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Continuity of Posets via Function Spaces[1]

Luoshan Xu

Department of Mathematics, Yangzhou University
Yangzhou Jiangsu 225002, P.R. China

Abstract

In this paper some kind of function spaces between posets are consedered. Characterizations of continuity of posets via this kind of function spaces are given. Main results are: (1) The function space from a poset to a CD-lattice L forms an L-fuzzy topology; (2) A poset is a continuous poset iff for some non-singleton CD-lattice L, the function space from the poset to L in pointwise order forms a CD-lattice iff for all CD-lattices L, function spaces from the poset to L are all CD-lattices.

Mathematics Subject Classifications: 06A11; 06B35; 54C35; 54D45

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1 Introduction

In 1972, Dana Scott introduced the notion of continuous lattices in order to provide models for the semantics of programming languages (see [9]). Later, a more general notion of continuous posets was introduced and extensively studied (see [2]-[10]). It should be noted that a distinctive feature of the theory of continuous posets is that many of the considerations are closely interlinked with topological ideas. The Scott topology, as an order-theoretical topology, is of fundamental importance in domain theory. Xu in [10] gave an interesting characterization that a poset L is continuous iff the lattice $\sigma(L)$ of all Scott-open subsets of L is a CD-lattice. With this understanding and some more mathematical considerations, we introduce a new concept of function spaces from posets to CD-lattices / L-domains. With these concepts we can draw

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more links among mathematical structures. It is well known that the unit interval $\mathbb{I} = [0,1]$ is a CD-lattice and the set $\mathbb{I}^X$ of all maps from a set X to $\mathbb{I}$ in pointwise order is also a CD-lattice. Now a question arises: the set $[\mathbb{I} \to \mathbb{I}]$ of all Scott continuous maps from $\mathbb{I}$ to $\mathbb{I}$ in pointwise order is a CD-lattice or not? The answer to this question turns out to be yes. To justify this, we develop some ideals and successfully give a comprehensive characterization theorem for the continuity of the underlying poset.

2 Preliminaries

We quickly recall some basic notions and results (see, for example, [2, 8]).

Let P be a poset. A *principal ideal* (*resp., principal filter*) is a set of the form $\downarrow x = \{y \in P : y \leq x\}$ (resp., $\uparrow y = \{x \in P : y \leq x\}$). A subset A of P is said to be *bounded above* if A has an upper bound in P. A poset in which every directed set has a sup is called a *directed complete poset* (in short, *dcpo*). A complete lattice which is completely distributive is called a *CD-lattice*. A poset in which every principal ideal is a complete lattice is called an L-poset.

An order-reversing involution on a poset P is a map $' : P \to P$ which is anti-order preserving with $x'' = x$ for all $x \in P$. On the unit interval $\mathbb{I}$, there is a canonical involution $' : \mathbb{I} \to \mathbb{I}$ defined by $x' = 1 - x$.

We say that x *approximates* y in a poset P, written $x \ll y$ if whenever D is directed and $\sup D \geq y$, then $x \leq d$ for some $d \in D$. The poset P is said to be *continuous* if every element is the directed sup of elements that approximate it.

A continuous dcpo is also called a domain. A domain which is also an L-poset is called an L-domain. It is well known that in a continuous poset, the approximating relation $\ll$ has the interpolation property

(INT): $x \ll z \Rightarrow \exists y \in P$ such that $x \ll y \ll z$.

An upper set U of P is said to be Scott open if for any directed set $D \subseteq P$, $\vee^{\uparrow} D \in U$ implies $U \cap D \neq \emptyset$. All the Scott open sets of P form a topology, called the *Scott topology*, denoted $\sigma(P)$. The complement of a Scott open set is called *a Scott closed set*. If a map $f : P \to Q$ is continuous from spaces $(P, \sigma(P))$ to $(Q, \sigma(Q))$, then f is said to be *Scott continuous*.

The following two propositions are well known and can be found in [2].

Proposition 2.1. *In a continuous poset P, for each $x \in P$, the set $\twoheaduparrow x = \{y \in P : x \ll y\}$ is a Scott open set, and these sets form a basis for the Scott topology of P.*

Proposition 2.2. *A map $g : P \to Q$ between posets is Scott continuous iff g preserves all existing directed sups.*

The poset of all Scott continuous maps from posets P to Q in pointwise order will be denoted by $[P \to Q]$. The topology generated by all the principal filters $\uparrow y$ (resp., all the principal ideals $\downarrow x$) of P as subbasic closed sets is called the *lower topology* (resp., *upper topology*) and denoted $\omega(P)$ (resp., $\nu(P)$). The common refinement $\sigma(P) \vee \omega(P)$ of $\sigma(P)$ and $\omega(P)$ is called the *Lawson topology*, denoted $\lambda(P)$.

