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Some Properties of Hamiltonian Ordered Semirings

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Abstract

In this paper, we study the congruence extension properties of Hamiltonian ordered semirings and four kinds of congruence extension properties are discussed. Finally, we also discuss the differences between Hamiltonian semirings and Hamiltonian ordered semirings.

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Keywords: semiring, Hamiltonian semirings, Hamiltonian ordered semirings, congruence extension property

1 Introduction and Preliminaries

Congruence extension has been researched by various authors, both from the point of semigruop and some specific structure, in this article we investigate the congruence extension properties for Hamiltonian ordered semiring, a specific algebra structure. Our research is closely related to those which was carried out in [1]-[10]. Furthermore, we give some congruence extension theorems based on convex subsemirings of Hamiltonian ordered semirings, it is a different view between Hamiltonian semirings and Hamiltonian ordered semirings.

An *ordered semiring* is a partially ordered set $(R, \leq)$ and a semiring $(R, +, \cdot)$ such that for all $a, a' \in R, a \leq a'$ implies $r + a \leq r + a', a + r \leq a' + r, ra \leq ra'$ and $ar \leq a'r$ for all $r \in R$.

An ordered semiring $(S, +, \cdot, \leq_S)$ is called a *subsemiring* of ordered semiring $(R, +, \cdot, \leq_R)$ if $S \subseteq R$ and it is closed with respect to addition and multiplication, and $\leq_S = \leq_R \bigcap (S \times S)$.

We call a quasiorder σ on an ordered semiring R a *compatible quasiorder* if it is compatible with operations and extends the order of R.

Given a quasiorder σ on a poset $(R, \leq)$ and $a, a' \in R$, we write

$$a \underset{\sigma}{\leq} a' \Leftrightarrow (\exists n \in \mathbb{N})(\exists a_1, a_2, \cdots a_n \in R)(a \leq a_1 \sigma a_2 \leq a_3 \sigma \cdots \sigma a_n \leq a').$$

An *order-congruence* on an ordered semiring $R = (R, +, \cdot, \leq)$ is a congruence θ of the semiring $(R, +, \cdot)$ such that the following condition is satisfied:

$$(\forall a, a' \in R)(a \underset{\theta}{\leq} a' \underset{\theta}{\leq} a \Rightarrow a \theta a').$$

If σ is a compatible quasiorder on R then $\theta = \sigma \bigcap \sigma^{-1}$ is an order-congruence. Let us denote the sets of all order-congruences and compatible quasiorders of R by $Con(R)$ and $Cqu(R)$, respectively.

For every subset $H \subseteq R \times R$ there is a smallest order-congruence Θ_H (resp. compatible quasiorder $\sum_H$) on R containing H, which is the intersection of all order-congruences (resp. compatible quasiorders) containing H. This relation Θ_H (resp. $\sum_H$) is called the order-congruence generated by H (resp. the compatible quasiorder generated by H).

Lemma 1.1. *Let R be an ordered semiring, $H \subseteq R \times R$. Then for any $c, c' \in R, c \underset{\Theta_H}{\leq} c'$ if and only if $c \leq c'$ or there exist $x_i, y_i \in R^1, u_i, v_i \in R^0$ and $(a_i, a_i') \in H \bigcup H^{-1}$ for $i \in \{1, 2, \cdots, n\}$ such that we have one sequence*

$$c \leq p_1(a_1) \quad p_2(a_2) \leq p_3(a_3) \quad \cdots \qquad p_n(a_n) \leq c',$$

$$p_1(a_1') \leq p_2(a_2') \cdots \quad p_{n-1}(a_{n-1}') \leq p_n(a_n')$$

for $p_i(a_i) = u_i + x_i a_i y_i + v_i$, $p_i(a_i') = u_i + x_i a_i' y_i + v_i$ from c to c'.

Lemma 1.2. *Let R be an ordered semiring, $H \subseteq R \times R$. Then for any $c, c' \in R, c \Theta_H c'$ if and only if $c = c'$ or there exist $x_i, y_i \in R^1, u_i, v_i \in R^0$ and $(a_i, a_i') \in H \bigcup H^{-1}$ for $i \in \{1, 2, \cdots, n\}$ such that we have one sequence*

$$c \leq p_1(a_1) \quad p_2(a_2) \leq p_3(a_3) \quad \cdots \qquad p_n(a_n) \leq c'.$$

$$p_1(a_1') \leq p_2(a_2') \cdots \quad p_{n-1}(a_{n-1}') \leq p_n(a_n')$$

for $p_i(a_i) = u_i + x_i a_i y_i + v_i$, $p_i(a_i') = u_i + x_i a_i' y_i + v_i$ from c to c' and another similar sequence from c' to c.

