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A Connection Between Stochastic Differential Equations and Uncertain Differential Equations

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Abstract

According to current research, both stochastic differential equations (SDEs) and uncertain differential equations (UDEs) are adopted on modeling asset prices in financial markets. When applying SDEs we often rely on theories such as Itô calculus, martingale theory, etc. But for UDEs we usually need to start from the α-paths. Despite the obvious differences, is there a connection between the two methods? We compared the SDEs with UDEs of similar forms, thereby a connection is found: when α takes 0.5, under certain conditions the value of this special α-path of a UDE equals to the expectation of the corresponding SDE's solution at any moment. This connection gives us a deeper understanding of non-deterministic processes described by UDEs and the Yao-Chen formula helps us explain that the expectation of a SDE's solution is exactly the "prospect" that people expect for the process.

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1 Introduction

Currently there exist at least two ways to describe non-deterministic processes: one is to use stochastic differential equations (SDEs) and the other is to use uncertain differential equations (UDEs). They are all applied to financial modeling. The stochastic finance theory has a longer history, Bachelier [1] pioneered the quantitative work in this area. The research results of Black and Scholes [2], Merton [3] are widely accepted.,

In 2008, Liu [4] proposed the UDE based on uncertainty theory. In uncertain finance theory, the basic assumption is that the stock price follows a geometric Liu process. Based on this, European option [5], American option [6] and Asian option [7, 8] pricing formulas are derived. Furthermore, there are other UDE models applied to finance, such as mean-reverting model [9] and exponential Ornstein-Uhlenbeck model [10, 11, 12].

But it is worth noting that there is a problem with UDEs: over time, it may not be able to calculate the expected values of their solutions. For instance [13], the expected value of the geometric Liu process $\mathrm{d}Y_t = eY_t\mathrm{d}t + \sigma Y_t\mathrm{d}C_t$ is:

$$E[Y_t] = \begin{cases} Y_0 \exp(et)\dfrac{\sigma t\sqrt{3}}{\sin(\sigma t\sqrt{3})}, & t < \dfrac{\pi}{\sigma\sqrt{3}}, \\ +\infty, & t \geq \dfrac{\pi}{\sigma\sqrt{3}}. \end{cases}$$

Once t increases beyond a certain value determined by the parameters, the expected value tends to infinity. Or if it is required to calculate the expected value within a certain period of time, then the volatility needs to meet certain conditions. However, as a comparison, the expectation of geometric Wiener process can be calculated in any finite time. This problem cannot be ignored when applying UDEs. For a fixed strike price K, the above problem will cause us not always being able to calculate European options' price which is $P_{call} = e^{-rt}E[(Y_t - K)^+]$. For other UDEs, there is no guarantee that this phenomenon will not occur. The root cause is that the variance of Liu process increases extremely fast with time. So we turn to focus on the α-paths of UDEs, then found a connection between the 0.5-path of a UDE and the expectation of the corresponding SDE. Through this relationship, we have a deeper understanding to the shape of the UDE's solution: it is more suitable for describing peoples subjective expectations rather than objective changes in the processes.

The content of this paper is arranged as follows: Some basic theorems are introduced in section 2; Linear SDEs and similarly structured UDEs are compared in section 3; The connection is examined in a more general situation in section 4; Section 5 is the conclusion.

2 Preliminary

In this paper, we focus on Itô SDEs of the form:

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}W_t, X_0 = x_0, \tag{1}$$

and UDEs of the form:

$$\mathrm{d}Y_t = f(t, Y_t)\mathrm{d}t + g(t, Y_t)\mathrm{d}C_t, Y_0 = y_0, \tag{2}$$

where $f, g : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ are Borel-measurable functions, W_t is a Wiener process and C_t is a Liu process. Set $X_0 = Y_0 = c, c \in \mathbb{R}$.

Suppose that the functions f, g satisfy the Lipschitz and linear growth conditions:

$$|f(t, x) - f(t, y)| + |g(t, x) - g(t, y)| \leq K|x - y|, \forall x, y \in \mathbb{R}, t \geq 0, \tag{3}$$

$$|f(t, x)| + |g(t, x)| \leq K(1 + |x|), \forall x \in \mathbb{R}, t \geq 0, \tag{4}$$

where K is a positive constant. Then there exists a continuous, adapted process X_t which is the unique strong solution of (1).

Theorem 2.1 *(Chen and Liu [14]) The UDE (2) has a unique solution if the coefficients $f(t, x)$ and $g(t, x)$ satisfy the Lipschitz condition (3) and linear growth condition (4).*

Definition 2.2 *(Yao and Chen [15]) Let α be a number with $0 < \alpha < 1$. UDE (2) is said to have an α-path X_t^α if it solves the corresponding ordinary differential equation (ODE):*

$$\mathrm{d}X_t^\alpha = f(t, X_t^\alpha)\mathrm{d}t + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t, \tag{5}$$

where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.$$

Theorem 2.3 *(Yao-Chen Formula [15]) Let X_t and X_t^α be the solution and α-path of (2) respectively. Then*

$$\mathcal{M}\{X_t \leq X_t^\alpha, \forall t\} = \alpha, \quad \mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha.$$

3 Linear differential equation

Let the functions f, g in (1) or (2) satisfy conditions (3) and (4). Then it is ensured that each of (1) and (2) has its own unique solution.