Lemma 2.3. (see [2]) *The Scott topology is equal to the upper topology on every CD-lattice.*

Proof. By Proposition VII-3.5 in [2], every CD-lattice is a hypercontinuous lattice. And by Theorem VII-3.4 in [2], the Scott topology is the upper topology in a hypercontinuous lattice. The assertion of the lemma is thus clear. $\square$

Lemma 2.4. (see [8]) *Let X be a set and L a CD-lattice. Then the set L^X of all maps from X to L in pointwise order is a CD-lattice and order isomorphic to the product lattice $\prod_{x \in X} L$.*

Definition 2.5. (see [8]) Let L be a complete lattice.

(1) *The complete way-below relation* "$\lhd$" in L is defined for all a, $b \in L$, $a \lhd b \Leftrightarrow (\forall S \subseteq L, b \leq \sup S \Rightarrow \exists s \in S,\ a \leq s)$;

(2) If $a \in L$ and $B \subseteq \{t \in L : t \lhd a\}$ with $\sup B = a$, then B is called a *minimal set* of a in L. We will use $\beta(a)$ to denote any one of the minimal sets of $a \in L$.

(3) A non-top element $p \in L$ is called a *prime* if $\forall s$, $t \in L$ with $s \wedge t \leq p$ implies $s \leq p$ or $t \leq p$.

(4) A non-zero element $x \in L$ is called a *join irreducible element* (also called a *co-prime*) if $\forall s$, $t \in L$ with $x \leq s \vee t$ implies $x \leq s$ or $x \leq t$. The set of all join irreducible elements of L is denoted by $M(L)$. If a minimal set $\beta(a) \subseteq M(L)$, then the minimal set is called a *molecular minimal set*. We will use $\beta^*(a)$ to denote one of the molecular minimal sets of $a \in L$.

Lemma 2.6. (see [8]) *Let L be a complete lattice. Then L is a CD-lattice iff for all $a \in L$, a has a minimal set iff for all $a \in L$, a has a molecular minimal set.*

Lemma 2.7. (see [10]) *A poset P is continuous iff its Scott topology is a CD-lattice.*

3 Scott function spaces and intrinsic fuzzy topologies

Definition 3.1. Let P be a poset and L an L-domain (with an order-reversing involution). All the Scott continuous maps $f : P \to L$ in pointwise order is denoted by $[P \to L]$, and called the *Scott function space* of P to L.

The following two propositions give operational properties with respect to joins and meets for the Scott function space $[P \to L]$.

Proposition 3.2. *Let P be a poset and L a CD-lattice (with an order-reversing involution). If $f_i : P \to L$ ($i \in J$, an index set) is a family of Scott continuous maps, then $f : P \to L$ defined for all $x \in P$, $f(x) = \bigvee f_i(x)$, is also a Scott continuous map.*

Proof. Since L is a CD-lattice, by Lemma 2.3, the Scott topology $\sigma(L)$ is equal to the upper topology $\nu(L)$. And to show the continuity of f, it suffices to show for any $t \in L$, $f^{-1}(\downarrow t)$ is a Scott closed set in P. It is easy to check that $f^{-1}(\downarrow t) = \bigcap {f_i}^{-1}(\downarrow t)$. As a meet of some Scott closed sets of P, $f^{-1}(\downarrow t)$ is Scott closed, as desired. □

Proposition 3.3. *Let P be a poset and L a CD-lattice (with an order-reversing involution). If f and $g : P \to L$ are Scott continuous, then $h : P \to L$ defined for all $x \in P$, $h(x) = f(x) \wedge g(x)$, is also Scott continuous.*

Proof. Let A be a directed set of P. Then by the definition of h, continuity of f and g, as well as the completely distributivity of L, we have

$$\begin{aligned} h(supA) &= f(\sup A) \wedge g(\sup A) \\ &= (\sup f(A)) \wedge (\sup g(A)) \\ &= sup_{(a,b)\in A\times A}(f(a) \wedge g(b)) \\ &= sup_{c\in A}(f(c) \wedge g(c)) \\ &= sup_{c\in A} h(c) \end{aligned}$$

So, $h(\sup A) = \sup h(A)$. By Proposition 2.2, h is Scott continuous, as desired. □