Lemma 1.3. *Let R be an ordered semiring and let $H \subseteq R \times R$ be a symmetric relation on R. Then $\Theta_H = \Sigma_H \bigcap (\Sigma_H)^{-1}$.*

2 Congruence Extension Properties of Hamiltonian Ordered Semirings

Definition 2.1. *An ordered semiring R is said to have the congruence extension property (CEP) with respect to a subsemiring S if every order-congruence θ on S is the restriction of some order-congruence Θ on R, i.e., $\Theta \bigcap (S \times S) = \theta$.*

Definition 2.2. *An ordered semiring R is said to have the congruence extension property (QEP) with respect to a subsemiring S if for any compatible quasiorder σ on S is the restriction of some compatible quasiorder Σ on R, i.e., $\Sigma \bigcap (S \times S) = \sigma$.*

Definition 2.3. *An ordered semiring R is said to have the strong congruence extension property (SCEP) with respect to a subsemiring S if for any order-congruence θ on S there is an order-congruence Θ on R such that $\Theta \bigcap (S \times R) = \theta$.*

Definition 2.4. *An ordered semiring R is said to have the strong quasiorder extension property (SQEP) with respect to a subsemiring S if for any compatible quasiorder σ on S there is a compatible quasiorder Σ on R such that $\Sigma \bigcap (S \times S) = \sigma$ and $\Sigma \bigcap \Sigma^{-1} \bigcap (S \times R) = \sigma \bigcap \sigma^{-1}$.*

Proposition 2.5. *Let R be an ordered semiring. We have*
(1) *If R has SQEP, then R has QEP and SCEP;*
(2) *If R has QEP or SCEP, then R has CEP.*

Of course, not every ordered semiring has congruence extension property, the example is as follows.

Example 2.6. *Let $R = \{1, 2, 3, 4\}, (R, +)$ is a left zero semigroup, $(R, \cdot)$ defined by the Calay table:*

$\cdot$	1	2	3	4
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3
4	1	2	2	4

Easily, we can prove that R is a semiring. Defined an order

$$\leq = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2)\}$$

on R. We can prove that R is an ordered semiring. Let $S = \{1, 2, 3\}$. Then S a subsemiring of R, the relation ρ defined $\rho = \{(1, 3), (3, 1)\} \bigcup \Delta_S$ is an order-congruence on S, and the congruence on R generated by ρ contains the pair $(1, 2) = (4, 4)(1, 3)$. Thus its restriction to S is not ρ.

From proposition 2.5 we have the following observation.

Lemma 2.7. *Let S be a subsemiring of an ordered semiring R, θ be an order-congruence on S and Θ be an order-congruence on R. Then $\Theta \bigcap (S \times R) = \theta$ if and only if $[a]_\Theta = [a]_\theta$ for all $a \in S$.*

Definition 2.8. *A subset B of a poset A is up-closed if, for every $b \in A$ and $a \in B, a \leq b$ implies that $b \in B$.*

A subset B of a poset A is down-closed if, for every $a \in A$ and $b \in B, a \leq b$ implies that $a \in B$.

A subset B of a poset A is convex if, for every $a \in A$ and $b, b' \in B, b \leq a \leq b'$ implies that $a \in B$.

Definition 2.9. *We say that an ordered semiring R is*

(1)Hamiltonian if every convex subsemiring S of R is a class of some order-congruence on R.

(2)strongly Hamiltonian if every subsemiring S of R is a class of some order-congruence on R.

Also strongly Hamiltonian ordered semiring is not productive.

Example 2.10. *Let $R = \{1, 2, 3\}$ with addition and multiplication defined by the Cayley table:*

$+$	1	2	3
1	1	2	3
2	2	2	3
3	3	3	3

$\cdot$	1	2	3
1	1	2	3
2	1	2	3
3	3	3	3

Easily, we can prove that R is a semiring. Defined an order

$$\leq = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2)\}$$

on R. We can prove that R is an ordered semiring. To see that R is not a strongly Hamiltonian ordered semiring, consider the subsemiring $S = \{1, 3\}$ and $\rho = \{(1, 3), (3, 1)\} \bigcup \Delta_R$. But $(1, 3)(2, 2) = (2, 3)$, so ρ is not a order-congruence on R, which is a contradiction. Then S is not a class of any order-congruence on R.

