

Expected Value-Based Power Options Pricing Under Uncertain Circumstance

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Abstract

Power options provide market participants with great flexibility and extensive leverage over ordinary options. In this paper, we investigate the power option pricing problem based on uncertainty theory, in which the price of the underlying asset follows the Ornstein-Uhlenbeck model involving an uncertain fractional differential equation, and we derive the pricing formulas of European power options. Finally, some numerical experiments are performed to illustrate the results.

Mathematics Subject Classification: 91B25

Keywords: Power option, Uncertainty theory, Ornstein-Uhlenbeck model, Uncertain fractional differential equation (UFDE)

1 Introduction

With the rapid development of financial markets, people's demand for financial products is increasing. As the core of financial derivatives, an option contract is a right rather than an obligation that gives the investors of the contract to purchase the underlying asset at a specific price, the strike price and a certain period of time. Power options gave the market participants great flexibility and leverage over ordinary options.

At present, mathematics play an increasingly important role in the financial field. The theory of option pricing can be traced back to the beginning of the 20th century. Black and Scholes [1] used the Ito's formula in options

pricing according to the theory of stochastic analysis. They promoted the application and development of mathematics in the finance markets. In the modern stock markets, investors evaluate the potential of a stock based on its development trend. Liu [4] proposed a mathematical theory called uncertainty theory to study the uncertainty. Based on the uncertainty theory, Zhu [10] gave an essential result which was called the equation of optimality in uncertain optimal control. Sheng and Zhu [8] studied an optimistic value model for a class of uncertain optimal control problems, and applied the model to solve the portfolio problem.

Nowadays, the application of differential equation in options pricing is increasing. Zhang et al. [9] studied the pricing of power options and they deduced the pricing formulas of power options. Dai et al. [2] calculated the option pricing formulas based on the Ornstein-Uhlenbeck exponential model. Zhu [11] introduced the concepts of the Riemann-Liouville type and Caputo type of uncertain fractional differential equations (UFDEs). UFDEs are used to describe the memory and historical properties of systems. Inspired by the above researches, we will use the UFDEs to study the pricing of power options under the Ornstein-Uhlenbeck model.

The layout of this paper is as follows. The second part mainly introduces some basic concepts and results. In the third part, we will introduce the Ornstein-Uhlenbeck model with an UFDE. In the fourth part, we present the option pricing formulas based on the UFDEs, and we give examples to calculate the options price. In the fifth part, a brief conclusion will be given.

2 Preliminary notes

Liu [4] introduced some basic concepts in uncertainty theory. The most fundamental concepts of these are four basic axioms, the expected value and the definition of an uncertain differential equation. UFDEs have different definitions, the most common ones are Caputo type and Riemann-Liouville type. Zhu [11] introduced the concepts of the two types of UFDEs based on the uncertainty theory. The Caputo type of UFDE with initial value conditions is defined by

$$\begin{cases} {}^c D^p X_t = f(t, X_t) + g(t, X_t) \frac{dC_t}{dt}, & t \in [0, T] \\ X_t^{(k)}|_{t=0} = x_k, & k = 0, 1, \dots, n-1 \end{cases} \quad (1)$$

where C_t is a Liu process, ${}^c D^p X_t$ represents the Caputo fractional derivative of p ($n-1 < p \leq n$). The definitions of fractional derivative can be seen in [11]. The above UFDE with initial value conditions is equivalent to an uncertain fractional integral equation. Therefore, the Caputo type of UFDE (1) has a

solution as follows

$$X_t = \sum_{k=0}^{n-1} \frac{x_k t^k}{\Gamma(k+1)} + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, X_s) ds + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} g(s, X_s) dC_s. \quad (2)$$