Consider a linear stochastic differential equation (LSDE) of the form:

$$\mathrm{d}X_t = [A(t)X_t + a(t)]\mathrm{d}t + [S(t)X_t + \sigma(t)]\mathrm{d}W_t, \tag{6}$$

where $X_0 = x_0$, and $A(t)$, $a(t)$, $S(t)$, $\sigma(t)$ are Borel-measurable functions bounded on every finite interval [0,T] without indeterminacy. The coefficients satisfy (3) and (4), thus the equation (6) has an unique solution X_t.

The expectation of the solution $E[X_t]$ can be found without solving the LSDE itself, only by using the properties of stochastic integrals:

$$\int_0^t [S(u)X_u + \sigma(u)]\mathrm{d}W_u$$

is a martingale, so we have

$$E\left[\int_0^t [S(u)X_u + \sigma(u)]\mathrm{d}W_u\right] = 0.$$

Take the expectations of both sides of the integral form of equation (6):

$$X_t = x_0 + \int_0^t [A(u)X_u + a(u)]du + \int_0^t [S(u)X_u + \sigma(u)]\mathrm{d}W_u.$$

Then

$$E[X_t] = x_0 + E\left[\int_0^t [A(u)X_u + a(u)]du\right] = x_0 + \int_0^t [A(u)E[X_u] + a(u)]du.$$

By differentiating we get the ODE:

$$\frac{\mathrm{d}E[X_t]}{\mathrm{d}t} = A(t)E[X_t] + a(t), E[X_0] = X_0 = c. \tag{7}$$

And the solution of (7) is:

$$E[X_t] = \exp\left[\int_0^t A(u)\mathrm{d}u\right]\left\{c + \int_0^t a(u)\exp\left[-\int_0^u A(s)\mathrm{d}s\right]\mathrm{d}u\right\}, t \geq 0. \tag{8}$$

Similarly, consider UDEs of the form:

$$\mathrm{d}Y_t = [A(t)Y_t + a(t)]\mathrm{d}t + [S(t)Y_t + \sigma(t)]\mathrm{d}C_t, \tag{9}$$

as counterparts of LSDEs, where $Y_0 = c$. An α-path Y_t^α of (9) solves the following ODE by (5):

$$\mathrm{d}Y_t^\alpha = [A(t)Y_t^\alpha + a(t)]\mathrm{d}t + |S(t)Y_t + \sigma(t)|\Phi^{-1}(\alpha)\mathrm{d}t. \tag{10}$$

Let $\alpha = 0.5$, then $\Phi^{-1}(\alpha) = 0$, so this 0.5-path of Y_t solves:

$$\mathrm{d}Y_t^{\alpha=0.5} = [A(t)Y_t^{\alpha=0.5} + a(t)]\mathrm{d}t. \tag{11}$$

Remeber that $Y_0^{\alpha=0.5} = Y_0 = c$, we have

$$Y_t^{\alpha=0.5} = \exp\left[\int_0^t A(u)\mathrm{d}u\right]\left\{c + \int_0^t a(u)\exp\left[\int_0^u -A(s)\mathrm{d}s\right]\mathrm{d}u\right\}, t \geq 0. \tag{12}$$

For $Y_t^{\alpha=0.5}$, according to the Yao-Chen formula, we also know that it has the following property:

$$\mathcal{M}\{Y_t \leq Y_t^{\alpha=0.5}, \forall t\} = \mathcal{M}\{Y_t > Y_t^{\alpha=0.5}, \forall t\} = 0.5.$$

From the perspective of belief degree, someone would believe $Y_t \leq Y_t^{\alpha=0.5}$ and $Y_t > Y_t^{\alpha=0.5}$ to the same degree if he use (9) to describe Y_t. We would like to call $Y_t^{\alpha=0.5}$ "the prospect" of Y_t, and write it as Y_t^P to avoid confusion.

Theorem 3.1 *If an LSDE (6) and an UDE (9) satisfy conditions (3), (4) and $X_0 = Y_0 = c$, $c \in \mathbb{R}$, then*

$$E[X_t] = Y_t^P, t \geq 0.$$

Proof. It comes from (8) and (12) directly.

Example 3.2 *Consider geometric Wiener process* $\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t$ *and geometric Liu process* $\mathrm{d}Y_t = \mu Y_t \mathrm{d}t + \sigma Y_t \mathrm{d}C_t$*,* $X_0 = Y_0 = c$*,* $c \in \mathbb{R}$*. We have*

$$E[X_t] = Y_t^P = ce^{\mu t}, t \geq 0.$$

Example 3.3 *Consider the LSDE* $\mathrm{d}X_t = (m - \alpha X_t)\mathrm{d}t + \sigma X_t \mathrm{d}W_t$ *and the UDE* $\mathrm{d}Y_t = (m - \alpha Y_t)\mathrm{d}t + \sigma Y_t \mathrm{d}C_t$*,* $X_0 = Y_0 = c$*,* $c \in \mathbb{R}$*. We have*

$$E[X_t] = Y_t^P = \frac{m}{\alpha} + e^{-\alpha t}(c - \frac{m}{\alpha}), t \geq 0.$$

For more details of these two models, see Peng and Yao [9], Black and Karasinski [16].