Proposition 2.11. *Let R be an ordered semiring. If σ is an order-congruence on R then every σ-class is a convex subset of R.*

From proposition 2.11 we have the following observation.

Proposition 2.12. *An ordered semiring is strongly Hamiltonian if and only if it is Hamiltonian and its subsemirings are convex.*

Theorem 2.13. *An ordered semiring has the SCEP if and only if it is strongly Hamiltonian and has the CEP.*

Proof. Necessity: If ordered semiring R has the SCEP then it clearly has the CEP. Let S be a subsemiring of R. Consider an order-congruence $\theta = S \times S$ of S. By SCEP and lemma 2.7, there exists an order-congruence Θ of R such that $S = [s]_\theta = [s]_\Theta$ for all $s \in S$. So S is a congruence class of Θ. Hence R is strongly Hamiltonian.

Sufficiency: Suppose that R is strongly Hamiltonian and has the CEP. Let S be a subsemiring of R and let θ be an order-congruence of S. By CEP, there exists an order-congruence Θ^* of R such that $\Theta^* \bigcap (S \times S) = \theta$. Since R is strongly Hamiltonian, there also exists an order-congruence Ψ of R such that $S = [s]_\Psi$ for all $s \in S$. So putting $\Theta = \Theta^* \bigcap \Psi$ is an order-congruence of R and we have

$$[s]_\Theta = [s]_{\Theta^*} \bigcap [s]_\Psi = [s]_{\Theta^*} \bigcap S = [s]_\theta$$

for all $s \in S$. By lemma 2.7, R has the SCEP. □

Corollary 2.14. *If an ordered semiring has the SCEP then all its subsemirings are convex.*

For compatible quasiorders we have the following analogue result of theorem 2.13.

Theorem 2.15. *An ordered semiring has the SQEP if and only if it is strongly Hamiltonian and has the QEP.*

Proof. Necessity: If R has the SQEP then it has the SCEP, so it is strongly Hamiltonian by theorem 2.13. Obviously SQEP implies QEP.

Sufficiency: Suppose that R is strongly Hamiltonian and has the QEP. Let S be a subsemiring of R and let σ be a compatible quasiorder of S. By QEP, there exists a compatible quasiorder Σ^* of R such that $\Sigma^* \bigcap (S \times S) = \sigma$. Since R is strongly Hamiltonian, there also exists order-congruence Ψ of R such that $S = [s]_\Psi$ for all $s \in S$. The relation $\underset{\Psi}{\leq}$ is a compatible quasiorder on R. We put $\alpha = \Sigma^* \bigcap \underset{\Psi}{\leq}$ is compatible quasiorder of R and verify the conditions of definition 2.4.

(1) If $(a, b) \in \alpha$, $a, b \in S$ then $(a, b) \in \sigma$. Hence $(a, b) \in \rho$. Then $\alpha \bigcap (S \times S) = \rho$. Conversely, if $(a, b) \in \rho$, $a, b \in S$ then $(a, b) \in \sigma$. Because $S = [a]_\theta$, we obtain $(a, b) \in \theta$. Thus $(a, b) \in \alpha$. Then $\alpha \bigcap (S \times S) = \rho$.

(2) If $(a, b) \in \alpha \bigcap \alpha^{-1} \bigcap (S \times R)$, $a \in S$, $b \in R$ then $(a, b) \in \sigma \bigcap \underset{\theta}{\leq}$. Hence $a \underset{\theta}{\leq} b \underset{\theta}{\leq} a$. Since θ is an order-congruence, we have $(a, b) \in \theta$. Because $S = [a]_\theta$, we obtain $b \in S$. Then $(a, b), (b, a) \in \alpha \bigcap (S \times S) = \rho$. Thus $(a, b) \in \rho \bigcap \rho^{-1}$. Conversely, if $(a, b) \in \rho \bigcap \rho^{-1}$, $a, b \in S$ then $(a, b) \in \sigma \bigcap \sigma^{-1}$, $(a, b) \in \theta \bigcap \theta^{-1}$. Thus $(a, b) \in \alpha \bigcap \alpha^{-1} \bigcap (S \times R)$. Hence $\alpha \bigcap \alpha^{-1} \bigcap (S \times R) = \rho \bigcap \rho^{-1}$. □

Lemma 2.16. *Let S be a subsemiring of an ordered semiring R, let Θ be an order-congruence of R, θ be a class of Θ and assume that θ is a reflexive, binary relation on S. Then for every $x, y, u, v \in R, s, s' \in S$, if $s\theta s'$, then*

$$(u + xsy + v, u + xs'y + v)\Theta(u + xsy + v, u + xsy + v)$$
$$(u + xsy + v, u + xs'y + v)\Theta(u + xs'y + v, u + xs'y + v).$$

If, in addition, $u + xsy + v \in S$ then also $(u + xsy + v, u + xs'y + v) \in \theta$.

