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$(L, M)-$Smooth Ideal Structures

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Abstract

In this paper we establish the concept of $(L, M)-$smooth ideal and $(L, M)-$smooth local map of $(L, M)-$smooth ideals with $(L, M)-$smooth topologies . We study a new sort of $(L, M)-$smooth ideal and $(L, M)-$smooth local map namely $(L, M)-$smooth ideal and $r-$smooth open local map. Many of its characterizations, properties and connections between it and other corresponding fuzzy notions are studied.

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Keywords: $(L, M)-$smooth topological spaces, $(L, M)-$ smooth ideal, $r-$smooth open local map, $r-$smooth quasi $H-$closed, $r-$smooth $L-$open, $r-$smooth $L-$closed and $F\mathcal{I}-$continuous

1 Introduction

. Šostak [16] introduced the notion of $(L, \wedge)-$ fuzzy topological spaces as a generalization of $L-$topological spaces. Höhle and Šostak [11] substitute a complete quasi-monoidal lattice (or GL-monoid) instead of a completely distributive lattice or a unit interval. Also they introduced the concept of $L-$filters for a complete quasi-monoidal lattice L which is the dual of fuzzy ideals. Many authors[1-10,12-15,17,18,19,20] studied the structures of fuzzy topology and the structures of fuzzy filters,.fuzzy ideals and fuzzy smooth ideals. Ramadan, Abdel-Sattar and Kim [14] studied the concept of a smooth ideals in [0,1]-smooth topological spaces. Abdel-Sattar [2] studied some structures of $(L, \odot)-$ smooth topological spaces and their properties. In [6] he studied the concept of $L-$smooth ideals and L-smooth ideal bases in $(L, \odot)-$ smooth topological spaces. In [19] we study some results of fuzzy ideals since the lattice is the closed interval $I = [0, 1]$. In this paper we establish the structures of $(L, M)-$smooth ideal by $(L, M)-$ smooth topological spaces [22].The concept of $r-$smooth open local map, new space namely $(L, M)-$ smooth ideal topological spaces and many of its characterizations are studied.

2 Preliminaries

Throughout this paper, let X be a nonempty set. $L = (L, \leq, \vee, \odot, ', 0, 1)$ denotes a completely distributive lattice with order-reversing involution $'$ which has the least and greatest elements,say 0 and 1, respectively. Let L^X be the family of all L-fuzzy subsets of X. For $\alpha \in L$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$. A

fuzzy point, x_t for $t \in L$ is an element of L^X such that, for $y \in X$,

$$x_t(y) = \begin{cases} t & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

The set of all fuzzy points in X is denoted by $Pt(X)$. A fuzzy point $x_t \in \lambda$ iff $t \leq \lambda(x)$. A fuzzy set λ is quasi-coincident with μ, denoted by $\lambda q\ \mu$, if there exists $x \in X$ such that $\lambda(x)+\mu(x) > 1$. If λ is not quasi-coincident with μ, we denote $\lambda\ \overline{q}\ \mu$. All the other notations and the other definitions are standard in fuzzy set theory.

Definition 2.1. [11] A triple $(L, \leq, \odot)$ is called a strictly two-sided, commutative quantale (stsc-quantale, for short) iff it satisfies the following properties:

(L1) $L = (L, \leq, 1, 0)$ is a complete lattice.

(L2) $(L, \odot)$ is a commutative semigroup.

(L3) $a = a \odot 1$, for each $a \in L$.

(L4) $\odot$ is a distributive over arbitrary joins, i.e. $(\bigvee_{i\in\Gamma} a_i\ ) \odot b = \bigvee_{i\in\Gamma}\ (a_i\ \ \odot b)$.

Example 2.2 [11] (1) Each frame is a stsc- quantale. In particular, the unite interval $(\ [0, 1], \leq, \vee, \wedge, 0, 1)$ is a stsc-quantale.

(2) The unit interval with a left-continuous t-norm t, $(\ [0, 1], \leq, t)$,is a stsc-quantale.

(3) Every GL-monoid is a stsc-quantale.

(4) Define a binary operation $\odot$ on $[0, 1]$ by $x \odot y = \max\{0, x+y-1\}$.Then $(\ [0, 1], \leq, \odot)$ is a stsc-quantale.

Lemma 2.3 [**21**] **For each** $x, y, z, x_i, y_i, w \in L$,we have the following properties:

(1) $\top \to x = x,\ \bot \odot x = \bot$,

(2) if $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z, x \to y \leq x \to z$ and $z \to x \leq y \to x$,

(3) $x \leq y$ iff $x \to y = \top$,

(4) $(\wedge_i y_i)^* = \vee_i y_i^*, (\vee_i y_i)^* = \wedge_i y_i^*$,

(5) $x \to (\wedge_i y_i) = \wedge_i (x \to y_i)$,

(6) $(\vee_i x_i) \to y = \wedge_i (x_i \to y)$,

(7) $x \odot (\vee_i y_i) = \vee_i (x \odot y_i)$,

(8) $(\wedge_i x_i) \oplus y = \wedge_i (x_i \oplus y)$,

(9) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,

(10) $(x \odot y) = (x \to y^*)^*, x \oplus y = x^* \to y$ and $x \to y = y^* \to x^*$,

(11) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$,

(12) $x \to y \leq (x \odot z) \to (y \odot z)$ and $(x \to y) \odot (y \to z) \leq x \to z$,

(13) $(x \to y) \odot (z \to w) \leq (x \oplus z) \to (y \oplus w)$,

(14) $x \odot (x \to y) \leq y$ and $y \leq x \to (x \odot y)$,

(15) $(x \vee y) \odot (z \vee w) \leq (x \vee z) \vee (y \odot w) \leq (x \oplus z) \vee (y \odot w)$,

(16) $\vee_{i \in \Gamma} x_i \to \vee_{i \in \Gamma} y_i \geq \wedge_{i \in \Gamma} (x_i \to y_i),\ \wedge_{i \in \Gamma} x_i \to \wedge_{i \in \Gamma} y_i \geq \wedge_{i \in \Gamma} (x_i \to y_i)$,

(17) $(x \odot y) \odot (z \oplus w) \leq (x \odot z) \oplus (y \odot w)$,

(18) $z \to x \leq (x \to y) \to (z \to y)$ and $(y \to z) \leq (x \to y) \to (x \to z)$.

(19) $(x^*)^* = x$

All algebraic operations on L can be extended pointwise to the set L^X as follows: $\forall x \in X, \forall \lambda, \mu \in L^X$

(1) $\lambda \leq \mu \Longleftrightarrow \lambda(x) \leq \mu(x)$,

(2) $(\lambda \odot \mu)(x) = \lambda(x) \odot \mu(x)$,

(3) $(\lambda \to \mu)(x) = \lambda(x) \to \mu(x)$.

Definition 2.4. [2,11,12] A mapping $\mathcal{T} : L^X \to M$ is called $(L, M)-$smooth topology on X if it satisfies the following conditions:

(O1) $\mathcal{T}(\underline{0}) = \mathcal{T}(\underline{1}) = 1$, where $\underline{0}(x) = 0$ and $\underline{1}(x) = 1$ for all $x \in X$.

(O2) $\mathcal{T}(\mu_1 \odot \mu_2) \geq \mathcal{T}(\mu_1) \odot \mathcal{T}(\mu_2)$, for any $\mu_1, \mu_2 \in L^X$.

(O3) $\mathcal{T}(\bigvee_{i \in \Gamma} \mu_i) \geq \bigwedge_{i \in \Gamma} \mathcal{T}(\mu_i)$, for any $\{\mu_i\}_{i \in \Gamma} \subset L^X$.

An $(L, M)-$smooth topological spaces is called enriched if

(P) $\mathcal{T}\ (\alpha \odot \mu) \geq \mathcal{T}(\mu)$,for any $\mu \in L^X$, and $\alpha \in M$.

The pair $(X, \mathcal{T})$ is called $(L, M)-$ smooth topological spaces (resp. enriched $(L, M)-$ smooth topological spaces)

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be two $(L, M)-$smooth topological spaces and $f : X \to Y$ be a mapping. Then f is said to be smooth *continuous* iff $\mathcal{T}'(\mu) \leq \mathcal{T}(f^{-1}(\mu))$ for each $\mu \in L^Y$.

