

The Difficult Road toward the Insight

Hristo Manev

University of Architecture, Civil Engineering and Geodesy
bul. Hristo Smirnenski No. 1 Sofia 1164, Bulgaria

This article is distributed under the Creative Commons by-nc-nd Attribution License.
Copyright © 2021 Hikari Ltd.

Abstract

The development of the idea of measuring the cardinality of different types and subtypes of number sets as mathematical magnitudes is traced.

Mathematics Subject Classification: 03Exx; 03E99

Keywords: Physical magnitudes, Mathematical magnitudes, Units of measurements for cardinality

1. Introduction

The measurement in objective reality is determination of an attitude between objects. The process of measurement is always connected with a conception of the objects, which are correlating. This process becomes a completed whole, called measuring, only when the obtained result is interpreted. Every object is describing by some substantial symptoms, called its qualities. The physical magnitudes are characterizations of the objective reality that are responding to determined qualities. At measurement, some physical magnitudes exhibit a comparable by degree property, called quantity y of the measured magnitude Y . The magnitude Y is a primary scar by which we can generally distinguish one thing from other different things, whereas the quantity y is a secondary scar by which we are able to distinguish one from other different things about the one and the same magnitude when they exhibit themselves as a comparable in degree property of this magnitude. In general, quantity is an idealized comparable in degree property of some magnitude.

The initial notion for number x arises with the universal designating of the ratio between one quantity y , which is being determined, and another determined by the same quantity u , pre-selected as a unit of measurement

$$(1) \quad x = \frac{y}{u}.$$

In that, a size of determined quantity y is its expression xu , where the measure number x is called value of the quantity y , which is presented by unit of measurement u . The remaining two letters in Equation (1) are also idealized as numbers in the theory of the numbers.

Basic topic in the subjective reality is the set as a mentally created unity that contains well determined and distinct one from other things, called elements on the set. Main quality of such standardly defined set is the containing of well determined and distinct one from other elements, which belong to the set according to a chosen by thinker their property. Each kind of infinity represents a concrete determinateness that is idealized as unlimited, and the set of the countal (cardinal) natural numbers $\mathbf{N}$ is based on all of kinds of simplest idealized quantitative regularities. (The set of ordinal natural numbers $\mathbf{N}_\alpha$ is other kind number set.) Because of that, the properties of the standardly defined number sets and the properties of their elements are most easily revealed when the elements of the set are countal natural numbers. According to the qualities of their countability, number sets are divided into three main different types: a) enumerable finites – such as the enumerable finite subsets of the set of countable infinite natural numbers $\mathbf{N}$; b) countable infinities – such as the countable infinite set of the natural numbers $\mathbf{N}$ and its countable infinite proper subsets, as well as the countable infinite set of the rational numbers $\mathbf{Q}$ and its countable infinite proper subsets; c) uncountable infinities – such as the one-dimensional uncountable infinite set of the real numbers $\mathbf{R}$ and its uncountable infinite proper subsets, as well as the next two-dimensional, three-dimensional and in general with a finite number of dimensions uncountable infinite sets and their respective uncountable infinite proper subsets.

Like physical magnitudes, the number sets are mathematical magnitudes that have a size or their measure, called cardinality. More Galileo Galilei in his book "Two New Sciences" [1] showed the dependence of the size and structure of the bodies of living organisms on the laws of physics, because of which the described by Jonathan Swift worlds of Lilliput's and of Giant's in "Gulliver's Travels" are impossible. The achievements of modern physics clearly delineate the boundary between the laws of the microworld with sizes smaller than the size of an atom, the laws of the mesoworld with sizes between the size of an atom and the size of a galaxy, and the laws of megaworld with sizes larger than the size of a galaxy. Because of that, the dependence of units for cardinality on the type of

different number sets considered here is of special interest. At that one thing are the different number sets as different mathematical magnitudes and other thing are their cardinalities as the most general their characterization. Not to mention the characterizations of the different elements of each of the sets, as well as the characterizations of the different relationships between the elements themselves.

