers, the number of the added such numbers is increased by two, while the numbers in themselves become bigger and bigger.

Bolzano's mentioned work shows that the relation between the cardinalities of the proper subsets of the uncountable infinite one-dimensional, two-dimensional and three-dimensional number sets is determined by the relation between the sizes of lengths, areas, and volumes, which they cover respectively over, on or in the proportionate to them extent: line, surface or three-dimensional space. At that each of them is comparable according to cardinality only with a cardinality of a set with dimension as is its. Therefore, the Archimedean property for comparability between the sizes of two such mathematical magnitudes, as are the sets, is valid only at availability of uniform unit of measurement for the quantities of elements and of the two magnitudes. In conjunction with this we must distinguish the unambiguous and reversible correspondence between the sizes of the different uncountable infinite sets, such as are: the set of points in the interval from 0 to 1 and the set of points over an infinite number line or the set of points in some other finite interval of it; the set of points over an infinite number line and the set of points on an infinite surface; the set of points on an infinite surface and the set of points in an infinite three-dimensional space; and as well as at the remaining cases for every space with more finite dimensions.

From the considered examples with countable infinite number sets it follows that their initial comparing according to cardinality is based on the kind of regularity, which determines them in relation to the chosen for unit of measurement countable infinite cardinality $\aleph_0$. The countable infinite cardinality $\aleph_0$ is incomparable with an enumerable finite cardinality n of any enumerable finite set, similar of the incomparable with a finite sum $1 + 1 + 1 + \ldots < \omega$ hyperreal numbers in contemporary non-standard analysis. Generally because the essential difference among the properties of the enumerable finite sets, of the countable infinite sets and of the uncountable infinite sets, their cardinalities are measured respectively by incomparable with each other units of measurements, namely: with the countal unit 1, with the countable infinite cardinality $\aleph_0$, and, with the uncountable infinite cardinality of the chosen for unit of measurement with a respective dimention spatial extent.

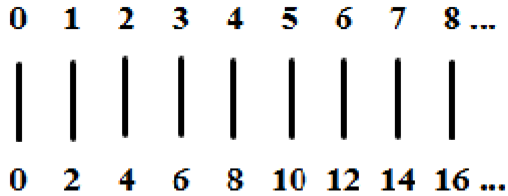

Fig. 5. Illustration of incongruous mutually countable correspondence.

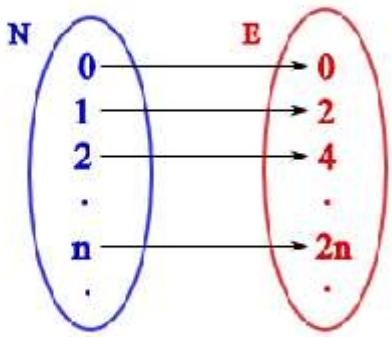

Fig. 6. Illustration of incorrectly understood bijection.

Of Fig. 5 is shown the incongruous mutually countable correspondence between the elements of the whole set of the natural numbers **N** with the elements of its proper subset of the even numbers. Of Fig. 6 the same correspondence is illustrated as a unilateral pairing of the elements of the whole set **N** with the elements of its proper subset **E**. The presented of Fig. 6 incongruous mutually countable correspondence between the elements of the sets **N** and **E** is now given as an example for bijection, that is, as a criterion for equivalence between infinite quantities of their elements. However the definition for proper subset states that all elements of the proper subset X should belong to the whole set Y and the proper subset X should be different from the whole set Y, i.e. $(X \subset Y) \Leftrightarrow (\forall x(x \in X \rightarrow x \in Y)) \wedge (X \neq Y)$. From this it follows, that the proper subset X must have smaller quantity of elements than the quantity of elements of the whole set Y. In this instance the subset **E** of the even numbers should have a smaller quantity of elements than the whole quantity of elements of the set **N**. In this way one reaches to the obvious contradiction $(x \wedge \neg x)$ that the sets **N** and **E** now have an equal quantity of elements, now that one of them has smaller quantity of elements than the other. That is why it is incorrectly to accept for bijection the depicted of Fig. 6 incongruous mutually countable correspondence between the elements of the sets **N** and **E**. Besides, above already is shown a definition for bijection at the countable infinite sets which is similar to the definition for bijection at the enumerable finite sets. Because of the indicated existing hitherto absurdity it is incorrectly considered, that each one of the countable infinite proper subsets of the set **N** has the same cardinality as **N**. From here also follows the incorrect definition: one set is infinite *if it has a proper subset with the same cardinality*, instead of the correct definition: if it is in incongruous mutually countable correspondence with a proper subset.

Paradoxically in the case is how it is possible up to now to be neglected the congruous mutually countable correspondence at determining the cardinality of the countable infinite sets, even after strictly determining of the countable infinite cardinality $\aleph_0$ of the set of the countable infinite natural numbers **N**. For the set **N** consists of an infinite many countable infinite proper subsets, which because of the determining them different kinds of regularities may contain as many as we want of more and more smaller part of the countable infinite cardinality $\aleph_0$ of the set **N**. An example for such countable infinite proper subsets of the set **N** are the subsets being presented by the sequences with gradually decreasing countable infinite cardinality:

a_1) 1, 2, 3, …, *n*, … $\aleph_0$
a_2) 2, 4, 6, …, 2*n*, … $\aleph_0/2$
a_3) 3, 6, 9, …, 3*n*, … $\aleph_0/3$
----
a_k) k, k2, k3, …, k*n*, … $\aleph_0/k$
----

where k is a natural number bigger than unit.