Definition 2.5. [11] Let $(L,*)$ and $(L,\odot)$ be a stsc-quantale. An operation $\odot$ dominates $*$ if it satisfies: $\forall x_1, x_2, y_1, y_2 \in L \Rightarrow (x_1 * y_1) \odot (x_2 * y_2) \geq (x_1 \odot x_2) * (y_1 \odot y_2)$

Definition 2.6. [11] Let $(L,*)$ and $(L,\odot)$ be a stsc-quantale. An operation $\odot$ dominates $*$ if it satisfies: $\forall x_1, x_2, y_1, y_2 \in L \Rightarrow (x_1 * y_1) \odot (x_2 * y_2) \geq (x_1 \odot x_2) * (y_1 \odot y_2)$

Example 2.7. [11]

(1) For any left- continuous t-norm $*, \wedge$ dominates $*$ because $(x_1 * y_1) \wedge (x_2 * y_2) \geq (x_1 \wedge x_2) * (y_1 \wedge y_2)$.

(2) Define t-norm as $x \odot y = \frac{x\ y}{x\ + y\ - x\ y}$ and $x * y = x\ \ y$. Then $\odot$ dominates $*$.

Definition 2.8. [14] If X is a set, then an ideal on X is a nonempty $D^* \subset 2^X$ satisfying the following conditions:

1- $X \notin D^*$.

2- If $A, B \in D^* \Rightarrow A \cup B \in D^*$.

3- If $B \in D^*$ and $A \subset B \Rightarrow A \in D^*, i.e., D^*$ is a lower set.

Definition 2.9. [14] If X is a set, then a preideal on X is a nonempty $D \subset I^X$ satisfying the following conditions:

1- $\underline{1} \notin D$.

2- If $\lambda, \mu \in D \Rightarrow \lambda \vee \mu \in D$.

3- If $\mu \in D$ and $\lambda \leq \mu \Rightarrow \lambda \in D$.

Definition 2.10. A map $I : L^X \times M \to L^X$ is called $(L,M)-$smooth interior operator on X iff I satisfies the following condations :

$(I1)$ $I(\underline{1}\ , r) = \underline{1}$ for all $r \in M$.

$(I2)$ $I(\lambda, r) \leq \lambda$ for all $r \in M$.

$(I3)$ If $\lambda \leq \mu$ and $r \leq s$, then $I(\lambda, s) \leq I(\mu, r)$.

$(I4)$ $I(\lambda \odot \mu,\ r \odot s) \geq\ I(\lambda, r) \odot I(\mu, s)$.

The pair $(X, I\ )$ is called $(L,M)-$smooth interior space.

The $(L,M)-$ smooth interior operator I is called topological if

$$I\ (I\ (\lambda, r)) \geq I(\lambda, r),\ \forall \lambda \in L^X,\ r \in M.$$

Let I_1 and I_2 be two $(L,M)-$smooth iterior operators on X. We say that I_1 is finer than I_2 (I_2 is coarser than I_1), denoted by $I_2 \leq I_1$, if $I_2(\lambda, r) \leq I_1(\lambda, r)$ for all $\lambda \in L^X, r \in M$.

Theorem 2.11. Let (X, τ) be $(L,M)-$ smooth topological spaces .Then for each $r \in M, \mathcal{A} \in L^X$, we define the operator $C_\tau : L^X \times M \to L^X$ as follows:

$$C_\tau(\mathcal{A}, r) = \bigwedge \{\mathcal{B} \in L^X | \mathcal{A} \leq \mathcal{B}, \tau(\underline{1} - \mathcal{B}) \geq r\}.$$

For each $\mathcal{A},\mathcal{B}\in L^X$ and $r,s\in M$, the operator C_τ satisfies the following conditions:

(C1) $C_\tau(\underline{0},r)=\underline{0}$.

(C2) $\mathcal{A}\leq C_\tau(\mathcal{A},r)$.

(C3) $C_\tau(\mathcal{A},r)\vee C_\tau(\mathcal{B},r)=C_\tau(\mathcal{A}\vee\mathcal{B},r)$.

(C4) $C_\tau(\mathcal{A},r)\leq C_\tau(\mathcal{A},s)$ if $r\leq s$.

(C5) $C_\tau(C_\tau(\mathcal{A},r),r)=C_\tau(\mathcal{A},r)$.

(C6) If $s=\bigvee\{r\in L_0|C_\tau(\mathcal{A},r)=\mathcal{A}\}$, then $C_\tau(\mathcal{A},s)=\mathcal{A}$.

Definition 2.12. Let (X,τ) be $(L,M)-$ smooth topological spaces, for each $\mathcal{A}\in L^X, r\in M$. Then:

(1) $\mathcal{A}$ is called r-smooth regular open (**r-SRO**, for short) iff $\mathcal{A}=int_\tau(C_\tau(\mathcal{A},r),r)$.

(2) $\mathcal{A}$ is called r-smooth preopen (**r-SPO**, for short) iff $\mathcal{A}\leq int_\tau(C_\tau(\mathcal{A},r),r)$..

Definition 2.13. An $(L,M)-$smooth topological spaces (X,τ) is called $r-$smooth regular iff for each $\tau(\mathcal{A})\geq r$ and $r\in M$, $\mathcal{A}=\bigvee\{\mathcal{B}\in L^X|\tau(\mathcal{B})\geq r, C_\tau(\mathcal{B},r)=\mathcal{A}\}$.

Definition 2.14. Let (X,τ) be $(L,M)-$smooth topological spaces, for each $\mathcal{A}\in L^X$, $x_t\in P_t(X)$ and $r\in M$. Then, $\mathcal{A}$ is called $r-$open Q_τ-neighborhood of x_t (for short, $Q_\tau(x_t,r)$) if $x_tq\mathcal{A}$ with $\tau(\mathcal{A})\geq r$.

Definition 2.15. Let $\underline{0}\notin\Theta$ be a subset of L^X. A mapping $\beta:\Theta\to M$ is called $(L,M)-$smooth base on X if it satisfies the following conditions:

(1) $\beta(\underline{1})=1$,

(2) $\beta(\mathcal{A}_1\odot\mathcal{A}_2)\geq\beta(\mathcal{A}_1)\odot\beta(\mathcal{A}_2)$, for all $\mathcal{A}_1,\mathcal{A}_2\in\Theta$.

Definition 2.16. Let $f:(X,\tau)\to(Y,\eta)$ be a mapping. Then,

(1) f is called smooth continuous (S-continuous, for short) iff $\eta(\mathcal{A})\leq\tau(f^{-1}(\mathcal{A}))$, for each $\mathcal{A}\in L^Y$.

(2) f is called smooth precontinuous (SP-continuous, for short) iff $f^{-1}(\mathcal{A})$ is **r-SPO** set for each $\eta(\mathcal{A})\geq r$.

(3) f is called smooth almost continuous iff $\tau(f^{-1}(\mathcal{A}))\geq r$, for each $\mathcal{A}\in L^Y$ with $\mathcal{A}=int_\eta(C_\eta(\mathcal{A},r),r)$.

(4) f is called smooth weakly continuous iff $f^{-1}(\mathcal{A})\leq int_\tau(f^{-1}(C_\eta(\mathcal{A},r)),r)$ for each $\mathcal{A}\in L^Y$ and $r\in M$ with $\eta(\mathcal{A})\geq r$.

(5) f is called smooth strongly continuous iff $f(C_\tau(\mathcal{A},r))\leq f(\mathcal{A})$ for each $\mathcal{A}\in L^X$ and $r\in M$.

3 $(L, M)-$ Smooth ideal and $r-$smooth open local map

Definition 3.1. A mapping $\mathcal{I} : L^X \to M$ is called $(L, M)-$smooth ideal on X if it satisfies the following conditions:

(I_1) $\mathcal{I}(\underline{0}) = 1$, $\mathcal{I}(\underline{1}) = 0$.

(I_2) If $\mathcal{A} \leq \mathcal{B}$, then $\mathcal{I}(\mathcal{B}) \leq \mathcal{I}(\mathcal{A})$, for each $\mathcal{A}, \mathcal{B} \in L^X$.

(I_3) $\mathcal{I}(\mathcal{A} \vee \mathcal{B}) \geq \mathcal{I}(\mathcal{A}) \odot \mathcal{I}(\mathcal{B})$, for $\mathcal{A}, \mathcal{B} \in L^X$.

If $\mathcal{I}_1$ and $\mathcal{I}_2$ are $(L, M)-$smooth ideals on X, we say that $\mathcal{I}_1$ is finer than $\mathcal{I}_2$ ($\mathcal{I}_2$ is coarser than $\mathcal{I}_1$), denoted by $\mathcal{I}_2 \leq \mathcal{I}_1$, iff $\mathcal{I}_1(\mathcal{A}) \leq \mathcal{I}_2(\mathcal{A})$ for $\mathcal{A} \in L^X$.

The triple $(X, \tau, \mathcal{I})$ is called $(L, M)-$smooth ideal topological space ($(L, M)-$sits, for short). For $\alpha \in M$, $(X, \tau_\alpha, \mathcal{I}_\alpha)$ is fuzzy ideal topological space in the sense of Sarkar [18].

