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Why Question-Based Reasoning Leads to Constructive Approach to Knowledge

Olga Kosheleva and Vladik Kreinovich

University of Texas at El Paso
500 W. University
El Paso, TX 79968, USA

Abstract

Once we have partial knowledge, what next question do we usually pursue? Empirical study shows, e.g., that if we know that $A \vee B$ is true, but we do not know whether A is true or B is true, then the usual next step is to ask whether A is true or B is true. This selection of the next step is in line with the constructive approach to knowledge, in which when $A \vee B$ is true, this means that we either know that A is true, or we know that B is true. In this paper, we provide a possible explanation for this empirical selection of the future question-to-ask.

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1 Formulation of the Problem

Question-based reasoning: a brief description. On each stage of our knowledge, many things are still unknown. There are many open questions that we can ask, many directions to pursue. Based on what we know so far, we select a question to pursue first. This is how we gain knowledge.

Which of the possible questions do we usually pursue? Which questions should we pursue? The corresponding field of study is known as *question-based reasoning*; see, e.g., [4, 5].

Empirical fact. One of the discoveries of research in question-based reasoning was that, if we know that $A \vee B$ is true, but we do not know whether A is true or B is true, then most people start checking whether A is true or B is true.

How this is related to constructive mathematics and constructive logic. One can see that this usual selection of the next question is related to *constructive approach* in mathematics. To explain this relation, let us recall what this constructive approach is about.

In the traditional mathematics, if we prove that there exists an object that satisfied a given property, this does not mean that we actually know an example of such an object – the proof can be by reduction to contradiction. Similarly, if we know that the statement $A \vee B$ is true, this does not necessarily mean that we know which of the two statements A and B is true. For example, for each statement A, we know that $A \vee \neg A$ is true, but we do not necessarily know whether A is true or its negation is true.

In practice, we often need to know the object satisfying the given property. To formalize this need, mathematicians invented the notions of *constructive logic* and *constructive mathematics*; see, e.g., [1, 2, 6]. In constructive mathematics, $\exists x\, P(x)$ means that we actually know an object that satisfies the property $P(x)$ – to be more precise, we know an algorithm for constructing such an object. Similarly, in constructive mathematics, the truth of a composite statement $A \vee B$ means that we either know A or know B.

From this viewpoint, the above feature of our question-based reasoning can be naturally reformulated in terms of constructive logic: if it so happens that our knowledge is not constructive, i.e., that we know that $A \vee B$ is true but we do not know whether A is true or B is true, then our next move is to try to make our knowledge more constructive, by trying to decide whether A is true or B is true.

How can we explain this empirical fact? In this paper, we provide a possible explanation for the above empirical fact.

2 Analysis of the Problem and the Resulting Explanation

General description of the state of knowledge. In general, we have some elementary ("atomic") statements $S_1, \ldots, S_n$. From this viewpoint, to fully describe the state of the world, we need to know the truth values of all n statements.

For each of n statements, there are two possible cases – when this statement is true and when this statement is false. If we have two elementary statements ($n = 2$), then each situation of truth or falsity of S_1 leaves to two different situations depending on whether S_2 is true or not. So, overall, we have $2 \cdot 2 = 2^2$ possible combinations of truth values. Similarly, one can show that in principle, we can have 2^n possible worlds.

When we have some information about the statements S_i, then not all worlds are possible – only the worlds in which our information is true. For example, if we know that $S_1 \vee S_2$ is true, and we have no information about S_3, then out of $2^3 = 8$ possible worlds, only the following six worlds are possible:

$$S_1 \,\&\, S_2 \,\&\, S_3, \quad S_1 \,\&\, S_2 \,\&\, \neg S_3, \quad S_1 \,\&\, \neg S_2 \,\&\, S_3,$$

$$S_1 \,\&\, \neg S_2 \,\&\, \neg S_3, \quad \neg S_1 \,\&\, S_2 \,\&\, S_3, \quad \neg S_1 \,\&\, S_2 \,\&\, \neg S_3.$$

In general, if we know that $S_1 \vee S_2$ is true, and we have no information about any other statements $S_3, \ldots, S_n$, then we have $3 \cdot 2^{n-2}$ possible worlds: indeed, we have three possible combinations of truth values of S_1 and S_2:

$$S_1 \,\&\, S_2, \quad S_1 \,\&\, \neg S_2, \quad \neg S_1 \,\&\, S_2;$$

for each of these three cases, there are 2^{n-2} possible combinations of truth values of statements S_3, $\ldots$, S_n.

Possible questions. As we have mentioned, the full description of the world means describing which of the n elementary statement is true and which is false. If our knowledge is incomplete, this means that for some of the elementary statement, we do not know whether they are true or false. In this case, a reasonable idea is to select one of such elementary statements and start analyzing whether the selected elementary statement is true or false.

Which of the possible questions should we select? Which of the elementary statements should we select? Acquiring new knowledge is not easy. Progress is mostly made by small steps – since large steps are difficult to do. It is therefore reasonable to concentrate on questions which are the easiest to solve – in the sense that their answer will bring in the smallest amount of information.

How can we gauge this amount? Suppose that in the current state of knowledge, there are N possible worlds. We have no reason to believe that any of these possible worlds is more probable than others. Thus, it is reasonable to assign to each of these worlds the same probability $\dfrac{1}{N}$. For each possible question – i.e., for each question about the statement S_i:

- we have M_i worlds in which the i-th elementary statement is true, and
- we have $N - M_i$ worlds in which this statement S_i is false.

Thus, the probability that S_i is true is equal to $p_i = \dfrac{M_i}{N}$. Thus, according to the Shannon's formula (see, e.g., [3]), the information gained after finding the answer to the question about S_i is equal to

$$I_i = -p_i \cdot \log_2(p_i) - (1 - p_i) \cdot \log_2(1 - p_i). \tag{1}$$

We should select the question i for which this amount of information is the smallest possible.

Let us apply this criterion to the $S_1 \vee S_2$ situation. Let us apply this criterion to the situation when the only information that we have is that $S_1 \vee S_2$ is true.

If we select the question about S_i for any $i > 2$, then, as one can easily see, exactly half of the worlds have the property S_i. In this case, $p_i = 1/2$, so $\log_2(p_i) = -1$, and the formula (2) leads to $I_i = 1$.

On the other hand, if we select the question about S_1 (or S_2), then S_1 holds in 2/3 of the worlds: namely, out of three groups of worlds

$$S_1 \,\&\, S_2 \,\&\, \ldots, \quad S_1 \,\&\, \neg S_2 \,\&\, \ldots, \quad \neg S_1 \,\&\, S_2 \,\&\, \ldots,$$

the statement S_1 is true in two groups:

$$S_1 \,\&\, S_2 \,\&\, \ldots, \quad S_1 \,\&\, \neg S_2 \,\&\, \ldots$$

In this case,

$$I_1 = -\frac{2}{3} \cdot \log_2\left(\frac{2}{3}\right) - \frac{1}{3} \cdot \log_2\left(\frac{1}{3}\right).$$

This value is smaller than 1, since it is well known that Shannon's entropy attains its largest value when the probabilities are equal [3], i.e., in this case, when $p_i = 1 - p_i = 1/2$.

Thus, the above minimum-information criterion leads to the selection of the questions about S_1 or S_2 – exactly what is empirically observed.

Conclusion. Thus, we have indeed explained the above empirical fact about how people usually choose future research questions. Since this usual way corresponds to the constructive approach, we have thus indeed explained why question-based reasoning indeed leads to constructive approach to knowledge.

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