International Mathematical Forum, Vol. 16, 2021, no. 4, 171 - 178
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2021.912297

New Explanation of Two Paradoxes

Hristo Manev

University of Architecture, Civil Engineering and Geodesy
bul. Hristo Smirnenski No. 1 Sofia 1164, Bulgaria

This article is distributed under the Creative Commons by-nc-nd Attribution License.
Copyright © 2021 Hikari Ltd.

Abstract

The notion of a standard defined set is introduced on the basis of which the reasons for the emergence of Russell's paradox are explained.

The current description of Hilbert's paradox has been compared with two new descriptions made on substantially different other grounds.

Mathematics Subject Classifications: 03Bxx; 03B99

Keywords: Set, Contain, Belong, Cardinality of set

Russell’s paradox

The notion set in mathematics is now defined as follows:

1. In Bulgarian: **Множеството** представлява съвкупност от различни обекти, наричани още *елементи*, която се разглежда като едно цяло. (**The set** represents aggregate of distinct objects, also called *elements*, which is considered as one whole.)

2. In Russian: **Множество** это математический объект, сам являющийся набором, **совкупностью**, собранием каких-либо объектов, которые называются **елементами** этого множества и обладают общим для всех их характеристическим свойством. (**The set** is mathematical object, which itself is a composition, **an aggregate**, a collection of any objects, that are called **elements** of this set and have a common characteristic property for all of them.)

3. In English: A **set** is a well-defined collection of distinct objects, considered as an object in its own right.

4. In German: Eine **Menge** is ein Verbund, eine Zusammenfassung von einzelnen Elementen. (A **set** is an association, a combination of individual elements.)

5. In French: En mathématiques, un **ensemble** désigne *intuitivement* une collection d'objets (les éléments de l'ensemble), "une multitude qui peut être comprise comme un tout" (au sens d'*omnis*). (In mathematics, a **wholeness** *intuitively* determined a collection of objects (the elements of the wholeness), "a multitude that can be understood as a totality" (in a sense of *omnis*).)

6. Cantor's definition: A set is a gathering together into a whole of definite, distinct objects of our perception or our thought — which are called elements of the set.

The widely used word *multitude* usually serves to denote the unity of significantly more than one different things, established on some attribute, paying attention to the attribute that unites these things without determining their quantity. The cited definitions try to formulate a more strictly defined notion of **set** as a designation of the abstract unity itself of more than one different things. An important shortcoming of these definitions is that they do not clearly emphasize the essential qualities that determine this unity, at which behind the word "objects" they conceal the subjective basis of the unity in question. Only Georg Cantor notes the subjective nature of such unity. The next improvement in this direction requires the introduction of a notion of **a standard defined set** as *a mentally created unity which contains well determined and different from each other things, called elements of the set*. Due to the finite determinateness of man, our thinking can begin any definition with only finite things. Because of that, the set thus defined is finite, based on a combination of the qualities embedded in it: **being contained something, belonging to something, being things which are different from each other**. The creation of any unity consists in determining a new whole, which is differentiated as self-sufficiently for self-existence not until after the completion of a certain stage of this process. The improved definition represents the first completed stage of the formation of a finite set as a unity of only two or more different from each other things. Between these things there cannot be only their unity as an element of this unity because of the finite speed of flowing of the thinking which creates it. Not until after the creation of the unity of the different from each other things can the primary standard defined finite set make its own element and this unity of its own. With this act, however, the primary standard defined finite set changes becoming another finite set with more content than the content of any one of its elements. Because of that, such a self-related finite set again does not contain itself, because it is already different from its new element, which is not a self-related set and has less content than the content of the self-related set. Because of that, it is not possible for a finite set to contain and itself as its element by making of itself one's own element. Because of that, the formation of a set of all finite sets leads to a vicious logical circle.