Definition 3.2. Let $(X, \tau, \mathcal{I})$ be $(L, M)-$sits and $\mathcal{A} \in L^X$. Then the r-smooth open local map $\mathcal{A}^\star_r(\tau, \mathcal{I})$ of $\mathcal{A}$ is the union of all fuzzy points x_t such that if $\mathcal{B} \in Q(x_t, r)$ and $\mathcal{I}(\mathcal{C}) \geq r$ then there is at least one $y \in X$ for which $\mathcal{B}(y) + \mathcal{A}(y) - 1 > \mathcal{C}(y)$.
there exists $\mathcal{B} \in Q(x_t, r)$ such that for every $y \in X$, $\mathcal{B}(y) + \mathcal{A}(y) - 1 \leq \mathcal{C}(y)$, for some $\mathcal{I}(\mathcal{C}) \geq r$. $\mathcal{A}^\star_r(\tau, \mathcal{I})$ is the set of fuzzy points at which $\mathcal{A}$ does not have the property $r-$fuzzy open locally.

We will occasionally write $\mathcal{A}^\star_r$ or $\mathcal{A}^\star_r(\mathcal{I})$ for $\mathcal{A}^\star_r(\tau, \mathcal{I})$ and it will cause no ambiguity.

Example 3.3. Let $(X, \tau, \mathcal{I})$ be $(L, M)-$sits. The simplest fuzzy ideal on X is $\mathcal{I}^0 : L^X \to M$ where

$$\mathcal{I}^0(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} = \underline{0}, \\ 0, & \text{otherwise.} \end{cases}$$

If we take $\mathcal{I} = \mathcal{I}^0$, for each $\mathcal{A} \in L^X$ we have $\mathcal{A}^\star_r = C_\tau(\mathcal{A}, r)$.

Theorem 3.4. Let (X, τ) be $(L, M)-$ smooth topological space and $\mathcal{I}_1$, $\mathcal{I}_2$ be two $(L, M)-$smooth ideals of X. Then for each $r \in M$ and $\mathcal{A}, \mathcal{B} \in L^X$.

(1) If $\mathcal{A} \leq \mathcal{B}$, then $\mathcal{A}^\star_r \leq \mathcal{B}^\star_r$.

(2) If $\mathcal{I}_1 \leq \mathcal{I}_2$, then $\mathcal{A}^\star_r(\mathcal{I}_1, \tau) \geq \mathcal{A}^\star_r(\mathcal{I}_2, \tau)$.

(3) $\mathcal{A}^\star_r = C_\tau(\mathcal{A}^\star_r, r) \leq C_\tau(\mathcal{A}, r)$.

(4) $(\mathcal{A}^\star_r)^\star_r \leq \mathcal{A}^\star_r$.

(5) $(\mathcal{A}^\star_r \vee \mathcal{B}^\star_r) = (\mathcal{A} \vee \mathcal{B})^\star_r$.

(6) If $\mathcal{I}(\mathcal{B}) \geq r$, then $(\mathcal{A} \vee \mathcal{B})^\star_r = \mathcal{A}^\star_r \vee \mathcal{B}^\star_r = \mathcal{A}^\star_r$.

(7) If $\tau(\mathcal{B}) \geq r$, then $(\mathcal{B} \odot \mathcal{A}^\star_r) \leq (\mathcal{B} \odot \mathcal{A})^\star_r$.

(8) $(\mathcal{A}^\star_r \odot \mathcal{B}^\star_r) \geq (\mathcal{A} \odot \mathcal{B})^\star_r$.

Proof. (1) Suppose there exist $\mathcal{A} \in L^X$ and $r \in M$ such that $\mathcal{A}^\star_r \not\leq \mathcal{B}^\star_r$, there exist $x \in X$ and $t \in M$ such that

$$\mathcal{A}^\star_r(x) \geq t > \mathcal{B}^\star_r(x).$$

Since $\mathcal{B}^\star_r(x) < t$, there exists $\mathcal{D} \in Q(x_t, r)$ with $\mathcal{I}(\mathcal{C}) \geq r$ such that for every $y \in X$, we have,

$$\mathcal{D}(y) + \mathcal{B}(y) - 1 \leq \mathcal{C}(y).$$

Since $\mathcal{A} \leq \mathcal{B}$, $\mathcal{D}(y)+\mathcal{A}(y)-1 \leq \mathcal{C}(y)$. So, $\mathcal{A}^\star_r(x) < t$, and this is a contradiction . Thus, $\mathcal{A}^\star_r \leq \mathcal{B}^\star_r$.

(2) Suppose that, $\mathcal{A}^\star_r(\mathcal{I}_1, \tau) \ngeq \mathcal{A}^\star_r(\mathcal{I}_2, \tau)$, then there exist $x \in X$ and $t \in M$ such that

$$\mathcal{A}^\star_r(\mathcal{I}_1, \tau)(x) < t \leq \mathcal{A}^\star_r(\mathcal{I}_2, \tau)(x).$$

Since $\mathcal{A}^\star_r(\mathcal{I}_1, \tau)(x) < t$, there exists $\mathcal{D} \in Q(x_t, r)$ with $\mathcal{I}_1(\mathcal{C}) \geq r$ such that for every $y \in X$, we have, $\mathcal{D}(y) + \mathcal{A}(y) - 1 \leq \mathcal{C}(y)$. Since $\mathcal{I}_2(\mathcal{C}) \geq \mathcal{I}_1(\mathcal{C}) \geq r$, $\mathcal{D}(y) + \mathcal{A}(y) - 1 \leq \mathcal{C}(y)$. Thus, $\mathcal{A}^\star_r(\mathcal{I}_2, \tau)(x) < t$. It is a contradiction . Thus, $\mathcal{A}^\star_r(\mathcal{I}_1, \tau) \geq \mathcal{A}^\star_r(\mathcal{I}_2, \tau)$.

(3) We show that $\mathcal{A}^\star_r \leq C_\tau(\mathcal{A}, r)$. Suppose that, $\mathcal{A}^\star_r \not\leq C_\tau(\mathcal{A}, r)$, then there exist $x \in X$ and $t \in M$ such that

$$\mathcal{A}^\star_r(x) \geq t > C_\tau(\mathcal{A}, r)(x).$$

Since $\mathcal{A}^\star_r(x) \geq t$, $x_t \in \mathcal{A}^\star_r$. So there is at least one $y \in X$ for each $\mathcal{D} \in Q(x_t, r)$ and $\mathcal{I}(\mathcal{C}) \geq r$ such that $\mathcal{D}(y) + \mathcal{A}(y) > \mathcal{C}(y) + 1$. Therefore, $x_t \in C_\tau(\mathcal{A}, r)$. It is a contradiction. Hence, $\mathcal{A}^\star_r \leq C_\tau(\mathcal{A}, r)$.

Now we show that $\mathcal{A}^\star_r \geq C_\tau(\mathcal{A}^\star_r, r)$. Suppose that, $\mathcal{A}^\star_r \ngeq C_\tau(\mathcal{A}^\star_r, r)$, then there exist $x \in X$ and $t \in M$ such that

$$\mathcal{A}^\star_r(x) < t < C_\tau(\mathcal{A}^\star_r, r)(x).$$

Since $C_\tau(\mathcal{A}^\star_r, r)(x) > t$, $x_t \in C_\tau(\mathcal{A}^\star_r, r)$. So, there at last one $y \in X$ with $\mathcal{B} \in Q(x_t, r)$ such that $\mathcal{B}(y)+\mathcal{A}^\star_r(y) > 1$. Therefore, $\mathcal{A}^\star_r(y) \neq 0$. Let $s = \mathcal{A}^\star_r(y)$. Then $y_s \in \mathcal{A}^*_r$ and $s + \mathcal{B}(y) > 1$, so that $\mathcal{B} \in Q(y_s, r)$. Now $y_s \in \mathcal{A}^\star_r$ implies there is at least one $x^{'} \in X$ such that $\mathcal{D}(x^{'})+\mathcal{A}(x^{'})-1 > \mathcal{C}(x^{'})$ for all $\mathcal{I}(\mathcal{C}) \geq r$ and $\mathcal{D} \in Q(y_s, r)$. This is also true for $\mathcal{B}$. So there is at least one $x^{''} \in X$ such that $\mathcal{B}(x^{''}) + \mathcal{A}(x^{''}) - 1 > \mathcal{C}(x^{''})$. Since $\mathcal{B}$ is an arbitrary and $\mathcal{B} \in Q(x_t, r)$, then, $\mathcal{A}^\star_r(x) > t$. It is a contradiction. Thus $\mathcal{A}^\star_r \geq C_\tau(\mathcal{A}^\star_r, r)$.

(4) Form (3), we have $(\mathcal{A}_r^\star)_r^\star = C_\tau((\mathcal{A}_r^\star)_r^\star, r) \leq C_\tau(\mathcal{A}_r^\star, r) = \mathcal{A}_r^\star$.