In order to solve the UFDEs, Lu and Zhu [5] investigated the relations between the solution and α -path of an UFDE. Assume that $\alpha \in (0, 1)$, a Caputo type of UFDE (1) with initial value conditions is called to have an α -path X_t^α which solves the associated fractional differential equation

$${}^c D^p X_t^\alpha = f(t, X_t^\alpha) + |g(t, X_t^\alpha)| \Phi^{-1}(\alpha) \quad (3)$$

with the same initial value conditions, where $\Phi^{-1}(\alpha)$ is an inverse standard normal uncertainty distribution, that is

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}. \quad (4)$$

If the function $J(x_1, x_2, \dots, x_l)$ increases strictly with respect to $x_1, x_2, \dots, x_m$ and decreases strictly with respect to $x_{m+1}, x_{m+2}, \dots, x_l$, then, by Lu and Zhu [5], we have

$$E[J(X_1, X_2, \dots, X_l)] = \int_0^1 J(X_1^\alpha, X_2^\alpha, \dots, X_m^\alpha, X_{m+1}^{1-\alpha}, \dots, X_l^{1-\alpha}) d\alpha \quad (5)$$

where X_i is the solution of (1) and X_i^α is the corresponding α -path.

3 Uncertain stock model

According to the long-term trend of stock prices, Peng and Yao [7] proposed an uncertain mean-reverting model. In order to prevent the influence of excessive growth of stock prices on the financial market, Dai et al. [2] use the logarithmic function to convert it as a nonlinear model, which is called Ornstein-Uhlenbeck model.

The research of the above model is based on uncertain differential equations. However, in the actual market, future asset price are not only related to current price, but also related to the price for a quite long period of time. Fractional differential derivatives can describe the process of memory and genetic characteristics well. Lu et al. [6] studied the pricing problem of Asian options in which the underlying asset price follows an UFDE. However, the change of the underlying asset price is very complex. The asset price under

the Ornstein-Uhlenbeck stock model based on uncertain fractional differential equation of the Caputo type is as follows:

$$\begin{cases} dX_t = rX_t dt \\ {}^c D^p Y_t = \mu(1 - a \ln(Y_t))Y_t + \sigma Y_t \frac{dC_t}{dt} \\ Y_t^{(k)}|_{t=0} = y_k, \quad k = 0, 1, \dots, n-1. \end{cases} \quad (6)$$

According to Eq. (2) the solution of model (6) is as the followings

$$\begin{aligned} Y_t = & \sum_{k=0}^{n-1} \frac{y_k t^k}{\Gamma(k+1)} + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \mu(1 - a \ln(Y_s)) ds \\ & + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \sigma Y_s dC_s. \end{aligned} \quad (7)$$

Because the model (6) is nonlinear, there is no analytical solution to the UFDE. Lu and Zhu [5] proposed a numerical algorithm for solving an UFDE with initial value conditions.

4 Expected value-based model

In the section, we introduce the pricing problem of European power options according to expected value-based model, which reflects the average trend of Ornstein-Uhlenbeck model (6). According to the idea of option pricing in Zhang et al. [9], the price of the European power call option with an order m , an expiration time T and a strike price K is $f_c = \exp(-rT)E[(Y_T^m - K)^+]$ and the price of the European power put option is $f_p = \exp(-rT)E[(K - Y_T^m)^+]$. Then we can get the following theorems.

Theorem 4.1. *Assume that a European power call option with an order m of the stock model (6) has an expiration time T and a strike price K . Then the European power call option pricing formula is*

$$f_c = \exp(-rT) \int_0^1 [(Y_T^\alpha)^m - K]^+ d\alpha \quad (8)$$

Proof. According to Eq. (3), we can get the following equation

$${}^c D^p Y_t^\alpha = \mu(1 - a \ln(Y_t^\alpha))Y_t^\alpha + \sigma Y_t^\alpha \Phi^{-1}(\alpha) \quad (9)$$

According to Eq. (5), the pricing formula of power call option is

$$f_c = E[J(Y_T)] = \int_0^1 J(Y_T^\alpha) d\alpha = \exp(-rT) \int_0^1 [(Y_T^\alpha)^m - K]^+ d\alpha. \quad (10)$$

Thus, the pricing formula (8) for model (6) can be verified.

