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Other Generalization of Pairwise Expandable Spaces

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Abstract

The purpose of this paper is study the other generalizations of pairwise expandibility such as pairwise boundedly expandable spaces and pairwise discretely expandable spaces.

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1. Introduction

In [7] Katetove introduced a sufficient and necessary condition under which every locally-finite collection of closed subsets of a topological space$(X\ ,\ \tau)$ can be expanded to a locally-finite collection of open subsets of X . L. Krajewski in [6] called such a space to be expandable.
In [3] J .C.Smith and L.L . Krajewski are studied generalizations of this topological space which are Boundedly and discretely expandable spaces as :

Definition 1.1. [3] A collection subset $\tilde{F}\ = \{F_\alpha : \alpha \in \Delta\}$ of a topological space (X, τ) is said to be boundedly locally finite if there exist a natural number n such that each $x \in X$ is contained in an open set U such that U intersects at most n of members of $\tilde{F}$.

Definition 1.2. [3] Let m be an infinite cardinal number , a topological space (X, τ) is called boundedly expandable space , if for every boundedly locally finite $\tilde{F}\ = \{F_\alpha :\ \alpha \in \Delta\}$ of subsets of X with $|\triangle| \leq m$, there exist boundedly locally finite collection $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for all $\alpha \in \Delta$ and for every infinite cardinal number m .

Definition 1.3. [3] Let m be an infinite cardinal, then a bitopological space (X, τ) is called discretely expandable space , if for every discrete collection $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ of subsets of X with $|\triangle| \leq m$, there exist locally finite collection $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for all $\alpha \in \Delta$ and for every infinite cardinal number m .

Also they are proved important theorems in this topic which are :

Theorem 1.4. [3] *If a topological space (X, τ) is expandable , then X is boundedly expandable.*

Theorem 1.5. [3] *If a topological space (X, τ) is expandable , then X is discretely expandable.*

In [4] , Kelly introduced the notion of a bitopological space, i.e. a triple (X, τ_1, τ_2) where X is a non-empty set and τ_1, τ_2 are two topologies on X. He also defined pairwise regular ($P-$regular), pairwise normal ($P-$normal), and obtained generalization of several standard results such as Urysohn′s lemma and Tietze extension theorem. Several authors have since considered the problem of defining compactness for such spaces, see Kim in [5] and Fletcheret in [1]. Also Fletcher in [1] gave the definitions of $\tau_1\tau_2-$open and $P-$open covers in bitopological spaces.
Also Reilly in [8] introduced a characterization of pairwise normal space as in the following theorem :

Theorem 1.6. [8]. *A bitopological space$X = (X, \tau_1, \tau_2)$ is pairwise normal space($P-$normal) if and only if for each τ_i-closed set A and τ_j-open set V containing A , there is a τ_j-open set U such that$A \subset U \subset \overline{U} \subset V$ for $i \neq j, i, j = 1, 2$.*

In [2] jamal oudetallah defined a pairwise expandable space as :

Definition 1.7. [2] Let m be an infinite cardinal , then a bitopological space (X, τ_1, τ_2) is called $\tau_\imath - m-$ expandable space with respect to τ_j if for every $\tau_{\imath-}$ locally finite $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ with $|\triangle| \leq m$, there exist τ_j - locally finite collection $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$for all $\alpha \in \Delta$ and for $i \neq j, i, j = 1, 2$. A bitopological space (X, τ_1, τ_2) is called $\tau_\imath -$ expandable with respect to τ_j.If it is $\tau_\imath -m-$ expandable for every cardinal m and $i \neq \jmath, i, \jmath = 1, 2$. A bitopological space (X, τ_1, τ_2) is called a pairwise expandable ($P-$ expandable) proved that it is $P-$ T_2- space and it is τ_1- expandable with respect to τ_2 and τ_2- expandable with respect to τ_1.

Also jamal oudetallah In [2] study their properties and their relations with other bitopological spaces . several examples are discussed and many will known theorems are generalized concerning pairwise expandable spaces . and we shall investigate subspaces of pairwise expandable space and also bitopological spaces which are related to pairwise expandability .

2. Boundedly and discretely pairwise expandable spaces

Definition 2.1. A collection subset $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ of a bitopological space (X, τ_1, τ_2) is said to be pairwise boundedly locally finite($P-$ boundedly locally finite) if there exist a natural number n such that each $x \in X$ is contained in a τ_1- open set U such that U intersects at most n of members of $\tilde{F}$, or there exist τ_2- open V containing x such that V intersects at most$n$of members of $\tilde{F}$. If n=1 then the $P-$ boundedly locally finite of subsets of a bitopological space (X, τ_1, τ_2) is called a pairwise discretely collection ($P-$ discretely collection).