(5) ($\Rightarrow$) Since $\mathcal{A}, \mathcal{B} \leq \mathcal{A} \vee \mathcal{B}$. By (1), we have $\mathcal{A}_r^\star \leq (\mathcal{A} \vee \mathcal{B})_r^\star$ and $\mathcal{B}_r^\star \leq (\mathcal{A} \vee \mathcal{B})_r^\star$. Hence $\mathcal{A}_r^\star \vee \mathcal{B}_r^\star \leq (\mathcal{A} \vee \mathcal{B})_r^\star$.

($\Leftarrow$) Suppose that $(\mathcal{A}_r^\star \vee \mathcal{B}_r^\star) \ngeq (\mathcal{A} \vee \mathcal{B})_r^\star$, then there exist $x \in X$ and $t \in M$ such that

$$(\mathcal{A}_r^\star \vee \mathcal{B}_r^\star)(x) < t \leq (\mathcal{A} \vee \mathcal{B})_r^\star(x).$$

Since $(\mathcal{A}_r^\star \vee \mathcal{B}_r^\star)(x) < t$, $\mathcal{A}_r^\star(x) < t$ or $\mathcal{B}_r^\star(x) < t$. So, there exists $\mathcal{D}_1 \in Q(x_t, r)$ such that for every $y \in X$ and for some $\mathcal{I}(\mathcal{C}_1) \geq r$ we have,

$$\mathcal{D}_1(y) + \mathcal{A}(y) - 1 \leq \mathcal{C}_1(y).$$

Similarly there exists $\mathcal{D}_2 \in Q(x_t, r)$ such that for every $y \in X$ and for some $\mathcal{I}(\mathcal{C}_2) \geq r$ we have,

$$\mathcal{D}_2(y) + \mathcal{B}(y) - 1 \leq \mathcal{C}_2(y).$$

Since $\mathcal{D} = \mathcal{D}_1 \odot \mathcal{D}_2 \in Q(x_t, r)$ and by (I_3), $\mathcal{I}(\mathcal{C}_1 \vee \mathcal{C}_2) \geq r$. Thus, for every $y \in X$, $\mathcal{D}(y) + (\mathcal{A} \vee \mathcal{B})(y) - 1 \leq (\mathcal{C}_1 \vee \mathcal{C}_2)(y)$. Therefore, $(\mathcal{A} \vee \mathcal{B})_r^\star(x) < t$. It is a contradiction. Hence $\mathcal{A}_r^\star \vee \mathcal{B}_r^\star \geq (\mathcal{A} \vee \mathcal{B})_r^\star$.

(6), (7) and (8) are obvious. $\square$

Example 3.5. [19]. Define $\tau, \mathcal{I} : L^X \to M$, where $L = M = I = [0, 1]$ as follows:

$$\tau(\mathcal{B}) = \begin{cases} 1, & \text{if } \mathcal{B} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if } \mathcal{B} = \underline{0.8}, \\ \frac{1}{2}, & \text{if } \mathcal{B} = \underline{0.7}, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\mathcal{C}) = \begin{cases} 1, & \text{if } \mathcal{C} = \underline{0}, \\ \frac{1}{2}, & \text{if } \mathcal{C} = \underline{0.3}, \\ \frac{2}{3}, & \text{if } \underline{0} < \mathcal{B} < \underline{0.3}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, $\underline{0} = (\underline{0.4}_{\frac{1}{2}}^\star)_{\frac{1}{2}}^\star \neq \underline{0.4}_{\frac{1}{2}}^\star = \underline{0.2}$.

Theorem 3.6. Let $(X, \tau, \mathcal{I})$ be $(L, M)-$sits, and $\{\mathcal{A}_i : i \in J\} \subset L^X$. Then:

(1) $(\bigvee (\mathcal{A}_i)_r^\star : i \in J) \leq (\bigvee \mathcal{A}_i : i \in J)_r^\star$.

(2) $(\bigwedge \mathcal{A}_i : i \in J)_r^\star \leq (\bigwedge (\mathcal{A}_i)_r^\star : i \in J)$.

Proof. (1) Since $\mathcal{A}_i \leq \bigvee \mathcal{A}_i$, for each $i \in J$, by Theorem 3.4(1), we have $(\mathcal{A}_i)_r^\star \leq (\bigvee \mathcal{A}_i)_r^\star$, for each $i \in J$. This implies $(\bigvee (\mathcal{A}_i)_r^\star : i \in J) \leq (\bigvee \mathcal{A}_i : i \in J)_r^\star$.

(2) Since $\bigwedge \mathcal{A}_i \leq \mathcal{A}_i$, $(\bigwedge \mathcal{A}_i)_r^\star \leq (\mathcal{A}_i)_r^\star$, for each $i \in J$. Thus, $(\bigwedge \mathcal{A}_i : i \in J)_r^\star \leq (\bigwedge (\mathcal{A}_i)_r^\star : i \in J)$. $\square$

Remark 3.7. For each $(X, \tau, \mathcal{I})$ and $\mathcal{A} \in L^X$, we can defines

$$Cl^\star(\mathcal{A}, r) = \mathcal{A} \vee \mathcal{A}^\star_r, \quad int^\star(\mathcal{A}, r) = \mathcal{A} \odot [\underline{1} - (\underline{1} - \mathcal{A})^\star_r].$$

Clearly, $Cl^\star$ is a fuzzy closure operator and $\tau^\star(\mathcal{I})$ is the $(L, M)-$ smooth topology generated by Cl^*. i.e.,

$$\tau^\star(\mathcal{I})(\mathcal{A}) = \bigvee\{r| \; Cl^\star(\underline{1} - \mathcal{A}, r) = \underline{1} - \mathcal{A}\}.$$

Now if, $\mathcal{I} = \mathcal{I}^0$ then $Cl^\star(\mathcal{A}, r) = \mathcal{A} \vee \mathcal{A}^*_r = \mathcal{A} \vee C_\tau(\mathcal{A}, r) = C_\tau(\mathcal{A}, r)$, for $\mathcal{A} \in L^X$. So, $\tau^\star(\mathcal{I}^0) = \tau$.

Theorem 3.8. Let $(X, \tau, \mathcal{I})$ be $(L, M)-$sits, $r \in M$ and $\mathcal{A} \in L^X$. Then
(1) $int^\star(\mathcal{A} \vee \mathcal{B}, r) \leq int^\star(\mathcal{A}, r) \vee int^\star(\mathcal{B}, r)$.
(2) $int_\tau(\mathcal{A}, r) \leq int^\star(\mathcal{A}, r) \leq \mathcal{A} \leq Cl^\star(\mathcal{A}, r) \leq C_\tau(\mathcal{A}, r)$.
(3) $Cl^\star(\underline{1} - \mathcal{A}, r) = \underline{1} - int^\star(\mathcal{A}, r)$ and $\underline{1} - Cl^\star(\mathcal{A}, r) = int^\star(\underline{1} - \mathcal{A}, r)$.
(4) $int^\star(\mathcal{A} \odot \mathcal{B}, r) = int^\star(\mathcal{A}, r) \odot int^\star(\mathcal{B}, r)$.

Proof. (1) and (2) Follows directly from definition $Cl^\star$, $int^\star$ and C_τ.
(3) Since

$$\begin{aligned} & Cl^\star(\underline{1} - \mathcal{A}, r) = \underline{1} - \mathcal{A} \vee (\underline{1} - \mathcal{A})^\star_r = \underline{1} - \mathcal{A} \vee [\underline{1} - (\underline{1} - (\underline{1} - \mathcal{A})^\star_r)] \\ = \; & \underline{1} - [\mathcal{A} \odot (\underline{1} - (\underline{1} - \mathcal{A})^\star_r)] = \underline{1} - int^\star(\mathcal{A}, r). \end{aligned}$$

(4) From Theorem 3.4 (5), we have

$$\begin{aligned} & int^\star(\mathcal{A} \odot \mathcal{B}, r) = (\mathcal{A} \odot \mathcal{B}) \odot [\underline{1} - (\underline{1} - (\mathcal{A} \odot \mathcal{B}))^\star_r] \\ = \; & (\mathcal{A} \odot \mathcal{B}) \odot [\underline{1} - [(\underline{1} - \mathcal{A}) \vee (\underline{1} - \mathcal{B})]^\star_r] \\ = \; & (\mathcal{A} \odot \mathcal{B}) \odot [\underline{1} - [(\underline{1} - \mathcal{A})^\star_r \vee (\underline{1} - \mathcal{B})^\star_r]] \\ = \; & (\mathcal{A} \odot [\underline{1} - (\underline{1} - \mathcal{A})^\star_r]) \odot (\mathcal{B} \odot [\underline{1} - (\underline{1} - \mathcal{B})^\star_r]) \\ = \; & int^\star(\mathcal{A}, r) \odot int^\star(\mathcal{B}, r). \end{aligned}$$

□

Theorem 3.9. Let $(X, \tau_1, \mathcal{I})$ and $(X, \tau_2, \mathcal{I})$ be $(L, M)-$sits's and $\tau_1 \leq \tau_2$. Then
(1) $\mathcal{A}^\star_r(\tau_2, \mathcal{I}) \leq \mathcal{A}^\star_r(\tau_1, \mathcal{I})$.
(2) $\tau^\star_1(\mathcal{I}) \leq \tau^\star_2(\mathcal{I})$.