2. Units of measurements for cardinality

Around 1302 year Duns Scot compared the infinite set of the even numbers with the whole infinite set of the natural numbers.

In the mentioned book [1] Galileo also discusses the problem of comparing the infinite quantities of points over lines of different length. He explains the difficulty for it solving with our uncritically ascribed properties on infinity which are established at the finite and the limited. As an example of such a difficulty, Galileo gives us our attempt to have being comparing the infinite set of the squares of natural numbers with the infinite set of all natural numbers. On the one hand, not all numbers are squares of the natural numbers, because of which we ought to accept that all natural numbers are more than the squares of the natural numbers. On the other hand, however, each natural number has as square other natural number and represents square root of other natural number, because of which we ought to accept that the squares of the natural numbers are as much as the natural numbers. Despite the fact that as we move away from the beginning of the sequence of the set of natural numbers, the percentage of their squares relative to the number of all natural numbers up to a selected finite distance from the beginning of this series decreases very rapidly. In these circumstances, Galileo came to the conclusion that only at finite sets does it make sense to distinguish between "less," "equal," or "greater," while for infinite sets we can only say that they are infinite.

The next profound insight into the difference between the properties of the enumerable finite sets and the properties of the uncountable infinite sets is that of Bernard Bolzano in the posthumously published book "Paradoxes of the infinite" [2]. It is shown in §20, §21, §22, §23, and §24 of this book, that the unambiguous and reversible correspondence between the elements of two enumerable finite sets is enough for proving of equality between the quantities of elements in every one of them, called cardinality n of the respective enumerable finite set, where n is the number of its elements. At the uncountable infinite sets however the availability of such a correspondence is not enough for proving of equality between their cardinalities. For the attitude between the cardinalities of the uncountable infinite sets is measured by means of the attitude between the sizes of the occupied of

them extends in a space with a dimension like theirs. For example, for the one-dimensional extent of the uncountable infinite set of the real numbers $\mathbf{R}$, such sizes are the lengths that its proper subsets cover over the number line. Thus from the equation $5y = 12x$ and its corresponding Fig. 1 in §20 it is seen, that the infinite subset of the real numbers

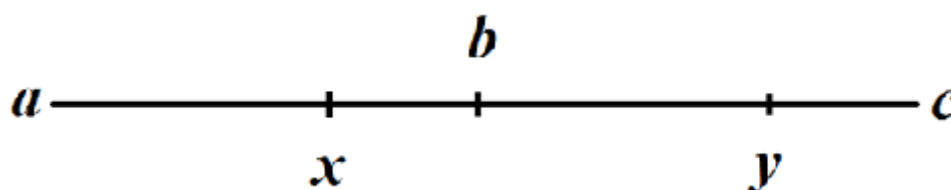


Fig. 1. A graph corresponding to the equation $5y = 12x$.

in the interval from 0 to 5 is in unambiguous and reversible correspondence with the infinite subset of the real numbers in the interval from 0 to 12. However the first subset of $\mathbf{R}$ undoubtedly represents a proper subset of the second subset of $\mathbf{R}$ because the length covered by it over the number line is only part of the length covered by the second subset of $\mathbf{R}$. At the end of §28, Bolzano summarizes his discovery in such a way: "... the correct calculation of the infinite aims at not ... calculating the infinite set in itself ..., but determining of an attitude between one infinite and another...". Moreover in §40 and §48 it is shown, with examples, the incomparability of the set of points over an infinite number line with the set of points on an infinite surface, and of the set of points on an infinite surface with the set of points in an infinite three-dimensional space. With this in mind, the cardinalities of these three subtypes of sets, which occupy extents with different dimensions, are measured with qualitatively different units of measurement for length, for area, and for volume, respectively.