Definition 2.2. Let m be an infinite cardinal, then a bitopologcal space(X, τ_1, τ_2) is called $\tau_\imath - m-$ boundedly expandable space with respect to τ_j. if for every $\tau_{\imath-}$ boundedly locally finite $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ of subsets of X with $|\triangle| \leq m$, there exist τ_j - boundedly locally finite collection $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$for all $\alpha \in \Delta$ and for $i \neq j, i, j = 1, 2$.
A bitopological space (X, τ_1, τ_2) is called $\tau_\imath -$ boundedly expandable with respect to τ_j. If it is $\tau_\imath -m-$ boundedly expandable for every cardinal m and $i \neq \jmath, i, \jmath = 1, 2$.
A bitopological space (X, τ_1, τ_2) is called a pairwise boundedly expandable($P-$ boundedly expandable,proved that it is $P-$ T_2- space and it is τ_1- boundedly expandable with respect to τ_2 and τ_2- boundedly expandable with respect to τ_1.
It is clearly that every $P-$ boundedly locally finite collection is $P-$ locally finite. Thus , we have the following theorem.

Theorem 2.3. *If a bitopological space*X=(X, τ_1, τ_2) *is* $P-$ *expandable , then* X *is* $P-$ *boundedly expandable.*

Proof. Let $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be a $P-$ boundedly locally finite collection of subsets of X=(X, τ_1, τ_2).Therefore , $\tilde{F}$ is a $P-$ locally finite collection , so by assumption X is τ_i-expandable with respect to τ_j , for $i \neq j, i, j = 1, 2$. So $\tilde{F}$ can be expanded to a τ_j- locally finite collection $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ of open subsets of X , such that $F_\alpha \subset G_\alpha$ for all $\alpha \in \Delta$, hence X is τ_i-boundedly expandable with respect to τ_j , for $i \neq j, i, j = 1, 2$, and so X is $P-$ boundedly expandable. □

Definition 2.4. Let m be an infinite cardinal, then a bitopological space(X, τ_1, τ_2) is called $\tau_\imath - m-$ collectionwise normal space with respect to τ_j , if for every $\tau_{\imath-}$ discrete collection $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ of closed subsets of X with $|\triangle| \leq m$, there exist τ_j - discrete collection $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$for all $\alpha \in \Delta$ and for $i \neq j, i, j = 1, 2$.

A bitopological space (X, τ_1, τ_2) is called $\tau_\imath -$ collectionwise normal with respect to τ_j. If it is $\tau_\imath -m-$ collectionwise normal for every cardinal m and $i \neq \jmath, i, \jmath = 1, 2$.

A bitopological space (X, τ_1, τ_2) is called a pairwise collectionwise normal ($P-$ collectionwise normal) proved that it is $P-$ T_2- space and it is τ_1- collectionwise normal with respect to τ_2 and τ_2- collectionwise normal with respect to τ_1.

Definition 2.5. Let m be an infinite cardinal , then a bitopological space(X, τ_1, τ_2) is called $\tau_\imath - m-$ discretely expandable space with respect to τ_j. if for every $\tau_\imath$-discrete collection $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ of subsets of X with $|\triangle| \leq m$, there exist τ_j - locally finite collection $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$for all $\alpha \in \Delta$ and for $i \neq j, i, j = 1, 2$.

A bitopological space (X, τ_1, τ_2) is called $\tau_\imath$ $-$ discretely expandable with respect to τ_j.If it is $\tau_\imath$ $-m-$ discretely expandable for every cardinal m and $i \neq \jmath, i, \jmath = 1, 2$.

A bitopological space (X, τ_1, τ_2) is called a pairwise discretely expandable($P-$ discretely expandable),proved that it is $P-$ T_2- space and it is τ_1- discretely expandable with respect to τ_2 and τ_2- discretely expandable with respect to τ_1.

Theorem 2.6. *If a bitopological space$X$=(X, τ_1, τ_2) is $P-$ expandable , then X is $P-$ discretely expandable.*
The Proof follows immediately from definition (2.5) .