Theorem 4.2. *Assume that a European power put option with an order m of the stock model (6) has an expiration time T and a strike price K . Then the European power put option pricing formula is*

$$f_p = \exp(-rT) \int_0^1 [K - (Y_T^\alpha)^m]^+ d\alpha. \quad (11)$$

Proof. The proof of the theorem is similar to that of Theorem 4.1.

Theorem 4.3. *The price f_c of the European power call option is increasing with respect to the power m when $Y_T^\alpha > 1$, decreasing when $Y_T^\alpha < 1$, and decreasing with respect to the riskless interest rate r and the strike price K .*

Proof. (i) Noting that as the power m increases, the value of the $(Y_T^\alpha)^m$ increases when $Y_T^\alpha > 1$, the f_c of formula (10) increases respect to m . When $Y_T^\alpha < 1$, the value of the $(Y_T^\alpha)^m$ decreases with respect to the power m , that is to say f_c decreases with respect to the power m . (ii) Note that the value of the integration $\int_0^1 [(Y_T^\alpha)^m - K]^+ d\alpha$ keeps unchanged as the riskless interest rate r changes. Because the $\exp(-rT)$ is decreasing with respect to the riskless interest rate r , so is the price f_c of the European power call option. (iii) As the strike price K increases, the value of the integral function $[(Y_T^\alpha)^m - K]^+$ decreases, so the price f_c of the European power call option decreases. The theorem is proved.

For Theorem 4.1, by using the inverse uncertainty distribution of Y_T , an algorithm is proposed to calculate the Y_T^α . Then according to the pricing formula, f_c can be obtained.

Algorithm: (The pricing of European power call option)

Step 1: Give T ; Let $\alpha = 0$ and the step length $\Delta\alpha$; Set $Z = 0$ and $M = 0$;

Step 2: Set $\alpha \leftarrow \alpha + \Delta\alpha$; $M = M + 1$;

Step 3: By using the predictor-corrector method proposed by Diethelm et al. [3] at each grid t_j , $j = 1, 2, \dots, N$ (Here we divide the interval $[0, T]$ into N pieces, and the step length of the method be $h = T/N$), calculate the following fractional differential equation with the initial conditions

$${}^c D^p Y_t^\alpha = f(t, Y_t^\alpha) + |g(t, Y_t^\alpha)| \Phi^{-1}(\alpha), \quad Y_t^{(k)}|_{t=0} = y_k, \quad k = 0, 1, \dots, n-1;$$

Step 4: Output iteration results Y_T^α ;

Step 5: If $(Y_T^\alpha)^m - K < 0$, set $Z(\alpha) = 0$; if $(Y_T^\alpha)^m - K > 0$, set $Z(\alpha) = (Y_T^\alpha)^m - K$; Set $Z = Z + Z(\alpha)$;

Step 6: If $\alpha + \Delta\alpha < 1$, return to step 2;

Step 7: Calculate f_c as following: $f_c = \exp(-rT)Z/M$.

Example 4.4. *Assume the uncertain stock model (6) has the parameters as follows: the power $m = 2$, the riskless interest rate $r = 0.08$, the log-diffusion*

$\sigma = 0.35$ and $\mu = 0.08$, $a = 2$, $p = 0.1$. Consider a European call option with a strike price $K = 4$, $Y_0 = 3$ and an expiration time $T = 2$. The parameters of the above algorithm are $N = 100$, $h = T/N = 0.02$. Then the European power call option price is $f_c = 5.49$.

For Example 4.4, we keep other parameters remain unchanged and then we can calculate the price of the European power call option with different m . The results are shown in Table 1.