Theorem 3.4. *Let P be a poset and L a CD-lattice (with an order-reversing involution). Then the Scott function space forms a stratified L-fuzzy topology on P in the sense of [8] that all the constant maps, joins of families of opens and meet of two opens are open, called the (intrinsic) Scott L-fuzzy topology of P.*

Proof. It follows from Proposition 3.2 and 3.3 □

Theorem 3.5. *Let P be a poset and L a CD-lattice (with an order-reversing involution). Then the Scott L-fuzzy topology $[P \to L]$ is just the induced L-fuzzy topology of the Scott topology (in the sense of [8]) of P, hence the name Scott L-fuzzy topology.*

Proof. It is easy to check by Lemma 2.3 that $[P \to L] = \omega_L(\sigma(P)) := \{f | f : P \to L, \forall t \in L, f^{-1}(\downarrow t) \in \sigma^*(P), \}$, where $\sigma^*(P)$ is the lattice of all Scott closed sets of P. □

4 Continuity of posets via Scott function spaces

It follows from [7] that for a set X and an L-domain L, the set $[X \to L] = L^X$ in pointwise order is an L-domain. If X is replaced with a continuous poset P, then $[P \to L]$ is also an L-domain. If in addition, the L-domain L is a CD-lattice, is $[P \to L]$ a CD-lattice? To answer this question we need some preparations.

The following result (see [1]) is due to Raney.

Lemma 4.1. ([1, P.248, Theorem 20] and [2, Ex. IV-3.32]) *A poset P is a CD-lattice iff there is an embedding $e : P \to \mathbb{I}^M$ preserving arbitrary sups and infs for some set M.*

Lemma 4.2. *Let P be a poset and $\mathbf{2}$ be the CD-lattice $\{0.1\}$ with $0 < 1$. Then the Scott function space $[P \to \mathbf{2}] \cong \sigma(P)$.*

Proof. Straightforward. □

Lemma 4.3. *Let* **POSET** *be the category of posets and Scott continuous maps. For a poset P, define $[P \to \cdot] : \mathbf{POSET} \to \mathbf{POSET}$ such that for $T \in ob(\mathbf{POSET})$, one has $[P \to \cdot](T) = [P \to T]$ and for $(f : T \to S) \in mor(\mathbf{POSET})$, one has $[P \to \cdot](f) = [id \to f]$, where $[id \to f] : [P \to T] \to [P \to S]$ is a map such that $\forall h \in [P \to T]$, $[id \to f](h) = f \circ h$. Then $[P \to \cdot]$ is a functor.*

Proof. Straightforward. □

For this functor, as projections are all Scott continuous, we immediately have

Lemma 4.4. (see [2, Lemma II-2.9]) *For a poset P, the functor $[P \to \cdot]$ preserves arbitrary products.*

Theorem 4.5. (The characterization theorem) *For a poset P, the following statements are equivalent:*

(1) *P is a continuous poset;*
(2) *$[P \to L]$ is a CD-lattice for all CD-lattice L;*
(3) *$[P \to L]$ is a CD-lattice for some nonsingleton CD-lattice L.*

Proof. (1) ⇒ (2): Let L be a CD-lattice. Then by Lemma 4.1, there is an embedding $e : L \to \mathbb{I}^M$ preserving arbitrary sups and infs for some set M. Define $E : [P \to L] \to [P \to \mathbb{I}^M]$ such that $\forall f \in [P \to L]$, $E(f) = e \circ f$. Then it is easy to see that E is an embedding preserving arbitrary sups and infs. By Lemma 4.4, the functor $[P \to \cdot]$ preserves products and $[P \to \mathbb{I}^M] \cong [P \to \mathbb{I}]^M$. So, by Lemma 2.4 and Lemma 4.1, it suffices to show that $[P \to \mathbb{I}]$ is a CD-lattice. By Lemma 2.6, we need to show that $\forall f \in [P \to \mathbb{I}]$, f has a minimal set. To show this, note that $\sigma(P)$ is a CD-lattice by (1) and Lemma 2.7. Thus every element of $\sigma(P)$ has a molecular minimal set. So, for any $a \in I$, pick a join irreducible element $U \in \sigma(P)$ such that $U \lhd f^{-1}(\Uparrow a)$. By [7, Lemma 2.4], we have $F(a, U, f) \ll f$, where $F(a, U, f)$ is defined by

$$F(a,U,f)(x) = \begin{cases} a, & \text{if } x \in U; \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if $g, h \in [P \to \mathbb{I}]$ with $g \vee h \geq f$, then it is easy to check that

$$\begin{aligned} h^{-1}(\Uparrow a) \cup g^{-1}(\Uparrow a) &= (h \vee g)^{-1}(\Uparrow a) \\ &\supseteq f^{-1}(\Uparrow a) \supseteq U. \end{aligned}$$