To the hitherto unawareness of the reasons for the emergence of a vicious logical circle at the set of all finite sets are added the problems from Cantor's attempt to increase the quantity of elements of a finite set unlimited. To justify such an increase, he derives the formula $2^n > n$, where n is the quantity of elements of a finite set and 2^n is the quantity of subsets of that set. In deriving this formula, the notion of a finite set extends and for the cases when the set has only one element, as well as when it has no any element – when it is an empty set, i.e. when it no longer has plurality in the proper sense of the word. Since it is assumed that the empty set is a subset of each finite set, then the set of one element together with the empty set in it can be considered as a standard defined finite set. But the set of only one empty set cannot be considered as a standard defined finite set, because it cannot concurrently contain itself and belong to itself, neither to be different from itself as an element of itself after being only one an empty set. As above it is shown and the last two kinds of primarily defined finite sets (by extending the notion of finite set) cannot be made to contain themselves as one's own element by making of themselves one's own element, i.e. by making themselves self-related. Due to the above circumstances, each of the three kinds of primarily defined sets does not contain itself as it one’s own element. In turn, the self-related sets are only secondarily obtained finite sets of a corresponding kind of primary defined set, whereby the secondarily obtained finite sets also cannot contain themselves as their element. In addition to the problems due to the extending of the notion of a finite set addressed here, the unlimited increase of the quantity of elements of a finite set leads also to the continuum hypothesis — see the article [2].

In all existing hitherto explanations of Russell's paradox, sets are divided into such, each of which does not have itself as its element, and such, each of which has itself as its element; of "ordinary" and "unusual"; of "normal" and "abnormal". As an example of a set which is not an element also to the in itself, a set of people is given, which in itself does not represent one person, at which the words denoting the sets in question with different contents are different; while as an example of a set which has and the itself as an element, a catalog of other catalogs is given, which is also a catalog, at which the word denoting the sets in question with different contents is the same as is and in the more abstract example with the set of all finite sets. It is very good when the meaning of a word corresponds as much as possible to some quality of something denoted by it. However, denoting of essentially different things with the same word cannot make them one and the same thing that is expected to be in the examples given. Therefore, the way in which the division of finite sets into such two types was announced is unfounded. The important thing is that the primarily defined sets have the quality of non-content and of themselves $R \notin R$, and the quality of content and of themselves $R \in R$ is unjustifiably attributed to the secondary obtained sets, because in fact they also do not contain themselves.

As early as the fourth century BC, the ancient Greeks pondered over the difference between the properties of one grain and the properties of a heap of such grains. In the nineteenth century, in analyzing the problem of the continuity of the set of real numbers, Bernard Bolzano came to the conclusion that the continuum can be explained only by accepting the fact that "… every whole has, and must have, many properties that do not are inherent in its components" – see §38 in the book [1]. Nevertheless, the set theory which emerged more than a century ago, used as the basis of all mathematics, continues to try to relegate something whole to the level of one of its components by not seeing the difference between the size of the content of the set as a whole and the size of the content of one its element.

Primary defined finite sets and secondary obtained finite sets have the common quality to contain their elements, which is implied by the unity in which each of them unites them. Russell's paradox arises by ignoring of this implication and composing a self-denying definition of "the set of all sets that do not contain themselves." In overt kind this definition reads "a containing of all containing's that do not contain themselves". Such a set is not possible, because the requirement for inclusion of all sets like it raises the question of applying of the chosen criterion and to the set, which by definition must contain and this, which must not contain, i.e. to not contain and itself. Thus the mentioned vicious logical circle is obviously closed, because if such a set does not contain and itself, it must also contain and itself, and if it contains and itself, it must not contain and itself, and so on to infinity.

So far, Russell's paradox has established only the impossibility of the existence of a set with such mutually exclusive qualities, without taking into account the difference between the sizes of its content when it is primarily defined and when it has already been secondarily obtained by making its element its previous definition. With this in mind, the current set theory considers the result of such a self-contradictory set as one equivalence of two sets with mutually exclusive definiteness only depending on whether or not they contain themselves $R \in R \Leftrightarrow R \notin R$, although the content of itself and for a secondarily obtained set is impossible.

To avoid of the in such a way resulting contradiction, Zermelo–Fraenkel's set theory changes its axioms, preserving the standard language of logic in which they are formulated, while Russell's theory of types of sets changes the standard language of logic itself.