Proof. (1) Suppose that $\mathcal{A}^\star_r(\tau_2, \mathcal{I}) \not\leq \mathcal{A}^\star_r(\tau_1, \mathcal{I})$, then there exist $x \in X$ and $t \in M$ such that

$$\mathcal{A}^\star_r(\tau_2, \mathcal{I})(x) \geq t > \mathcal{A}^\star_r(\tau_1, \mathcal{I})(x).$$

Since $\mathcal{A}_r^\star(\tau_1,\mathcal{I})(x) < t$, there exist $\mathcal{D} \in Q_{\tau_1}(x_t, r)$ with $\mathcal{I}(\mathcal{C}) \geq r$ such that for every $y \in X$, $\mathcal{D}(y) + \mathcal{A}(y) - 1 \leq \mathcal{C}(y)$. Since $\tau_1 \leq \tau_2$, $\mathcal{D} \in Q_{\tau_2}(x_t, r)$. Thus, $\mathcal{A}_r^\star(\tau_2,\mathcal{I})(x) < t$. It is a contradiction.

(2) Clearly, $\tau_1^\star(\mathcal{I}) \leq \tau_2^\star(\mathcal{I})$, as $\mathcal{A}_r^\star(\tau_2,\mathcal{I}) \leq \mathcal{A}_r^\star(\tau_1,\mathcal{I})$. □

Theorem 3.10. Let $(X,\tau,\mathcal{I}_1)$ and $(X,\tau,\mathcal{I}_2)$ be $(L,M)-$sits's and $\mathcal{I}_1 \leq \mathcal{I}_2$. Then

(1) $\mathcal{A}_r^\star(\mathcal{I}_1,\tau) \geq \mathcal{A}_r^\star(\mathcal{I}_2,\tau)$.

(2) $\tau_1^\star(\mathcal{I}_1) \leq \tau_2^\star(\mathcal{I}_2)$.

Proof. Obvious. □

Theorem 3.11. Define the mapping $\beta : \Theta \to L$ on X by

$$\beta(\mathcal{A}) = \bigvee\{\tau(\mathcal{B}) \odot \mathcal{I}(\mathcal{C})|\ \mathcal{A} = \mathcal{B} \odot (\underline{1} - \mathcal{C})\}.$$

The β is base for the $(L,M)-$smooth topology $\tau^\star$.

Proof. (1) Since $\mathcal{I}(\underline{0}) = 1$, $\beta(\underline{1}) = 1$.

(2) Suppose there exist $\mathcal{A}_1, \mathcal{A}_2 \in \Theta$ such that $\beta(\mathcal{A}_1 \odot \mathcal{A}_2) \not\geq \beta(\mathcal{A}_1) \odot \beta(\mathcal{A}_2)$. There exists $t \in M$ such that

$$\beta(\mathcal{A}_1 \odot \mathcal{A}_2) < t \leq \beta(\mathcal{A}_1) \odot \beta(\mathcal{A}_2).$$

Since $\beta(\mathcal{A}_1) \geq t$ and $\beta(\mathcal{A}_2) \geq t$, there exist $\mathcal{B}_1, \mathcal{B}_2, \mathcal{C}_1, \mathcal{C}_2 \in \Theta$ with $\mathcal{A}_1 = \mathcal{B}_1 \odot (\underline{1} - \mathcal{C}_1)$ and $\mathcal{A}_2 = \mathcal{B}_2 \odot (\underline{1} - \mathcal{C}_2)$ such that $\beta(\mathcal{A}_1) \geq \tau(\mathcal{B}_1) \odot \mathcal{I}(\mathcal{C}_1) \geq t$ and $\beta(\mathcal{A}_2) \geq \tau(\mathcal{B}_2) \odot \mathcal{I}(\mathcal{C}_2) \geq t$. Therefore,

$$\begin{aligned} & \mathcal{A}_1 \odot \mathcal{A}_2 = (\mathcal{B}_1 \odot (\underline{1} - \mathcal{C}_1)) \odot (\mathcal{B}_2 \odot (\underline{1} - \mathcal{C}_2)) \\ = \ & (\mathcal{B}_1 \odot \mathcal{B}_2) \odot ((\underline{1} - \mathcal{C}_1) \odot (\underline{1} - \mathcal{C}_2)) \\ = \ & (\mathcal{B}_1 \odot \mathcal{B}_2) \odot (\underline{1} - (\mathcal{C}_1 \vee \mathcal{C}_2)) \end{aligned}$$

Hence,

$$\begin{aligned} & \beta(\mathcal{A}_1 \odot \mathcal{A}_2) \geq \tau(\mathcal{B}_1 \odot \mathcal{B}_2) \odot \mathcal{I}(\mathcal{C}_1 \vee \mathcal{C}_2) \\ \geq \ & \tau(\mathcal{B}_1) \odot \tau(\mathcal{B}_2) \odot \mathcal{I}(\mathcal{C}_1) \odot \mathcal{I}(\mathcal{C}_2) \\ = \ & (\tau(\mathcal{B}_1) \odot \mathcal{I}(\mathcal{C}_1)) \odot (\tau(\mathcal{B}_2) \odot \mathcal{I}(\mathcal{C}_2)) \geq t. \end{aligned}$$

It is a contradiction. Thus, $\beta(\mathcal{A}_1 \odot \mathcal{A}_2) \geq \beta(\mathcal{A}_1) \odot \beta(\mathcal{A}_2)$. □

Theorem 3.12. Let (X,τ) be $(L,M)-$smooth topological space and $\mathcal{I}_1, \mathcal{I}_2$ be two $(L,M)-$smooth ideals on X. Then, for any $\mathcal{A} \in L^X$ and $r \in M$,

(1) $\mathcal{A}_r^\star(\mathcal{I}_1 \odot \mathcal{I}_2,\tau) = \mathcal{A}_r^\star(\mathcal{I}_1,\tau) \vee \mathcal{A}_r^\star(\mathcal{I}_2,\tau)$.

(2) $\mathcal{A}_r^\star(\mathcal{I}_1 \vee \mathcal{I}_2, \tau) = \mathcal{A}_r^\star(\mathcal{I}_1, \tau^\star(\mathcal{I}_2,)) \odot \mathcal{A}^\star(\mathcal{I}_2, \tau^\star(\mathcal{I}_1))$.

Proof. (1) Suppose that $\mathcal{A}_r^\star(\mathcal{I}_1 \odot \mathcal{I}_2, \tau) \not\leq \mathcal{A}_r^\star(\mathcal{I}_1, \tau) \vee \mathcal{A}_r^\star(\mathcal{I}_2, \tau)$, there exist $x \in X$ and $t \in M$ such that

$$\mathcal{A}_r^\star(\mathcal{I}_1 \odot \mathcal{I}_2, \tau)(x) \geq t > \mathcal{A}_r^\star(\mathcal{I}_1, \tau) \vee \mathcal{A}_r^\star(\mathcal{I}_2, \tau)(x).$$

Since $\mathcal{A}_r^\star(\mathcal{I}_1, \tau) \vee \mathcal{A}_r^\star(\mathcal{I}_2, \tau)(x) < t$, $\mathcal{A}_r^\star(\mathcal{I}_1, \tau)(x) < t$ and $\mathcal{A}_r^\star(\mathcal{I}_2, \tau)(x) < t$. Now, $\mathcal{A}_r^\star(\mathcal{I}_1, \tau)(x) < t$, implies there exist $\mathcal{D}_1 \in Q_\tau(x_t, r)$ and for some $\mathcal{I}_1(\mathcal{C}_1) \geq r$ such that for every $y \in X, \mathcal{D}_1(y) + \mathcal{A}(y) - 1 \leq \mathcal{C}_1(y)$. **Again**, $\mathcal{A}_r^\star(\mathcal{I}_2, \tau)(x) < t$, implies there exists $\mathcal{D}_2 \in Q_\tau(x_t, r)$ and for some $\mathcal{I}_2(\mathcal{C}_2) \geq r$ such that for each $y \in X, \mathcal{D}_2(y) + \mathcal{A}(y) - 1 \leq \mathcal{C}_2(y)$. Therefore, $(\mathcal{D}_1 \odot \mathcal{D}_2)(y) + \mathcal{A}(y) - 1 \leq (\mathcal{C}_1 \odot \mathcal{C}_2)(y)$, for every $y \in X$. Since $(\mathcal{D}_1 \odot \mathcal{D}_2) \in Q_\tau(x_t, r)$ and $(\mathcal{I}_1 \odot \mathcal{I}_2)(\mathcal{C}_1 \odot \mathcal{C}_2) \geq r$, $\mathcal{A}_r^\star(\mathcal{I}_1 \odot \mathcal{I}_2, \tau) < t$, and this is a contradiction. So that $\mathcal{A}_r^\star(\mathcal{I}_1 \odot \mathcal{I}_2, \tau) \leq \mathcal{A}_r^\star(\mathcal{I}_1, \tau) \vee \mathcal{A}^\star(\mathcal{I}_2, \tau)$.