Lemma 2.17. *Let S be a convex subsemiring of an ordered semiring R, where $R\times R$ is Hamiltonian. If ψ is either an order-congruence or a compatible quasiorder on S which is also a convex subset of $S \times S$, then the compatible quasiorder Σ on R generated by ψ satisfies the following conditions:*

$$\Sigma \bigcap \Sigma^{-1} \bigcap (S \times R) = \psi \bigcap \psi^{-1}.$$

Lemma 2.18. *An order-congruence θ on an ordered semiring R is a convex subset of $R \times R$ if and only if θ extends the order of R.*

Combining lemma 1.3 and 2.17 we immediately get the following result.

Corollary 2.19. *Let S be a convex subsemiring of an ordered semiring R, where $R \times R$ is Hamiltonian. If θ is an order-congruence on S which is a convex subset of $S\times S$ then there exists an order-congruence Θ on R such that $\Theta \bigcap (S \times R) = \theta$.*

If $(R, +, \cdot, \leq)$ is an ordered semiring, then a congruence on the underlying semiring $(R, +, \cdot)$ will be called an *algebraic congruence* on R.

Proposition 2.20. *Let S be an up-closed or down-closed subsemiring of an ordered semiring R. Assume that R has the algebraic SCEP with respect to S. Then R has the SCEP with respect to S as an ordered semiring.*

Theorem 2.21. *Let S be an up-closed subsemiring of an ordered semiring R, where $R \times R$ is Hamiltonian. If σ is a compatible quasiorder on S which is a convex subset of $S \times S$ then there exsits a compatible quasiorder Σ on R such that $\Sigma \bigcap (S \times S) = \sigma$ and $\Sigma \bigcap \Sigma^{-1} \bigcap (S \times R) = \sigma \bigcap \sigma^{-1}$.*

Proof. Let S be an up-closed subsemiring of an ordered semiring R and σ be a convex subset of $S \times S$. Take $\Psi \in Con(R \times R)$ to be an order-congruence such that σ is a congruence class of Ψ and let Σ be the compatible quasiorder on R generated by the subset $\sigma \in R \times R$. We can easily shown that $\Sigma \bigcap \Sigma^{-1} \bigcap (S \times R) = \sigma \bigcap \sigma^{-1}$ by lemma 2.17.

For the other equality we notice that $\sigma \subseteq \Sigma \bigcap (S \times S)$ is obvious. Take $(c, c') \in \Sigma \bigcap (S \times S)$. Then we have a sequence

$$c \leq u_1 + x_1 s_1 y_1 + v_1 \quad u_2 + x_2 s_2' y_2 + v_2 \leq \quad \cdots \qquad u_n + x_n s_n y_n + v_n \leq c'.$$

$u_1 + x_1 s_1' y_1 + v_1 \leq u_2 + x_2 s_2 y_2 + v_2 \cdots \quad u_{n-1} + x_{n-1} s_{n-1}' y_{n-1} + v_{n-1} \leq u_n + x_n s_n y_n + v_n$.
Since S is up-closed, $(u_1 + x_1 s_1 y_1 + v_1) \in S$, By lemma 2.16, $(u_1 + x_1 s_1 y_1 + v_1)\sigma(u_1 + x_1 s_1' y_1 + v_1)$. Analogously $(u_i + x_i s_i y_i + v_i)\sigma(u_i + x_i s_i' y_i + v_i)$ for all $i \in \{1, ..., n\}$. Since σ extends the order of S, we get

$$c\sigma(u_1 + x_1 s_1 y_1 + v_1)\sigma(u_1 + x_1 s_1' y_1 + v_1)\sigma(u_2 + x_2 s_2 y_2 + v_2)\sigma \cdots \sigma(u_n + x_n s_n' y_n + v_n)\sigma c',$$

and hence $c\sigma c'$. Thus $\Sigma \bigcap (S \times S) \subseteq \sigma$, as required. □

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