Table 1: The price of European power call options with different value of m

Methods	$m=0.8$	$m=1$	$m=1.3$	$m=1.5$	$m=1.8$
f_c	0.03	0.14	0.73	1.53	3.52
Methods	$m=2.1$	$m=2.3$	$m=2.5$	$m=2.8$	$m=3$
f_c	6.73	9.87	14.14	23.68	33.19

Then we keep the other variables unchanged and only change the value of order p . The options price are calculated and the results are shown in the following Table 2.

Table 2: The price of European power call options with different value of p

Methods	$p=0.1$	$p=0.2$	$p=0.3$	$p=0.4$	$p=0.5$
f_c	5.49	5.94	6.34	6.69	6.96
Methods	$p=0.6$	$p=0.7$	$p=0.8$	$p=0.9$	$p=1$
f_c	7.17	7.32	7.41	7.45	7.44

It can be seen that the price of the European power call option fluctuates greatly with the slight change of m in Table 1. When the stock price has big fluctuation in the market, the value of m can be adjusted appropriately to mitigate the impact of market fluctuations. Besides, the price of the options varies with the change of p in Table 2. Accurately, f_c increases with respect to p when $0 < p < 1$.

Then we will show the influence of some parameters (m, r, K) on the value of f_c by concrete figures. When we explore the relationship between f_c and the parameters, we keep the others parameters unchanged.

Example 4.5. Assume the uncertain stock model (6) has the parameters as follows: the power $m = 2$, the riskless interest rate $r = 0.08$, the log-diffusion $\sigma = 0.35$ and $\mu = 0.08$, $a = 2$, $p = 0.1$. Consider a European call option with a strike price $K = 0.8$, $Y_0 = 0.2$ and an expiration time $T = 2$. The parameters of the above algorithm are $N = 100$, $h = T/N = 0.02$. Then the European power call option price is $f_c = 0.049$.

Based on Example 4.5, we use the figures to verify the monotonicity between f_c and the parameters (m, r, K) . As shown in Figure 1. With the change of m , f_c firstly decreases and then increases, which is consistent with Theorem 4.3(i). Next, we verify the monotonicity of f_c with respect to r and K respectively, keeping the other parameters of Example 4.5 unchanged, and we get the Figure 2. We can see f_c decreases with respect to r and K , respectively, which is consistent with actual financial markets.

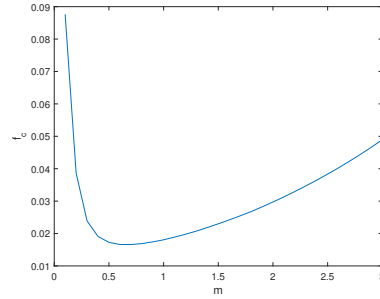


Figure 1: European power call option price f_c with respect to m

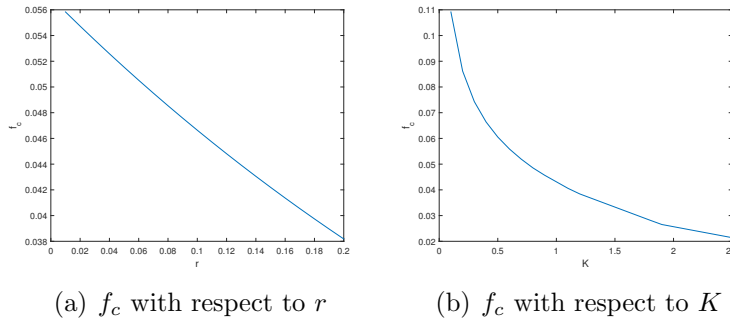


Figure 2: European power call option price f_c with respect to r and K

5 Conclusion

This paper mainly studies the pricing of European power options in uncertain financial markets. The exponential of power options can prevent the financial market changing caused by the violent fluctuations of the underlying assets. The UFED is used to describe the changing process of stock price, and the pricing formulas of European power options are calculated based on expected value criterion. In the future, we will explore whether this model can be applied to real financial markets.

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