International Mathematical Forum, Vol. 15, 2020, no. 6, 255 - 263
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2020.91279

Integrals in Gradshteyn and Ryzhik: Hyperbolic and Algebraic Functions

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Abstract

We derive certain integrals by Wolfgang Gröbner and David Bierens de Haan which are reported in Gradshteyn and Ryzhik. We evaluate several of these definite integrals of the form given by

$$\int_0^\infty R(\alpha,\beta,y)\left((\log(a)+iy)^k-(\log(a)-iy)^k\right)dy$$

Mathematics Subject Classification: Primary 30-02, 30D10, 30D30, 30E20, 11M35, 11M06, 01A55

Keywords: entries of Gradshteyn and Ryzhik, hyperbolic integrals, Gröbner, Bierens de Haan, Lerch function

1 Introduction

We will derive integrals as indicated in the abstract in terms of special functions. Some special cases of these integrals have been reported in Gradshteyn and Ryzhik [7]. In 1858, Gröbner and Hofreiter [3] and 1867 David Bierens de Haan derived hyperbolic integrals of the form

$$\int_0^\infty \frac{\cosh(\alpha y)}{\sinh(\beta y)} \left((\log(a)+iy)^k - (\log(a)-iy)^k\right) dy \tag{1}$$

In our case the constants in the formulas are general complex numbers subject to the restrictions given below. The derivations follow the method used by us in [8]. The generalized Cauchy's integral formula is given by

$$\frac{y^k}{k!} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}} dw. \tag{2}$$

This method involves using a form of equation (2) then multiplys both sides by a function, then takes a definite integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of equation (2) by another function and take the infinite sum of both sides such that the contour integral of both equations are the same.

2 Definite integral of the contour integral

We use the method in [8]. Here the contour is similar to Figure 2 in [8] except we replace the vertical lines ± 1 by $\pm\Re(\beta)$. Using a generalization of Cauchy's integral formula we first replace y by $iy+\log(a)$ for the first equation and then by y by $-iy+\log(a)$ to get the second equation. Then we subtract these two equations, followed by multiplying both sides by $\frac{\cosh(\alpha y)}{2i\sinh(\beta y)}$ to get

$$\begin{aligned} &\frac{\cosh(\alpha y)\operatorname{csch}(\beta y)\left((\log(a)+iy)^k - (\log(a)-iy)^k\right)}{2ik!} \\ &\qquad = \frac{1}{2\pi i}\int_C a^w w^{-k-1}\frac{\sin(wy)\cosh(\alpha y)}{\sinh(\beta y)} dw \end{aligned} \tag{3}$$

the logarithmic function is defined in equation (4.1.2) in [2]. We then take the definite integral over $y \in [0, \infty)$ of both sides to get

$$\begin{aligned}
&\int_0^\infty \frac{\cosh(\alpha y)\left((\log(a)+iy)^k-(\log(a)-iy)^k\right)}{2ik!\sinh(\beta y)}dy \\
&= \frac{1}{2\pi i}\int_0^\infty \int_C a^w w^{-k-1}\sin(wy)\cosh(\alpha y)\text{csch}(\beta y)dwdy \\
&= \frac{1}{2\pi i}\int_C \int_0^\infty a^w w^{-k-1}\sin(wy)\cosh(\alpha y)\text{csch}(\beta y)dydw \\
&= \int_C \frac{\tanh\left(\frac{\pi(w-i\alpha)}{2\beta}\right)+\tanh\left(\frac{\pi(w+i\alpha)}{2\beta}\right)}{8i\beta a^{-w}w^{k+1}}dw
\end{aligned} \tag{4}$$

from equation (2.7.24) in [1] and the integral is valid for α, a, k and β complex and $\Im(w)+\Re(\alpha+\beta)<0$ and $\Im(w)\leq\Re(\alpha+\beta)$ and $\Re(\beta)>0$. The hyperbolic tangent function can be expressed in terms of trigonometric and hyperbolic rational function given by:

$$\begin{aligned}
\tanh(x+iy) &= \frac{1}{i}\tan(i(x+iy)) \\
&= \frac{i}{i^2}\tan(ix-y) = -i\frac{\sin(ix-y)}{\cos(ix-y)} = -i\frac{\sin(ix-y)\cos(ix+y)}{\cos(ix-y)\cos(ix+y)} \\
&= -i\frac{\sin(2ix)-\sin(2y)}{\cos(2ix)+\cos(2y)} = \frac{\sinh(2x)+i\sin(2y)}{\cosh(2x)+\cos(2y)} \\
&= \frac{\sinh(2x)}{\cosh(2x)+\cos(2y)} + i\frac{\sin(2y)}{\cosh(2x)+\cos(2y)}
\end{aligned} \tag{5}$$

Since (4) involves the sum of tanh and tanh with the complex conjugate of its argument, the result of adding the two as in (4) is to remove the complex part in (5). Thus the hyperbolic tangent function can be simplified to the form in [1] using equation (5).

3 Infinite sum of the contour integral

In this section we will again use the generalized Cauchy's integral formula to derive equivalent contour integrals. First we replace y by $\pi(p+1)/\beta+\log(a)$

then multiply both sides by $\frac{\pi}{2\beta}(-1)^p e^{i\alpha(\pi(p+1)/\beta)}$ to get

$$\frac{\pi^{k+1}(-1)^p\left(\frac{1}{\beta}\right)^{k+1} e^{\frac{i\pi\alpha}{\beta}+\frac{i\pi\alpha p}{\beta}}\left(\frac{\beta\log(a)}{\pi}+p+1\right)^k}{2k!} = \int_C \frac{(-1)^p w^{-k-1}\exp\left(w\left(\log(a)+\frac{\pi(p+1)}{\beta}\right)+\frac{i\pi\alpha(p+1)}{\beta}\right)}{4i\beta}dw \quad (6)$$

then we take the infinite sum over $p \in [0,\infty)$ to get

$$\begin{aligned}\sum_{p=0}^{\infty}&\frac{\pi^{k+1}(-1)^p\left(\frac{1}{\beta}\right)^{k+1} e^{\frac{i\pi\alpha}{\beta}+\frac{i\pi\alpha p}{\beta}}\left(\frac{\beta\log(a)}{\pi}+p+1\right)^k}{2k!}\\ &=\sum_{p=0}^{\infty}\int_C \frac{(-1)^p w^{-k-1}\exp\left(w\left(\log(a)+\frac{\pi(p+1)}{\beta}\right)+\frac{i\pi\alpha(p+1)}{\beta}\right)}{4i\beta}dw\\ &=\int_C\sum_{p=0}^{\infty} \frac{(-1)^p w^{-k-1}\exp\left(w\left(\log(a)+\frac{\pi(p+1)}{\beta}\right)+\frac{i\pi\alpha(p+1)}{\beta}\right)}{4i\beta}dw\\ &=\int_C\left(\frac{a^w w^{-k-1}}{8i\beta}+\frac{a^w w^{-k-1}\tanh\left(\frac{\pi(w+i\alpha)}{2\beta}\right)}{8i\beta}\right)dw\end{aligned} \quad (7)$$

from equation (1.232.1) in [7]. Then we simplify the left-hand side to get the Lerch function equivalent to get

$$\frac{\pi^{k+1}e^{\frac{i\pi\alpha}{\beta}}\left(\frac{1}{\beta}\right)^{k+1}\Phi\left(-e^{\frac{i\pi\alpha}{\beta}},-k,\frac{\beta\log(a)}{\pi}+1\right)}{2k!} = \int_C\left(\frac{a^w w^{-k-1}}{8i\beta}+\frac{\pi a^w w^{-k-1}\tanh\left(\frac{\pi(w+i\alpha)}{2\beta}\right)}{8i\beta}\right)dw \quad (8)$$