Theorem 2.7. *Let X=(X, τ_1, τ_2) be $P-$ normal space . Then X is $P-$ collection normal if and only if it is $P-$ discretely expandable .*

Proof. Suppose X is $P-$ collection normal . Let $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be $\tau_\imath - m-$ discrete collection of subsets of X such that $|\triangle| \leq m$ and $i \neq j, i, j = 1, 2$ where m be an infinite cardinal . So by $P-$ collectionwise normality of X , there exist a τ_j - discrete collection $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for all $\alpha \in \Delta$. So $\tilde{G}$ is τ_j- bounded collection and hence it is τ_j-locally finite collection thus X is τ_i-m- discretely expandable for any infinite cardinal m, therefore X is τ_i- discretely expandable and so it is P-discretely expandable .
Conversely , let $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be $\tau_\imath - m-$ discrete collection of closed subsets of X such that $|\triangle| \leq m$ and $i \neq j, i, j = 1, 2$ where m be an infinite cardinal . Since X is P-discretely expandable , there exist a τ_j - locally finite $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for each $\alpha \in \Delta$.Define

$$K_\alpha = G_\alpha - \bigcup \{F_\beta : \beta \neq \alpha\}$$

So K_α is τ_j - open subset of X and $F_\alpha \subset K_\alpha$ for all $\alpha \in \Delta$. So by P- normality of a space X , there exist τ_j - collection $\tilde{V} = \{V_\alpha : \alpha \in \Delta\}$ of open subsets of X such that

$$F_\alpha \subset V_\alpha \subset \overline{V_\alpha} \subset K_\alpha$$

for each $\alpha \in \Delta$. Define

$$U_\alpha = V_\alpha - \bigcup\{\overline{V_\beta} : \beta \neq \alpha\}$$

for each $\alpha \in \Delta$.Then $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ is a τ_j - mutually disjoint collection of open subsets of X , so it is τ_j - discrete collection of open subsets of X , thus a space X is $\tau_\iota - m-$-collectionwise normal for any infinite cardinal m hence it is τ_i- collectionwise normal for all $i \neq j, i, j = 1, 2$. So X is P- collectionwise normal . □

Theorem 2.8. *A bitopological space* X=(X, τ_1, τ_2) *is* P- *boundedly expandable if and only if it is* P- *discretely expandable .*

Proof. in this proof consider m be an infinite cardinal and $i \neq j, i, j = 1, 2$. Suppose X is P-boundedly expandable.Let $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be $\tau_\iota - m-$ discrete collection of subsets of X such that $|\triangle| \leq m$. By assumption and since every $\tau_\iota - m-$ discrete collection is $\tau_\iota - m-$ bounded locally finite the result follows.

Conversely , the proof is by induction on n .

(i) For n=1 , let $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be $\tau_\iota - m - 1-$ bounded locally finite of closed subsets of X , so $\tilde{F}$ is a $\tau_\iota - m-$ discrete collection of subsets of X so by assumption the result is hold .

(ii) Assume that every $\tau_\iota - m-$ locally finite $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ of closed subsets of X with

$$\sup\{ord(x, \tilde{F}) : x \in X\} \leq K$$

can be expandable to τ_j - locally finite $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ of open subsets of X , for every $k = 1, 2, ..., n-1$.

(iii) Suppose $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be $\tau_\iota - m-$ locally finite with $sup\{ord(x, \tilde{F}) :$ x $\in$ X$\}$ = n. It is clear that for each x $\in$ X ,

$$N(x) = X - \bigcup\{F \in \tilde{F} : x \notin F\}$$

is a τ_i- open set containing x and intersects at most n members of $\tilde{F}$. Now define $\tilde{B_n} = \{B \subseteq \Delta : x \notin F\}$, and for each $B \in \tilde{B_n}$ let $K_B = \bigcap_{\beta \in B} F_\beta$, so K_B is a a τ_i- closed set and consider $\tilde{H_n} = \{K_B : B \in \tilde{B_n}\}$. Then $\tilde{H_n}$ is a $\tau_\iota - m - 1-$ discrete collection of closed subsets of X . For if $N(x)$ intersects two distinct members of $\tilde{H_n}$,then $N(x)$ must be intersect at least n+1distinct members of $\tilde{F}$. So by P-discretely expandibility of X ,there exist a τ_j- locally finite $\tilde{G_n} = \{G_B : B \in \tilde{B_n}\}$ of open subsets of X such that $K_B \subseteq G_B$ for all $B \in \tilde{B_n}$. Define $G = \bigcup_{B \in \tilde{B_n}} G_B$ and $F^* = \{F_\alpha - G : \alpha \in \Delta\}$, then G is τ_j- open set and