Since U is join irreducible, $U \subseteq h^{-1}(\Uparrow a)$ or $U \subseteq g^{-1}(\Uparrow a)$ holds. Thus $F(a, U, f) \leq h$ or $F(a, U, f) \leq g$. Combining this with $F(a, U, f) \ll f$, we see that $F(a, U, f) \lhd f$. Then it is straightforward to check that

$$\bigvee_{a \in I} \Big(\bigvee_{b \in \Downarrow a} \Big(\bigvee_{V \in \beta^*(f^{-1}(\Uparrow a))} F(b, V, f) \Big) \Big) = f.$$

This shows that $\{F(b, V, f) | b < a \in I, V \in \beta^*(f^{-1}(\Uparrow a))\}$ is a molecular minimal set of f. Thus $[P \to \mathbb{I}]$ is a CD-lattice and so is $[P \to L]$.
(2) $\Rightarrow$ (3): trivial.
(3) $\Rightarrow$ (1): Suppose that there is a nonsingleton CD-lattice L such that $[P \to L]$ is a CD-lattice. Define $e : \mathbf{2} = \{0, 1\} \to L$ such that $e(0) = 0$ and $e(1) = 1$, where 0 is the least and 1 is the greatest elements of $\mathbf{2}$ and L. Then e preserves arbitrary sups and infs. Define $E : [P \to \mathbf{2}] \to [P \to L]$ such that $\forall f \in [P \to \mathbf{2}]$, $E(f) = e \circ f$. Then E is an embedding preserving arbitrary sups and infs. By (3), $[P \to L]$ is a CD-lattice, and $[P \to \mathbf{2}]$ as a sub-complete-lattice of $[P \to L]$ is also a CD-lattice. By Lemma 4.2, $\sigma(P) \cong [P \to \mathbf{2}]$ is a CD-lattice. By Lemma 2.7, P is a continuous poset, as desired. □

It is easy to see by Lemma 4.2 that Theorem 4.5 is a generalization of Lemma 2.7.

As $\mathbb{Q}$ (the rational numbles), $\mathbb{R}$ and $\mathbb{I}$ in the original orders are all continuous posets, we have immediately the following

Corollary 4.6. *The Scott function spaces* $[\mathbb{Q} \to \mathbb{I}]$, $[\mathbb{R} \to \mathbb{I}]$ *and* $[\mathbb{I} \to \mathbb{I}]$ *of all Scott continuous functions from* $\mathbb{Q}$, $\mathbb{R}$ *and* $\mathbb{I}$, *respectively, to* $\mathbb{I}$, *are all CD-lattices.*

References

[1] R. Balbes and P. Dwinger, *Distributive Lattices*, University of Missouri Press, 1974.

[2] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, D. S. Scott, *Continuous Lattices and Domains*, Cambridge University Press, Cambridge, 2003. https://doi.org/10.1017/cbo9780511542725

[3] J. A. Goguen, L-fuzzy sets, *J. Math. Anal. Appl.*, **18** (1967), 145-174. https://doi.org/10.1016/0022-247x(67)90189-8

[4] J. D. Lawson, The duality of continuous posets, *Houston J. of Mathematics*, **5** (3) (1979), 357-386.

[5] J. D. Lawson and Luoshan Xu, When does the class $[\mathbf{A} \to \mathbf{B}]$ consist of continuous domains?, *Topology and Its Applications*, **130** (2003), 91-97. https://doi.org/10.1016/s0166-8641(02)00213-4

[6] J. D. Lawson and Luoshan Xu, Posets having continuous intervals, *Theoretical Computer Science*, **316** (2004), 89-103. https://doi.org/10.1016/j.tcs.2004.01.025

[7] Y. Liu and J. Liang, Solution to two problems of J. D. Lawson and M. Mislove, *Topology and Its Applications*, **69** (1996), 153-164. https://doi.org/10.1016/0166-8641(95)00083-6

[8] Guojun Wang, *Theory of L-fuzzy topological spaces*, Shaanxi Normal University Press, Xi'an, 1988. (in Chinese)

[9] D. S. Scott, Continuous Lattice, Lecture Notes in Mathematics 274, Springer-Verlag, Berlin, 1972, 97-136. https://doi.org/10.1007/bfb0073967

[10] Luoshan Xu, Continuity of Posets via Scott Topology and Sobrification, *Topology and Its Applications*, **153** (2006), 1886-1894. https://doi.org/10.1016/j.topol.2004.02.024

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