With using of $\aleph_0$ as a unit of measurement for countable infinite cardinality of the countable infinite sets, now we can without a problem determine as the faster decreasing countable infinite cardinality of the sets, being presented by the sequences with equal powers of the natural numbers, at the infinite succession of their increasing powers:

a) $1^2, 2^2, 3^2, \ldots, n^2, \ldots$
b) $1^3, 2^3, 3^3, \ldots, n^3, \ldots$
c) $1^4, 2^4, 3^4, \ldots, n^4, \ldots$
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so and the more faster decreasing countable infinite cardinality of the sets, being presented by the sequences with increasing powers of the prime numbers, at the infinite succession of these numbers:

d) $2^1, 2^2, 2^3, \ldots, 2^n, \ldots$
e) $3^1, 3^2, 3^3, \ldots, 3^n, \ldots$
f) $5^1, 5^2, 5^3, \ldots, 5^n, \ldots$

Contemporary factual logic indisputably establishes that at every finding of size x*u* of determined quantity $y = \mathrm{x}u$ of physical magnitude *Y*, where x is measure number, and *u* is a preliminary chosen for unit of measurement other determined quantity of the same magnitude *Y*, the determinateness of the measure number x always is limited. In physics this limitedness is expressed by the finite number *m* of its reliably determined digits, which begin with its first most reliably determined different from zero digit and end with its last reliably-enough determined digit, called the significant digits of the measure number. Idealizing as unlimited the processes with numbers increases the quantity of their determinable digits. The process of division, for example, finishes in some cases without remainder after a finite number of steps, at which the obtained number is presented with a finite quantity of determinable digits. In other cases, however, this process cannot be finished due to obtaining a remainder which is periodically reiterating. Therefore the obtained number is presented with its unlimited prolongation of infinite quantity of predictably distributed determinable digits of this remainder. We can call *completed* the first kind rational numbers as distinct from the second kind of *uncompleted* rational numbers. At many other kinds of processing with numbers, as for example at some root extractions, are obtained the so-called *irrational* numbers with infinite quantity of unpredictably distributed determinable digits. The first big shock in formal logic happens at the discovery of the incommensurability of the diagonal of the ideal square with its side, which is an example of an irrational number.

The infinite set **Q** of the rational numbers is presented in a positional numeral system by two qualitatively different infinite proper subsets: a) by a proper subset q_c of the completed numbers, which is countable and with finite quantity of predictably distributed digits of this numbers due to the regularity of successive alternation of a finite quantity of distinct one from other digit marks when adding a unit to every such preceding number, as is and at the countable infinite set **N**, and b) by a proper subset q_u of the uncompleted numbers, which is countable and with infinite quantity of predictably distributed digits of these numbers, which are formed of the eventualy unperiodically reiterating digits of these numbers unlimited prolongated with their periodically reiterating remainders. Therefore the infinite set **Q** of the rational numbers as a whole is also countable. However between the infinite quantity of elements of the set **N** and the infinite quantity of elements of the proper subset q_c of the completed rational numbers of the set **Q** there is a determined by the same regularity congruous mutually countable one-to-one (1–1) correspondence, whereas the elements of the proper subset q_u of the uncompleted rational numbers of the set **Q** remain without such a correspondence with the elements of the set **N**. Therefore the infinite cardinality $\aleph_q$ of the set **Q**, as a sum of the infinite cardinalities of the two kinds of its countable infinite proper subsets, is bigger than the infinite cardinality $\aleph_0$ of the set **N**. The very proper subset q_u of the uncompleted numbers of the set **Q** is an infinite set composed by the all sorts of terms of the infinite set of its proper subset q_c of completed numbers, every one of which is consecutively reproduced with the unlimited prolongations of all sorts of the periodically reiterating remainders. With this in mind the set **Q** represents a countable infinite set with the possibly greatest countable infinite cardinality $\aleph_q$. Cantor incorrectly accepts the incongruous mutual countability as a criterion for the same cardinality at the countable infinite sets and presents uniformly the two kinds subsets of the rational numbers through the relation $\frac{a}{b}$, in contrast to their evident presentation as different kinds in a positional numeral system. Thus, by the proved by him incongruous mutual countability of the set **Q** of the rational numbers with the set **N** of the natural numbers, is concealed the bigger countable infinite cardinality $\aleph_q$ of the set **Q** than the countable infinite cardinality $\aleph_0$ of the set **N**.

3. Inference

The finding of $\aleph_0$ as natural unit of measure for cardinality of the countable infinite countal sets open the way for unambiguous solving of the continuum hypothesis, shown in the paper “The logical paradoxes” [2].

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