F^* is a $\tau_i - m-$ locally finite of open subsets of X and

$$sup\{ord(x, F^*) : \mathrm{x} \in \mathrm{X}\} \leq \mathrm{n} - 1$$

so by assumption of induction , there exist a τ_j- locally finite $\tilde{V} = \{V_\alpha : \alpha \in \Delta\}$ such that $F_\alpha - G \subseteq V_\alpha$, for each $\alpha \in \Delta$. Now define

$$G_\alpha = V_\alpha \cup \{\bigcup G_B : K_B \subseteq F_\alpha\}$$

then G_α is τ_j- open set in X and $F_\alpha \subseteq G_\alpha$ for each $\alpha \in \Delta$. Since $\tilde{V}$ and $\tilde{G_n}$ are τ_j-locally finite collections , for each $x \in X$ there is τ_j- open set $U(x)$ containing x such that $U(x)$ intersects only finitely many members of $\tilde{V}$ and $\tilde{G_n}$. Also for a fixed $B \in \tilde{B_n}$, we have $K_B \subseteq F_\alpha$ occurs if and only if $\alpha \in B$. Therefore , $U(x)$ intersects only finitely many G_α and hence $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ τ_j-locally finite of open subsets of X, thus $\tau_i - m-$ locally finite collection $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ can be expandable to τ_j- locally finite $\tilde{G}$ for every infinite cardinal m . So X is τ_i- bounded expandable for all $i \neq j, i, j = 1, 2$, and hence it is $P-$ boundedly expandable.

□

Definition 2.9. A bitopological space (X, τ_1, τ_2) is called τ_i- countably paracompact space with respect to τ_j if every countable open cover $\tilde{U}$ of τ_i- subsets of X, has a τ_j- countable locally finite open refinement for $i \neq j, i, j = 1, 2$.

A bitopological space (X, τ_1, τ_2) is called a pairwise countably paracompact ($P-$ countably paracompact) proved that it is $P-$ T_2- space and it is τ_1- countably paracompact with respect to τ_2 and τ_2- countably paracompact with respect to τ_1 .

Theorem 2.10. *Let X=(X, τ_1, τ_2) be a $P-$ normal space . If it is $P-$ countably paracompact space ,then every countable cover $\{U_k\}$ of τ_i- subsets of X has a τ_j- refinement $\{V_k\}$ with $\overline{V_k} \subset U_k$ for all $k \in \mathrm{N}$.*

Proof. Let $\{U_k\}$ be any countable cover of τ_i- subsets of X , Since X is $P-$ paracompact then there is a τ_j- locally finite open refinement $\tilde{W}$, so $\tilde{W}$ is point finite refinement . Now for all $W \in \tilde{W}$ let $g(W) = U_k$. Then $W \subset U_k$, and let $G_k = \bigcup_{g(W)=U_K} W$. Then $\{G_k\}$ is a point finite cover of τ_i- subsets of X and $G_k \subset U_k$. Since X is $P-$ countably paracompact and $P-$ normal space then by theorem (1.6) , X has a τ_j- refinement $\{V_k\}$ such that $\overline{V_k} \subset G_k \subset U_k$, hence $\overline{V_k} \subset U_k$ for all $k \in \mathrm{N}$. □

Theorem 2.11. *Let X=(X, τ_1, τ_2) be a $P-$ normal space . If it is $P-$ countably paracompact space ,then for every decreasing sequence $\{F_k\}$ of τ_i-*

closed subsets of X with $\bigcap_{k=1}^{\infty} F_k = \phi$, there is a sequence $\{G_k\}$ of τ_j- open sets with $\bigcap_{k=1}^{\infty} G_k = \phi$ such that $F_k \subset G_k$ for all $k \in \mathrm{N}$.

Proof. Suppose a $P-$ normal space X is $P-$ countably paracompact space. Let $\{F_k\}$ be a decreasing τ_i- closed subsets of X with $\bigcap_{k=1}^{\infty} F_k = \phi$. Let $U_k = X - F_k$, then $\{U_k\}$ is a τ_i- countable covering of X . So by theorem (2.10) , there is a sequence of τ_j- open sets say $\{V_k\}$ with $\overline{V_k} \subset U_k$. Let $G_k = X - \overline{V_k}$, then $\{G_k\}$ is a sequence of τ_j- open sets in X such that $F_k \subset G_k$ and since $\bigcup_{k=1}^{\infty} \overline{V_k} = X$ then $\bigcap_{k=1}^{\infty} G_k = \phi$. □