4 Generalization

Now we try to generalize Theorem 3.1 to a more general situation.

Theorem 4.1 *If a SDE (1) and a UDE (2) satisfy conditions (3), (4),* $E[f(t, X_t)] = f(t, E[X_t])$ *under probability measure, and* $X_0 = Y_0 = c$*,* $c \in \mathbb{R}$*, then*

$$E[X_t] = Y_t^P, t \geq 0.$$

Proof. First of all, it is easier to calculate the α-path of a UDE of the form (2). Let $\alpha = 0.5$, then

$$\mathrm{d}Y_t^P = f(t, Y_t^P)\mathrm{d}t. \tag{13}$$

Next we need to calculate the expectation of the corresponding SDE's solution. We have

$$X_t = x_0 + \int_0^t f(u, X_u)\mathrm{d}u + \int_0^t g(u, X_u)\mathrm{d}W_u, \tag{14}$$

and the term

$$\int_0^t g(u, X_u)\mathrm{d}W_u$$

in (14) is also a martingale. Then

$$E[X_t] = x_0 + E\left[\int_0^t f(u, X_u)\mathrm{d}u\right] = x_0 + \int_0^t E[f(u, X_u)]\mathrm{d}u. \tag{15}$$

Differentiate time on both sides:

$$\frac{\mathrm{d}E[X_t]}{\mathrm{d}t} = E[f(t, X_t)]. \tag{16}$$

If the following formula can still be satisfied:

$$E[f(t, X_t)] = f(t, E[X_t]), \tag{17}$$

then both of $E[X_t]$ and Y_t^P will satisfy the same ODE by (13), (16) and (17), meanwhile $E[X_0] = X_0 = Y_0^P = Y_0 = c$, so we have $E[X_t] = Y_t^P$. The theorem is proved.

Example 4.2 *Consider the SDE* $\mathrm{d}X_t = (a - bX_t)\mathrm{d}t + \sigma\sqrt{X_t}\mathrm{d}W_t$ *(see Cox et al. [17], CIR model) and the UDE* $\mathrm{d}Y_t = (a - bY_t)\mathrm{d}t + \sigma\sqrt{Y_t}\mathrm{d}C_t$, $X_0 = Y_0 = c$, $c \in \mathbb{R}$. *We have*

$$E[X_t] = Y_t^P = \frac{a}{b} + e^{-bt}(c - \frac{a}{b}), t \geq 0.$$

Example 4.3 *Consider the SDE* $\mathrm{d}X_t = \mu(1 - k\ln X_t)\mathrm{d}t + \sigma X_t\mathrm{d}W_t$ *and the UDE* $\mathrm{d}Y_t = \mu(1 - k\ln Y_t)\mathrm{d}t + \sigma Y_t\mathrm{d}C_t$ *(see Dai et al. [10], Sun et al. [11], Gao et al. [12], Exponential Ornstein-Uhlenbeck model),* $X_0 = Y_0 = c$, $c \in \mathbb{R}$. *We have*

$$E[X_t] = \exp\left\{\ln(c)\exp(-\mu kt) + (\mu - \frac{1}{2}\sigma^2)\int_0^t e^{-\mu ku}\mathrm{d}u + \frac{1}{2}\sigma^2\int_0^t e^{-2\mu ku}\mathrm{d}u\right\}$$

$$\neq \exp\{\ln(c)\exp(-\mu kt) + \frac{1}{k}[1 - \exp(-\mu kt)]\} = Y_t^P, \forall t > 0.$$

Notice that $E[\mu(1 - k\ln X_t)] \neq \mu(1 - k\ln E[X_t])$.

5 Conclusion

Since sometimes we cannot obtain the expected value of an UDE's solution, we turn to the α-path as an alternative. With the study of α-paths, we can explore the shape of the solution effectively. There is a special α-path where the α is equal to 0.5. When we compare UDEs with SDEs of similar forms, a connection appears. If the two equations satisfy certain conditions, then the 0.5-path of the UDE will be equal to the expectation of corresponding SDE's solution.

For this particular path, it has another noteworthy property: $\mathcal{M}\{Y_t \leq Y_t^P, \forall t\} = \mathcal{M}\{Y_t > Y_t^P, \forall t\} = 0.5$. This is also why we would like to call it "the prospect" of a UDE. If there exists a number A such that $\mathcal{M}\{Y_t \leq A\} < \mathcal{M}\{Y_t > A\}$, it means that someone is more willing to believe $Y_t > A$ rather than $Y_t \leq A$, since the uncertain measure is used to handle belief degrees. In other words, he will think that A is smaller than the prospect of Y_t as long as he uses the UDE to describe Y_t. Therefore, we suggest UDEs are suitable for describing people's views on a dynamic process.

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