Also, $\mathcal{I}_1, \mathcal{I}_2 \geq \mathcal{I}_1 \odot \mathcal{I}_2$, so by Theorem 3.4(2), $\mathcal{A}_r^\star(\mathcal{I}_1 \odot \mathcal{I}_2) \geq \mathcal{A}_r^\star(\mathcal{I}_1) \vee \mathcal{A}^\star(\mathcal{I}_2)$. Then, $\mathcal{A}_r^\star(\mathcal{I}_1 \odot \mathcal{I}_2, \tau) = \mathcal{A}_r^\star(\mathcal{I}_1, \tau) \vee \mathcal{A}^\star(\mathcal{I}_2, \tau)$.

(2) Suppose that $\mathcal{A}_r^\star(\mathcal{I}_1 \vee \mathcal{I}_2, \tau) \not\geq \mathcal{A}_r^\star(\mathcal{I}_1, \tau^\star(\mathcal{I}_2,)) \odot \mathcal{A}^\star(\mathcal{I}_2, \tau^\star(\mathcal{I}_1))$, then there exist $x \in X$ and $t \in M$ such that

$$\mathcal{A}_r^\star(\mathcal{I}_1 \vee \mathcal{I}_2, \tau)(x) < t \leq \mathcal{A}_r^\star(\mathcal{I}_1, \tau^\star(\mathcal{I}_2,))(x) \odot \mathcal{A}^\star(\mathcal{I}_2, \tau^\star(\mathcal{I}_1))(x)$$

Since $\mathcal{A}_r^\star(\mathcal{I}_1 \vee \mathcal{I}_2, \tau)(x) < t$, there exists $\mathcal{D} \in Q_\tau(x_t, r)$, such that for every $y \in X$, and for some $(\mathcal{I}_1 \vee \mathcal{I}_2)(\mathcal{C}) \geq r$, $\mathcal{D}(y) + \mathcal{A}(y) - 1 \leq \mathcal{C}(y)$. Therefore, by heredity of $L-$smooth ideals and $\tau \leq \tau^\star$ we can find $\mathcal{D}_1 \in Q_{\tau^\star(\mathcal{I}_1)}(x_t, r)$ or $\mathcal{D}_2 \in Q_{\tau^\star(\mathcal{I}_2)}(x_t, r)$ such that for every $y \in X$, $\mathcal{D}_1(y) + \mathcal{A}(y) - 1 \leq \mathcal{C}_1(y)$, or $\mathcal{D}_2(y) + \mathcal{A}(y) - 1 \leq \mathcal{C}_2(y)$, for some $\mathcal{I}_2(\mathcal{C}_2) \geq r$ or $\mathcal{I}_1(\mathcal{C}_1) \geq r$. This implies $\mathcal{A}^\star(\mathcal{I}_2, \tau^\star(\mathcal{I}_1))(x) < t$ or $\mathcal{A}^\star(\mathcal{I}_1, \tau^\star(\mathcal{I}_2))(x) < t$. It is a contradiction. Thus, $\mathcal{A}_r^\star(\mathcal{I}_1 \vee \mathcal{I}_2, \tau) \geq \mathcal{A}_r^\star(\mathcal{I}_1, \tau^\star(\mathcal{I}_2,)) \odot \mathcal{A}^\star(\mathcal{I}_2, \tau^\star(\mathcal{I}_1))$.

Conversely, similarly $\mathcal{A}_r^\star(\mathcal{I}_1 \vee \mathcal{I}_2, \tau) \leq \mathcal{A}_r^\star(\mathcal{I}_1, \tau^\star(\mathcal{I}_2,)) \odot \mathcal{A}^\star(\mathcal{I}_2, \tau^\star(\mathcal{I}_1))$.

An important result follows from the above theorem that $\tau^\star(\mathcal{I})$ and $[\tau^\star(\mathcal{I})]^\star(\mathcal{I})$ (in short $\tau^{\star\star}$) are equal for any $(L, M)-$smooth ideal on X. □

Corollary 3.13. Let $(X, \tau, \mathcal{I})$ be $(L, M)-$sits. For any $\mathcal{A} \in L^X$ and $r \in M$. Then $\mathcal{A}_r^\star(\mathcal{I}) = \mathcal{A}_r^\star(\mathcal{I}, \tau^\star)$ and $\tau^\star(\mathcal{I}) = \tau^{\star\star}$.

Proof. Putting $\mathcal{I}_1 = \mathcal{I}_2$ in Theorem 3.12(2), we have required result. □

Corollary 3.14. Let (X, τ) be $(L, M)-$smooth topological space and $\mathcal{I}_1, \mathcal{I}_2$ be two $(L, M)-$smooth ideals on X. Then,

(1) $\tau^\star(\mathcal{I}_1 \vee \mathcal{I}_2) = [\tau^\star(\mathcal{I}_2)]^\star(\mathcal{I}_1) = [\tau^\star(\mathcal{I}_1)]^\star(\mathcal{I}_2)$.

(2) $\tau^\star(\mathcal{I}_1 \odot \mathcal{I}_2) = \tau^\star(\mathcal{I}_1) \odot \tau^\star(\mathcal{I}_2)$.

Proof. proof is easily by using Theorem 3.12(2). □

Definition 3.15. For $(L, M)-$smooth topological space (X, τ) with $(L, M)-$smooth ideal $\mathcal{I}$, τ is said to be smooth open compatible with $\mathcal{I}$, denoted by $\tau \sim \mathcal{I}$, if for each $\mathcal{A} \in L^X$, $x_t \in \mathcal{A}$, and $\mathcal{C} \in L^X$ with $\mathcal{I}(\mathcal{C}) \geq r$ there exists $\mathcal{D} \in Q_\tau(x_t, r)$ such that $\mathcal{D}(y) + \mathcal{A}(y) - 1 \leq \mathcal{C}(y)$ holds for every $y \in X$, then $\mathcal{I}(\mathcal{A}) \geq r$.

Definition 3.16. Let $\{\mathcal{B}_j : j \in J\}$ be a indexed family of smooth set of X such that $\mathcal{B}_j q \mathcal{A}$ for each $j \in J$ where $\mathcal{A} \in L^X$. Then $\{\mathcal{B}_j : j \in J\}$ is said to be a $r-$smooth quasi-cover of $\mathcal{A}$ iff $\mathcal{A}(y) + \bigvee_{j \in J}(\mathcal{B}_j)(y) \geq 1$ for every $y \in X$.

Further, let (X, τ) be $(L, M)-$smooth topological space, for each $\tau(\mathcal{B}_j) \geq r$. Then this $r-$smooth quasi-cover will be called smooth quasi open-cover of $\mathcal{A}$.

Theorem 3.17. Let (X, τ) be $(L, M)-$ smooth topological space with $(L, M)-$smooth ideal $\mathcal{I}$ on X. Then the following conditions are equivalent.

(1) $\tau \sim \mathcal{I}$.

(2) If for every $\mathcal{A} \in L^X$ has $r-$smooth quasi open-cover of $\{\mathcal{B}_j : j \in J\}$ such that for each j, $\mathcal{A}(y) + \mathcal{B}_j(y) - 1 \leq \mathcal{C}(y)$, for every $y \in X$ and for some $\mathcal{I}(\mathcal{C}) \geq r$, then $\mathcal{I}(\mathcal{A}) \geq r$.

(3) For every $\mathcal{A} \in L^X$, $\mathcal{A} \odot \mathcal{A}^\star_r = \underline{0}$ implies $\mathcal{I}(\mathcal{A}) \geq r$.

(4) For every $\mathcal{A} \in L^X$, $\mathcal{I}(\widetilde{\mathcal{A}}) \geq r$, where $\widetilde{\mathcal{A}} = \bigvee x_t$ such that $x_t \in \mathcal{A}$ but $x_t \notin \mathcal{A}^*_r$.