Theorem 2.12. *Let X=(X, τ_1, τ_2) is P- normal space , then X is $P-$countably paracompact if and only if it is $P - w_o-$expandable .*

Proof. Let $\tilde{A} = \{A_n : n \in \mathrm{N}\}$ be a τ_i- locally finite collection of subsets of X. We must find a τ_j- locally finite collection $\tilde{G} = \{G_n : n \in \mathrm{N}\}$ of open subsets of X such that $A_n \subset G_n$ for all $i \neq j, i, j = 1, 2$. Let

$$F_k = \bigcup \{\overline{A_p} : p \geq k\}$$

Since $\tilde{A}$ is τ_i- locally finite ,F_k is τ_i- closed set for each k ,$F_k \supset F_{k+1}$ and $\overline{A_k} \subset F_k$, then for $x \in X$, x belongs to only a finite number of the $\overline{A_n}$ say $\overline{A_{n_1}}$,....,$\overline{A_{n_{t_x}}}$. Set

$$m_x = 1 + Max\{i_1, i_2, ..., i_{t_x}\}$$

then for $k \geq m_x$, $x \notin \overline{A_k}$ and hence $x \notin F_{m_x}$ thus $\bigcap F_k = \phi$ and since X is $P-$ countably paracompact and $P-$ normal, then by theorem (2.11) there is a τ_j- sequence $\{H_k\}$ of open subsets of X such that $F_k \subset H_k$ for each k and $\bigcap H_k = \phi$. Now $F_1 \subset H_1$ and since X is $P-$ normal there is τ_j- open set G_1 such that $F_1 \subset G_1 \subset \overline{G_1} \subset H_1$. Proceeding inductive , suppose that for each positive integer $p \leq n$ τ_j open set G_p have been defined such that $G_1 \supset G_2 \supset .... \supset G_n$,and such that

$$F_p \subset G_p \subset \overline{G_p} \subset H_p, (p = 1, 2, ..., n)$$

Now $F_{n+1} \subset F_n \subset G_n$ and $F_{n+1} \subset H_{n+1}$. Therefore $F_{n+1} \subset G_n \cap H_{n+1}$. By the $P-$ normality of X from theorem (1.6) , there is a τ_j open set G_{n+1} such that

$$F_{n+1} \subset G_{n+1} \subset \overline{G_{n+1}} \subset G_n \cap H_{n+1}$$

consequently

$$F_{n+1} \subset G_{n+1} \subset \overline{G_{n+1}} \subset H_{n+1}$$

and $G_{n+1} \subset G_n$. We shall now show that $\{G_k\}$ is τ_j- locally finite . Let $x \in X$ and since $\bigcap H_k = \phi$ there is an t_x such that $x \notin H_{t_x}$. since $H_{t_x} \supset G_{t_x}$ then $x \notin \overline{G_{t_x}}$,there is a τ_j- open set V_x , such that $V_x \cap G_{t_x} = \phi$ because

$G_k \subset G_{t_x}$ thus V_x meets at most $G_1, G_2, ..., G_{t_x-1}$. Hence $\{G_k\}$ is locally finite for each k and since $A_k \subset \overline{A_k} \subset F_k \subset G_k$ then X is $\tau_i - w_o-$ expandable space for all $i \neq j, i, j = 1, 2$, so X is $P - w_o-$ expandable space .
conversely, let$\tilde{R} = \{ R_n : n \in \mathrm{N}\}$ be a $P-$ countably open covering of X. Put

$$\mathrm{S}_n = \bigcup\{R_p : p = 1, 2, ..., n\}$$

Let $A_1 = S_1$ and $A_n = S_n - S_{n-1}$, $n = 2, 3, ...$. Evidently $A_n \subset R_n$ for each n . If $x \in X$ then$x \in R_n$ for some n , If n_x is the smallest such n then $x \in A_{n_x}$. Hence $\tilde{A} = \{A_n : n \in \mathrm{N}\}$ is a refinement of $\tilde{R}$. Finally , observe that $R_n \cap A_k = \phi$ if $k > n$. Hence $\tilde{A}$ is locally finite . By $P - w_o-$ expandibility of X there is a τ_j- or τ_j- locally finite collection $\tilde{G} = \{G_n\}$ of open subsets of X such that$A_n \subset G_n$ for each n . let $U_n = R_n \cap G_n$. If $x \in X$ then for some n we have $x \in A_n$. But $A_n \subset R_n$ and $A_n \subset G_n$. Hence $x \in U_n = R_n \cap G_n$, then for $\tilde{U} = \{U_n\}$ is a parallel refinement of $\tilde{R}$. Moreover , $\tilde{U}$ is $P-$ locally finite so for $x \in X$ there is $N(x)$ open set containing x such that $N(x) \cap U_n \neq \phi$ which implies that $N(x) \cap G_n \neq \phi$,thus $\tilde{U}$ is a $P-$ locally finite open parallel refinement of $\tilde{R}$, thus X is $P-$ countably paracompact . □