Hilbert's paradox

Hilbert's paradox of the Grand Hotel is given as striking example for the difference between the properties of the enumerable finite subsets of the countable infinite set of the natural numbers **N** and the properties of its countable infinite proper subsets. The most general characteristic of a number set is the quantity of elements belonging to it, called cardinality of the set. According to existing hitherto comprehensions all countable infinite sets, what in addition to the set **N** are and its countable infinite proper subsets, as well as the set of the rational numbers **Q** and its countable infinite proper subsets, have one and the same cardinality $\aleph_0$ (aleph-null), such as is the cardinality of the set **N**. Because of that, despite the fact that all of infinitely many rooms $\aleph_0$ of the Hotel are already occupied, after a suitable relocation of tenants in them it can always accommodates not only another guest but also infinitely many new guests $\aleph_0$ at once, as well as all guests who arrive simultaneously by infinitely many vehicles $\aleph_0$, in every one of which there are infinitely many guests $\aleph_0$. To accommodate another new guest, one simultaneously moves the tenant currently in room 1 to room 2, the tenant currently in room 2 to room 3, and so on, moving every tenant from his current room n to room $n + 1$. After this room 1 is vacant and the new guest can be accommodated into this room. For accommodatin an infinite number of new guests at once, every one of the infinite many initial tenants $\aleph_0$ currently in room n is moved into room $2n$, and the infinite new guest's $\aleph_0$ are accommodated in the rooms with odd numbers. The third problem is solved most elegantly by moving every of the available tenants from room n in room 2^n, while the guests from the first vehicle are accommodated in the rooms 3^n, the guests from the second vehicle are accommodated in the rooms 5^n, the guests from the third vehicle are accommodated in the rooms 7^n, and so on to the infinity of the prime numbers, whereby the rooms 1, 6, 10, 12, 14, 15, 18, 20, 21, 22 and so on to the infinity of the numbers that are not powers of the prime numbers remain vacant. Although with many another details and generalizations, Hilbert's paradox has so far been considered only on the basis thus described.

As the countable infinite cardinality $\aleph_0$ of the countable infinite set of natural numbers **N**, so and the uncountable infinite cardinality c of the uncountable infinite set of real numbers **R** have so far been considered only as immeasurably large and incomparable with each other infinite cardinalitys. However, the article [2] shows that the countable infinite cardinality $\aleph_0$ of the countable infinite set **N**, as a strictly determined incomparably big cardinality in comparison with the enumerable finite cardinality *n* of any enumeruble finite set and incomparably small in relation to the uncountable infinite cardinality *c* of the uncountable infinite set **R**, represents the natural unite of measurement for cardinality of the mentioned countable infinite sets. Measured with $\aleph_0$ the cardanilities of the countable infinite proper subsets of the set **N**, as and the cardinality of the countable infinite set **Q** and the cardinality of the one of its basic two countable infinite proper subsets turns out different from $\aleph_0$. Because of this, the incomparable with each other infinitely big cardinalities $\aleph_0$ and *c* here are accepted as strictly determined in size. Withal, the enumeruble finite subsets of the countal natural numbers **N** are strictly determined in size with exactness up to one countal unit 1. The enumeruble finite subsets of the ordinal natural numbers $\mathbf{N}_\alpha$ are strictly determined in size with exactness up to one ordinal unit 1. The countable infinite subsets of natural numbers **N** can in some cases be strictly determined in size with exactness up to a finite part of the cardinality of $\aleph_0$, as is the example with the gradually decreasing countable infinite countal cardinality of the sets represented by the sequences:

a_1) 1, 2, 3, …, *n*, …	$\aleph_0$
a_2) 2, 4, 6, …, 2*n*, …	$\aleph_0/2$
a_3) 3, 6, 9, …, 3*n*, …	$\aleph_0/3$
--	
a_k) k, k2, k3, …, k*n*, …	$\aleph_0/k$
--	

where k is a natural number greater than one. And in other cases they can be strictly determined in size with exactness up to a different rate of decreasing of the countable infinite countal cardinality of the sets when it tends to zero relative to $\aleph_0$, when the sets are represented by the sequences:

а) $1^2, 2^2, 3^2, \ldots, n^2, \ldots$
б) $1^3, 2^3, 3^3, \ldots, n^3, \ldots$
в) $1^4, 2^4, 3^4, \ldots, n^4, \ldots$
----

as well as with other similar sequences, such for example:
г) $2^1, 2^2, 2^3, \ldots, 2^n, \ldots$
д) $3^1, 3^2, 3^3, \ldots, 3^n, \ldots$
е) $5^1, 5^2, 5^3, \ldots, 5^n, \ldots$