(5) For every $\tau^\star(\underline{1} - \mathcal{A}) \geq r$, $\mathcal{I}(\widetilde{\mathcal{A}}) \geq r$.

(6) For every $\mathcal{A} \in L^X$, if $\mathcal{A}$ contains no $\mathcal{B} \neq \underline{0}$ with $\mathcal{B} \leq \mathcal{B}^\star_r$, then $\mathcal{I}(\mathcal{A}) \geq r$.

Proof. We prove most of the equivalent conditions which ultimately prove all the equivalence.

(1)⇒(2): Let $\{\mathcal{B}_j : j \in J\}$ be a smooth quasi open-cover of $\mathcal{A} \in L^X$ such that for $j \in J$, $\mathcal{A}(y) + \mathcal{B}_j(y) - 1 \leq \mathcal{C}(y)$, for every $y \in X$ and for some $\mathcal{I}(\mathcal{C}) \geq r$. Therefore, as $\{\mathcal{B}_j : j \in J\}$ is $r-$smooth quasi open-cover of $\mathcal{A}$, for each $x_t \in \mathcal{A}$, there exist at least one $\mathcal{B}_{j\circ}$ such that $x_t q \mathcal{B}_{j\circ}$ and for every $y \in X$, $\mathcal{A}(y) + \mathcal{B}_{j\circ}(y) - 1 \leq \mathcal{C}(y)$, for some $\mathcal{I}(\mathcal{C}) \geq r$. Obviously, $\mathcal{B}_{j\circ} \in Q_\tau(x_t, r)$. By (1), we have $\mathcal{I}(\mathcal{A}) \geq r$.

(2)⇒(1): Clear from the fact that a collection of $\{\mathcal{B}_j : j \in J\}$ which contains at least one $\mathcal{B}_{j\circ} \in Q_\tau(x_t, r)$, of each fuzzy point of $\mathcal{A}$, constitutes a smooth quasi-open cover of $\mathcal{A}$.

(1)⇒(3): Let $\mathcal{A} \odot \mathcal{A}^\star_r = \underline{0}$, for every $y \in X$, $x_t \in \mathcal{A}$ implies $x_t \notin \mathcal{A}^\star_r$. Then there exists $\mathcal{D} \in Q(x_t, r)$ and $\mathcal{I}(\mathcal{C}) \geq r$ such that for every $y \in X$, $\mathcal{D}(y) + \mathcal{A}(y) - 1 \leq \mathcal{C}(y)$. Since $\mathcal{D} \in Q_\tau(x_t, r)$. By (1), we have $\mathcal{I}(A) \geq r$.

(3)⇒(1): For every $x_t \in \mathcal{A}$, there exist an $\mathcal{D} \in Q_\tau(x_t, r)$ such that for every $y \in X$, $\mathcal{D}(y) + \mathcal{A}(y) - 1 \leq \mathcal{C}(y)$, for some $\mathcal{I}(\mathcal{C}) \geq r$. This implies $x_t \notin \mathcal{A}^\star_r$. Now, there are two cases: either $\mathcal{A}^\star_r = \underline{0}$ or $\mathcal{A}^\star_r \neq \underline{0}$ but $t > \mathcal{A}^\star_r \neq \underline{0}$. Let, if possible, $x_t \in \mathcal{A}$ such that $t > \mathcal{A}^\star_r \neq \underline{0}$. Let $t' = \mathcal{A}^\star_r(x)$. Then $x_{t'} \in \mathcal{A}^\star_r(x)$. Also, $x_{t'} \in \mathcal{A}$. Thus, for every $\mathcal{V} \in Q_\tau(x_t, r)$ for every $\mathcal{I}(\mathcal{C}) \geq r$, there is at least one $y \in X$ such that $\mathcal{V}(y) + \mathcal{A}(y) - 1 > \mathcal{C}(y)$. Since $x_{t'} \in \mathcal{A}$, this contradicts the assumption for every fuzzy point of $\mathcal{A}$. So, $\mathcal{A}^\star_r = \underline{0}$. That means, $x_t \in \mathcal{A}$, implies $x_t \notin \mathcal{A}^*_r$. Now this is true for every $\mathcal{A} \in L^X$. So, for every $\mathcal{A} \in L^X$, $\mathcal{A} \odot \mathcal{A}^\star_r = \underline{0}$. Hence, by(3), we have $\mathcal{I}(\mathcal{A}) \geq r$, which implies $\tau \sim \mathcal{I}$.

(3)⇒(4): Let $x_t \in \widetilde{\mathcal{A}}$. Then, $x_t \in \mathcal{A}$ but $x_t \notin \mathcal{A}^\star_r$. So, there exist an $\mathcal{D} \in Q_\tau(x_t, r)$ such that for every $y \in X$, $\mathcal{D}(y) + \mathcal{A}(y) - 1 \leq \mathcal{C}(y)$, for some $\mathcal{I}(\mathcal{C}) \geq r$. Since $\widetilde{\mathcal{A}} \leq \mathcal{A}$, So for every $y \in X$, $\mathcal{D}(y) + \widetilde{\mathcal{A}}(y) - 1 \leq \mathcal{C}(y)$ for some $\mathcal{I}(\mathcal{C}) \geq r$. Therefore, $x_t \notin \widetilde{\mathcal{A}}^\star_r$ implies that $\widetilde{\mathcal{A}}^\star_r = \underline{0}$ or $\widetilde{\mathcal{A}}^\star_r \neq \underline{0}$ but $t > \widetilde{\mathcal{A}}^\star_r$. Let $x_{t'} \in P_t(X)$ such that $t' \leq \widetilde{\mathcal{A}}^\star_r(x) < t$, i.e., $x_{t'} \in \widetilde{\mathcal{A}}^\star_r$. Then, for each $\mathcal{V} \in Q_\tau(x_{t'}, r)$ and for each $\mathcal{I}(\mathcal{C}) \geq r$ there is at least one $y \in X$ such that $\mathcal{V}(y) + \widetilde{\mathcal{A}}(y) - 1 > \mathcal{C}(y)$ Since $\widetilde{\mathcal{A}} \leq \mathcal{A}$, then so for each $\mathcal{V} \in Q_\tau(x_{t'}, r)$ and for each $\mathcal{I}(\mathcal{C}) \geq r$ there is at least one $y \in X$ such that $\mathcal{V}(y) + \mathcal{A}(y) - 1 > \mathcal{C}$. This implies $x_{t'} \in \mathcal{A}^\star_r$. But as $t' < t$, $x_t \in \widetilde{\mathcal{A}}$ implies $x_{t'} \in \widetilde{\mathcal{A}}$, and therefore, $x_{t'} \notin \mathcal{A}^\star_r$. This is a contradiction. Hence, $\mathcal{A}^\star_r = \underline{0}$, so that $x_t \in \widetilde{\mathcal{A}}$ implies $x_t \notin \widetilde{\mathcal{A}}^\star_r$ with $\widetilde{\mathcal{A}}^\star_r = \underline{0}$. Thus, $\widetilde{\mathcal{A}} \odot \widetilde{\mathcal{A}}^\star_r = \underline{0}$, for every $\mathcal{A} \in I^X$. Hence, by (3), $\mathcal{I}(\widetilde{\mathcal{A}}) \geq r$.

(4)⇒(5): Straightforward.

(4)⇒(6): Let $\mathcal{A} \in L^X$, $\mathcal{A}$ contains $\mathcal{B} \neq \underline{0}$ with $\mathcal{B} \leq \mathcal{B}^\star_r$. Then, for every $\mathcal{A} \in L^X$, $\mathcal{A} = \widetilde{\mathcal{A}} \vee (\mathcal{A} \odot \mathcal{A}^\star_r)$. Therefore, $\mathcal{A}^\star_r = (\widetilde{\mathcal{A}} \vee (\mathcal{A} \odot \mathcal{A}^\star_r))^\star_r = \widetilde{\mathcal{A}}^\star_r \vee (\mathcal{A} \odot \mathcal{A}^\star_r)^\star_r$. by Theorem 3.4(5).
Now by (4), we have $\mathcal{I}(\widetilde{\mathcal{A}}) \geq r$, then $\widetilde{\mathcal{A}}^\star_r = \underline{0}$. Hence, $(\mathcal{A} \odot \mathcal{A}^\star_r)^\star_r = \mathcal{A}^\star_r$ but $\mathcal{A} \odot \mathcal{A}^\star_r \leq \mathcal{A}^\star_r$, then $\mathcal{A} \odot \mathcal{A}^\star_r \leq (\mathcal{A} \odot \mathcal{A}^\star_r)^\star_r$. This contradicts the hypothesis about every fuzzy $\mathcal{A} \in L^X$, if $\underline{0} \neq \mathcal{B} \leq \mathcal{A}$ with $\mathcal{B} \leq \mathcal{B}^\star_r$. Therefore, $\mathcal{A} \odot \mathcal{A}^\star_r = \underline{0}$, so that $\mathcal{A} = \widetilde{\mathcal{A}}$ by (4), we have $\mathcal{I}(\mathcal{A}) \geq r$.