As for unraveling the properties of the countable infinite sets, Bolzano did not have much success. Guided by his intuition, in the beginning of §29 he anticipates the existing of "aleph-null" by marking the infinite quantity of the set of the natural numbers $\mathbf{N}$ with $\mathbf{N}^{(0)}$, which called "number of all numbers". Obviously is too early to be realized that aleph-null can not be a number because it represents the size of the infinite cardinality of the countable infinite set of the natural numbers, whereas every number represents the size of the enumerable finite cardinality of corresponding finite set. At that he allows a mistake at comparing the cardinalities of countable infinite sets in §33. Really, at the initial estimating for the quantity of terms in the sequence of the squares of natural numbers in comparison with the quantity of terms in the sequence of their first

powers, he shows some hesitation. But in the end, he wrongly assumes that as infinities these quantities are equal. This acceptance does not take into account the fact established by Galileo of the growing distance between the terms in the sequence of the squares of the natural numbers in comparison with the compact following of the terms in the sequence of the natural numbers, where in addition to the squares of the natural numbers participate and the natural numbers which, are not exact squares of other natural numbers. Therefore, when summing up, Bolzano incorrectly considers the successive members of each of the sequences representing these sets as corresponding, which leads to a wrong conclusion about the ratio between their sums. This mistake shows only the importance of the countable infinite countal cardinality aleph-null as a unit of measurement for initial delimiting according to cardinality of the countable infinite countal sets.

At the end of 19th century Georg Cantor used the correspondence or lack of correspondence at juxtaposition between the elements of two sets for distinguish the cardinality of the countable infinite set of the natural numbers $\mathbf{N}$ as incomparably small in respect to the cardinality of the set of real numbers $\mathbf{R}$. However obviously he did not realize that and the finite cardinality n of any enumerable finite set is incomparably small toward the infinite cardinality of the countable infinite set $\mathbf{N}$. At that Cantor presumably did not know about Bolzano's quoted work, for he rashly accepted the sufficiency of the unambiguous and reversible correspondence between the elements of two finite sets for proving of equality between their cardinalities as valid and for the countable infinite sets, as well as for the uncountable infinite sets. Because of that Cantor failed to see the grandiosity of the world of the countable infinite sets discovered by him, as well as the even more complicated world of the uncountable infinite sets. That is why the presentation of these two worlds engenders many delusions. One of them is the delusion that in the interval from 0 to 1 there is so many points as are the points in every space with a finite number of dimensions. In a letter to Richard Dedekind Cantor exclaims ahead of his self-delusion in such a way: "I see it, but I don't believe it!"

Cantor's great insight with hitherto unconscious consequences consists in marking the size of the countable infinite cardinality of the set $\mathbf{N}$ with one sign, such as the letter aleph-null $\aleph_0$. With this marking, the letter $\aleph_0$ is transmuted into symbol of cardinality with strictly determined size, which by definition is incomparably big toward the size n of enumerable finite cardinality of any enumerable finite set and incomparably small in respect of the size c of the uncountable infinite cardinality of the uncountable infinite set $\mathbf{R}$. The difficulty to be conscious of this transmutation is due to not the standard way, in which is

defined the size *countable infinite cardinality* $\aleph_0$ of the mathematical magnitude countable infinite set $\mathbf{N}$, as strictly determined size.

The next essential insight is consisted in realizing of the fact that the so determined size of the countable infinite cardinality $\aleph_0$ of the countable infinite set $\mathbf{N}$ represents the natural unit of measurement for cardinality of the countable infinite sets, what in addition to the set $\mathbf{N}$ are and its countable infinite proper subsets as well as the set of the rational numbers $\mathbf{Q}$ and its countable infinite proper subsets – see article [4]. The countable infinite cardinality $\aleph_0$ is incomparable with an enumerable finite cardinality n of any enumerable finite set, similar of the incomparable with a finite sum $1 + 1 + 1 + \dots < \omega$ hyperreal numbers in contemporary non-standard analysis. Generally, because of the essential difference among the properties of the enumerable finite sets, of the countable infinite sets and of the uncountable infinite sets, their cardinalities are measured respectively by incomparable with each other units of measurements, namely: with the countal unit 1, with the countable infinite cardinality $\aleph_0$, and, with the uncountable infinite cardinality of the uncountable infinite set of a pleriminary chosen for unit of measurement part of spacial extent, which has equal number of dimentions with the number of dimentions of the spatial extent which is being occupying by the subtype of set which is being measuring.