In §40 and §48 of Bernard Bolzano's book [1], with examples, is shown the incomparability of the set of points over an infinite number line with the set of points on an infinite surface, and of the set of points on an infinite surface with the set of points in an infinite three-dimensional space. With this in mind, the cardinalities of these three uncountable infinite subtypes of sets and of their uncountable infinite subsets, which occupy spatial extents with different dimensions, are measured with qualitatively different units of measurement for length, for area, and for volume, respectively. Because of that, the strict determinateness according to size of the cardinality of each subtype of the uncountable infinite sets and according to size of the cardinality of its uncountable infinite subsets is determined with precision to the exactness, with which is determined the size of the spatial extents occupied by them in the space with dimensionality such as is and their dimensionality.

That is why the difference between the properties of the enumerable finite subsets of **N** and the properties of its countable infinite proper subsets ought to be described in the following way.

Since the different countable infinite sets are subsets of the uncountable infinite set **R**, the strictly determined countable infinite unite of measurement $\aleph_0$ for countable infinite cardinality of the countable infinite sets may be considered as a subcardinality of the strictly determined uncontable infinite cardinality c of the uncountable infinite set **R**. In the first case, at the accommodating of another new guest in Grand Hotel, one of its initial tenants will be sent in c. In the second case the one half from the arrived new guests $\aleph_0$ will remain in c not accommodated, the other half will be accommodated in the rooms with odd registration numbers of the Hotel, whereas in the rooms with the even registration numbers will remain the half of the initial tenants $\aleph_0$ of the Hotel, and the other half of them will be sent in c. In the third case, an initial tenants will remain in the Hotel in the rooms with the registration numbers of the terms of the set represented by the infinite sequence of the increasing powers of the first prime number 2, as well as in the rooms with the registration numbers, represented by the numbers, which are not powers of the prime numbers, but will be sent in c all initial tenants of the Hotel who have been in the rooms with the registration numbers of the terms of the sets represented by the infinite sequences of the increasing powers of the remaining odd prime numbers. At that, in the rooms of the Hotel with the registration numbers of the terms of the sets represented by the infinite series of increasing degrees of the other odd prime numbers will be accepted more and more insignificantly small parts of the new visitors arriving with the respective vehicle, as many as more insignificantly small part is the cardinality of the set represented by the infinite series of increasing degrees of the corresponding odd prime number with respect to cardinality $\aleph_0$. In the latter case, all others of the infinite number of newcomers will remain in c unaccepted.

The strict determinedness of the countable infinite unit of measurement $\aleph_0$ for cardinality of the mentioned countable infinite sets, as well as and the strict determinedness of the uncountable infinite cardinality *c* of the uncountable infinite set **R**, together with the differences between the cardinalities of the different countable infinite proper subsets of the countable infinite set **N**, can be apply and if it is possible the rooms of the Hotel to accommodate an unlimited number of visitors. Then the Grand Hotel will really be able to receive all the mentioned new guests without leaving some of them in *c* or sending in *c* initial tenants. In the first case, when accepting only one new guest, at least in one of the rooms of the Hotel, such as is for example the first, will have to accommodate two tenants. In the second case, when receiving at once so many infinitely many new guests $\aleph_0$, with an even distribution of tenants in each of the rooms of the Hotel will be accommodated in two tenants. In the third case, at receiving at once of all new guests who arrive simultaneously by infinitely many vehicles $\aleph_0$ with infinitely many guests $\aleph_0$ each, at even distribution of tenants in each of the Hotel rooms will be accommodated $\aleph_0 + 1$ tenants.

Inference

A more in-depth analysis of the notions is needed in order not to formulate a self-denying definition, as is the case with Russell's paradox.

Finding of $\aleph_0$ as natural unit of measurement for the cardinality of the countable infinite sets opens the way for an unambiguous solution of the continuum hypothesis.

References

[1] B. Bolzano, F. Přihonský, *Paradoxien des Unendlichen*, C.H. Reclam sen., Leipzig, Germany, 1851.

[2] H. Manev, Mathematical magnitudes, *International Mathematical Forum*, **16** (2021), no. 3, 137 – 146. https://doi.org/10.12988/imf.2021.912250

Received: October 11, 2021; Published: November 12, 2021