(6)⇒(4): Since, for every $\mathcal{A} \in L^X$, $\mathcal{A} \odot \mathcal{A}^\star_r = \underline{0}$ Therefore, by (6), as $\mathcal{A}$ contains no non-empty fuzzy subset $\mathcal{B}$ with $\mathcal{B} \leq \mathcal{B}^\star_r$, $\mathcal{I}(\mathcal{A}) \geq r$.

(5)⇒(1): For every $\mathcal{A} \in L^X$, $x_t \in \mathcal{A}$, there exist an $\mathcal{D} \in Q_\tau(x_t, r)$ such that $\mathcal{D}(y) + \mathcal{A}(y) - 1 \leq \mathcal{C}(y)$ holds for every $y \in X$ and for some $\mathcal{I}(\mathcal{C}) \geq r$. This implies $x_t \notin \mathcal{A}^\star_r$. Let $\mathcal{B} = \mathcal{A} \vee \mathcal{A}^*_r$. Then, $\mathcal{B}^*_r = (\mathcal{A} \vee \mathcal{A}^*_r)^\star_r = \mathcal{A}^\star_r \vee (\mathcal{A}^\star_r)^\star_r = \mathcal{A}^\star_r$ by Theorem 3.4(4). So, $Cl^\star(\mathcal{B}, r) = \mathcal{B} \vee \mathcal{B}^\star_r = \mathcal{B}$. That means $\tau^\star(\underline{1} - \mathcal{B}) \geq r$. Therefore, by (5), we have $\mathcal{I}(\mathcal{B}) \geq r$.

Again, For any $x_t \in P_t(X)$, $x_t \notin \widetilde{\mathcal{B}}^\star_r$ implies $x_t \in \mathcal{B}$ but $x_t \notin \mathcal{B}^\star_r = \mathcal{A}^\star_r$ So, as $\mathcal{B} = \mathcal{A} \vee \mathcal{A}^\star_r$, $x_t \in \mathcal{A}$. Now, by hypothesis about $\mathcal{A}$, we have for every $x_t \in \mathcal{A}^\star_r$. So, $\widetilde{\mathcal{B}} = \mathcal{A}$. Hence, $\mathcal{I}(\mathcal{A}) \geq r$, i.e., $\tau \sim \mathcal{I}$. □

Theorem 3.18. Let (X, τ) be $(L, M)-$ smooth topological space with $(L, M)-$ smooth ideal $\mathcal{I}$, on X. Then the following are equivalent and implied by $\tau \sim \mathcal{I}$.

(1) For every $\mathcal{A} \in L^X$, $\mathcal{A} \odot \mathcal{A}^\star_r = \underline{0}$ implies $\mathcal{A}^*_r = \underline{0}$.

(2) For every $\mathcal{A} \in L^X$, $\widetilde{\mathcal{A}}^\star_r = \underline{0}$.

(3) For every $\mathcal{A} \in L^X$, $\mathcal{A} \odot \mathcal{A}^\star_r = \mathcal{A}^\star_r$.

Proof. Clear from Theorem 3.17. □

An important consequence of Theorem 3.18 is the following corollary.

Corollary 3.19. Let $\tau \sim \mathcal{I}$. Then $\beta(\tau, \mathcal{I})$, a base for $\tau^\star$, and also $\beta(\tau, \mathcal{I}) = \tau^\star$.

Proof. Clear.

4 Conclusion

A smoothing by using fuzzy logic gives rather good results. In particular smoothing of ideals , $r-$smooth open local map. $(L, M)-$smooth ideal topological spaces seem to be a good examples and corresponding concepts trace back to the (classic) fuzzy ideal structures. We feeling that we can be build a new mathematical object ($(L, M)-$smooth structures) where $L = (L, \leq, \vee, \odot, ', 0, 1)$ denotes a completely distributive lattice with order-reversing involution $'$ which has the least and greatest elements,say 0 and 1, respectively. This approach could be a subject of further studies.

References

[1] M.A. Abdel-Sattar, $(L, \odot)-$ Smooth filters, *International Mathematical Forum*, **8** (2013), 633-646. https://doi.org/10.12988/imf.2013.13066

[2] M.A.Abdel-Sattar, Some structures of $(L, \odot)-$ Smooth topological spaces, *Int. J. Appl. Math. Stat.*, **34** (2013), 104-117.

[3] M.A. Abdel-Sattar, Some types of fuzzy filters in BL-algebras, *International Journal of Computational Cognition*, **9** (2011), 13-16.

[4] M.A. Abdel-Sattar, Some structures of L-fuzzy topology, *J. Fuzzy Math.*, **14** (2006), 751- 766.

[5] M.A.Abdel-Sattar, Basic Structures of $(L\ ,\odot)$-Smooth Syntopogenous, *Applied Mathematical Sciences*, **9** (2015), 6615 - 6628. https://doi.org/10.12988/ams.2015.510614

[6] M.A. Abdel-Sattar and Y. M. Saber, A Study of $L-$Smooth Ideals in $(L\ ,\odot)$-Smooth Topological Spaces, *International Mathematical Forum*, **9** (2014), 483 - 498. https://doi.org/10.12988/imf.2014.35100

[7] M.E. Abd El-Monsef, A.A. Nasef and A.A.Salama, Some fuzzy topological operators via fuzzy ideals, *Chaos Solitons & Fractals*, **12** (2001), 2509-2515. https://doi.org/10.1016/s0960-0779(00)00228-9

[8] C.L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.*, **24** (1968), 182-190. https://doi.org/10.1016/0022-247x(68)90057-7

[9] B. Hutton and I. Reilly, Separation axioms in fuzzy topological spaces., *Fuzzy Sets Syst.,* **3** (1980), 93-104.

[10] T.R. Hamlett and D. Jankovice, New topologies from old via ideals, *Amer. Math. Monthly*, **97** (4) (1990), 295-310. https://doi.org/10.1080/00029890.1990.11995593

[11] U. Höhle and A.P. Sostak, *Axiomatic foundation of fixed-basis fuzzy topology*, Chapter 3 in Mathematics of fuzzy Sets, Logic, Topology and Measure Theory, Handbook of Fuzzy Series, Kluwer Academic Publisher, Dordrecht Boston, 1999. https://doi.org/10.1007/978-1-4615-5079-2_5

[12] Y.C. Kim, The properties of $(L,\odot)-$ filters, *Int. J. Contemp. Math. Sciences*, **7** (2012), 631-640.

[13] R.A. Mahmoud, Fuzzy ideals, fuzzy local functions and fuzzy topology, *J. of Fuzzy Math.*, **5** (1) (1997), 165-172.

[14] A.A. Ramadan, M.A. Abdel-Sattar and Y.C. Kim, Some properties of smooth ideals, *Indian J. Pure Appl. Math.*, **34** (2003), 247-264.

[15] A. A. Ramadan, S.N. El-Deeb and M.A. Abdel-Sattar, On smooth topological spaces IV, *Fuzzy Sets and Systems*, **119** (2001), 473-482. https://doi.org/10.1016/s0165-0114(99)00083-4

[16] A.P. Šostak, On a fuzzy topological structure, *Suppl. Rend. Circ. Matem. Palermo Ser. II,* **11** (1985), 89-103.

[17] A.P. Šostak, On some modifications of fuzzy topologies, *Mathematicki Vesnik,* **41** (1989), 51-64.

[18] D. Sarkar, Fuzzy ideal theory, Fuzzy local function and generated fuzzy topology, *Fuzzy Sets and Systems*, **78** (1997), 117-123. https://doi.org/10.1016/s0165-0114(96)00032-2

[19] Y. M. Saber and M.A. Abdel-Sattar, Ideals on Fuzzy Topological Spaces, *Applied Mathematical Sciences*, **8** (2014), 1667-1691. https://doi.org/10.12988/ams.2014.33194

[20] Yasser Abdelghani and Ahmed Zahran, Ideal of fuzzy topological spaces: Fuzzy Topology Paperback, Noor Publishing (May 17, 2017).

[21] R. Bělohavek, *Fuzzy Relational Systems*, Kluwer Academic Publishers, New York, 2002.

[22] T. Kubiak and Śostak, Foundations of the theory of (L,M)-fuzzy topological spaces, Abstracts of the 30th Linz Semi-nar on Fuzzy Set Theory (U. Bodenhofer, B. De Baets, E.P. Klement and S. Saminger-Platz, eds), Johannes Kepler Universitat, Linz, 2009, pp. 70-73.

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