Corollary 2.13. *An $P-$ expandable space is $P-$ countably paracompact . The proof follows immediately from theorem (2.12)*

Theorem 2.14. *A bitopological space $X = (X, \tau_1, \tau_2)$ is $P-$ expandable space if and only if X is $P-$ discretely expandable and $P-$ countably paracompact space.*

Proof. Suppose X is $P-$ expandable space . By theorem (2.6) X is $P-$ discretely expandable , and by corollary (2.13) X is $P-$ countably paracompact. Conversely , let $i \neq j$, $i, j = 1, 2$ and let $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be a $\tau_i - m$ locally finite of closed subsets of X for any infinite cardinal m, we show that $\tilde{F}$ is expandable to τ_j- locally finite of open subsets of X. For each $n \geq 0$, define

$$S_n = \{x \in X : ord(x, \tilde{F}) \leq n\}$$

It is easy to show that $\{S_n : n = 0, 1, 2, ...\}$ is a $\tau_i - m-$ countable open cover of X. Since X is $P-$ countably paracompact and $S_n \subseteq S_{n+1}$, for each n, there exist a τ_j- locally finite open cover of X, say $\{U_n : n = 0, 1, 2, ...\}$, such that $U_n \subseteq \overline{U_n} \subseteq S_n$,for each$\tau_j-$locally finite open collection $\{V_n : n = 0, 1, 2, ...\}$such that

$$U_n \subseteq \overline{U_n} \subseteq V_n \subseteq S_n$$

for each n Now define

$$F_n = \{\overline{U_n} \cap F : F \in \tilde{F}\} = \{F(n, \alpha) : \alpha \in \Delta\}$$

So that F_n is a $\tau_i - m-$ bounded locally finite of closed subsets of X. Since X is $P-$ discretely expandable and hence $P-$boundedly expandable there exist a τ_j- locally finite collection $\tilde{G}_n = \{G(n, \alpha) : \alpha \in \Delta\}$ such that $F(n, \alpha) \subseteq G(n, \alpha)$, for each $\alpha \in \Delta$ and $G(n, \alpha) \subseteq V_n$, for each n. Now define $G_\alpha =$

$\bigcup_{n=1}^{\infty} G(n,\alpha)$ for each $\alpha \in \Delta$, so $F_\alpha \subseteq G_\alpha$ and $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ is a τ_j- locally finite of open subsets of X, thus $\tau_i - m$- locally finite collection $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ can be expandable to τ_j- locally finite $\tilde{G}$ for every infinite cardinal m. So X is τ_i- expandable for all $i \neq j$, $i, j = 1, 2$ and hence it is $P-$ expandable .

□

References

[1] Fletcher, P., Hoyle, H.B. and Patty, C.W., The Comparison of Topologies, *Duke Math. J.*, **36** (1969), 325-331. https://doi.org/10.1215/s0012-7094-69-03641-2

[2] Jamal Oudetallah, On expandability in bitopological spaces, *JP Journal of Geometry and Topology*, to appear.

[3] J. C. Smith and Krajewskiy, L.L., Expandability and collectionwise normality, *Trans. Amer. Math. Soc.*, **160** (1971), 437-451. https://doi.org/10.1090/s0002-9947-1971-0284966-5

[4] Kelly, J. C., Bitopological Spaces, *Proc. London Math. Soc.*,**13** (1963), 71-89. https://doi.org/10.1112/plms/s3-13.1.71

[5] Kim,Y. W., Pairwise Compactness, *Publ. Math. Debrecen*, **15** (1968), 87-90.

[6] L.L. Krajewski, Expanding locally finite collections, *Can. J. Math.*, **23** (1971), 58-68. https://doi.org/10.4153/cjm-1971-006-3

[7] M. Katetove, Extension of locally - finite covers, *Colloq. Math.*, **6** (1953) 145-151. (in Russian)

[8] Reilly, I., Bitopological Local Compactness, *Indagationes Mathematicae* (proceedings), **75** (5) (1972), 407-411. https://doi.org/10.1016/1385-7258(72)90037-6

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