3. Inference

The contribution of Bolzano for the contemporary grounding of the differential and integral calculus is rediscovered long ago. However, until the current 21st century, his contribution for the contemporary grounding of set theory has not been noticed. Certificate for this is the review book of Hardy Grant and Israel Kleiner “Turning Points in the History of Mathematics” [5], published during 2016 year. Its 9th chapter “The Infinite: From Potential to Actual” swarms with incorrect explanations of the treated problems due to the prevailing notion that the criterion availability of unambiguous and reversible correspondence between the elements of two sets is enough for proving the equality of their cardinality regardless of whether they are finite or infinite. Above already is mentioned where and how Bolzano grounds the validity of this criterion for the enumerable finite sets and the invalidity of the same criterion for the uncountable infinite sets. Bright illustration of the indicated invalidity there is and in §41 of the book [2]: “In two perfectly similar to each other extents the sets of their points must be in such an attitude namely, in what their sizes are”. The points for example of two concentric circumferences with different radii are in unambiguous and irreversible correspondence, which is established with constructing any radius

from the center to them. However if we suppose that the quantities of their points are equal, at infinite decreasing the radius of the fewer circumference it is coming to the ridiculous conclusion, that in their center there are as many infinite points, when it consists of only one point.

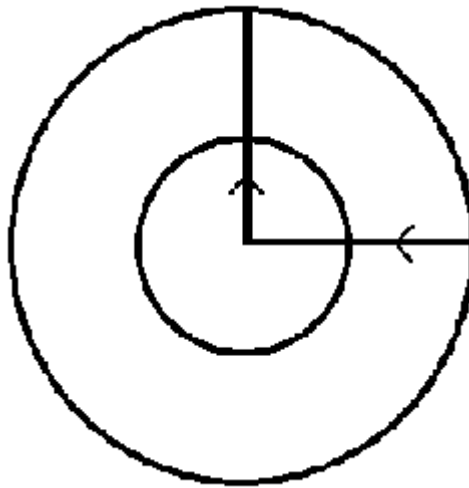


Fig. 2.

Therefore, Fig. 9.1 on page 76 in the book [5] with the explanation "Two concentric circles with unequal diameter but with the same number of points" marks a two-century lag of the current set theory in comparison with the achievement of Bernard Bolzano – see here Fig. 2, which is identical to Fig. 9.1. In articles [3] and [4] it is shown that the mentioned criterion is not valid also for the countable infinite sets.

References

- [1] G. Galilei, *Discorsi E Dimostrazioni Matematiche Intorno a Due Nuove Scienze*, 1st ed, Elsevier Press, Leiden, the Netherlands, 1638.
- [2] B. Bolzano, F. Přihonský, *Paradoxien des Unendlichen*, C.H. Reclam sen., Leipzig, Germany, 1851.
- [3] H. Manev, The logical paradoxes, *International Mathematical Forum*, **15** (2) (2020), 61 – 92. <https://doi.org/10.12988/imf.2020.91249>
- [4] H. Manev, Mathematical magnitudes, *International Mathematical Forum*, **16** (3) (2021), 137 – 146. <https://doi.org/10.12988/imf.2021.912250>

[5] H. Grant, I. Kleiner, *Turning Points in the History of Mathematics*, York University, Toronto, Canada, 2016.

Received: August 29, 2021; Published: September 23, 2021