Then we replace α with $-\alpha$ to get the second equation for the contour integral given by

$$\frac{\pi^{k+1}e^{-\frac{i\pi\alpha}{\beta}}\left(\frac{1}{\beta}\right)^{k+1}\Phi\left(-e^{-\frac{i\pi\alpha}{\beta}},-k,\frac{\beta\log(a)}{\pi}+1\right)}{2k!} = \int_C\left(\frac{a^w w^{-k-1}}{8i\beta}+\frac{a^w w^{-k-1}\tanh\left(\frac{\pi(w-i\alpha)}{2\beta}\right)}{8i\beta}\right)dw \quad (9)$$

Now we will derive the added contour using the generalized Cauchy's integral formula. Now we replace y by $\log(a)$ and multiply both sides by $\frac{\pi}{4\beta}$ simplify to get

$$\frac{\pi \log^k(a)}{4\beta k!} = \frac{1}{8i\beta}\int_C a^w w^{-k-1} dw \tag{10}$$

4 Definite integral in terms of the Lerch function

Since the right-hand sides of equation (4), (8) and (9) are equivalent we can equate the left-hand sides to get

$$\begin{aligned}\int_0^\infty \frac{\cosh(\alpha y)}{\sinh(\beta y)}\left((\log(a)+iy)^k - (\log(a)-iy)^k\right) dy \\ = i\pi^{k+1}e^{-\frac{i\pi\alpha}{\beta}}\left(\frac{1}{\beta}\right)^{k+1}\Phi\left(-e^{-\frac{i\pi\alpha}{\beta}}, -k, \frac{\beta\log(a)}{\pi}+1\right) \\ + i\pi^{k+1}e^{\frac{i\pi\alpha}{\beta}}\left(\frac{1}{\beta}\right)^{k+1}\Phi\left(-e^{\frac{i\pi\alpha}{\beta}}, -k, \frac{\beta\log(a)}{\pi}+1\right) - \frac{i\pi\log^k(a)}{\beta}\end{aligned} \tag{11}$$

from (9.550) in [7] where $\Phi(r, s, u)$ is the Lerch function. Note the left-hand side of equation (11) converges for all finite k. The integral in equation (11) can be used as an alternative method to evaluating the Lerch function. The Lerch function has a series representation given by

$$\Phi(z, s, v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n \tag{12}$$

where $|z|< 1, v \neq 0, -1, ..$ and is continued analytically by its integral representation given by

$$\Phi(z, s, v) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t - z}dt \tag{13}$$

where $Re(v) > 0$, or $|z|\leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$. A special case of the Lerch function is polylogarithm $\Phi(z, s, 1) = \frac{Li_s(z)}{z}$ given by equation (6) in [6].

5 Special cases and derivation of some Gröbner and Bierens de Haan integrals

In this section we will use equation (11) along with L'Hôpital's rule to derive some interesting definite integrals.

5.1 Derivation of entry 3.529.2 in [7]

Using equation (11) and setting $a = 1$ simplify we get

$$\begin{aligned}\int_0^\infty \frac{1}{2} i \left((iy)^k - (-iy)^k\right) \cosh(\alpha y) \operatorname{csch}(\beta y) dy \\ = \frac{1}{2}\pi^{k+1}\left(\frac{1}{\beta}\right)^{k+1} \operatorname{Li}_{-k}\left(-e^{-\frac{i\pi\alpha}{\beta}}\right) + \frac{1}{2}\pi^{k+1}\left(\frac{1}{\beta}\right)^{k+1} \operatorname{Li}_{-k}\left(-e^{\frac{i\pi\alpha}{\beta}}\right)\end{aligned} \tag{14}$$

where $\operatorname{Li}_k(z)$ is the polylogarithm function defined in equation (13) with $v = 1$. Now we will let $k = -1$ for the first equation to get

$$\int_0^\infty \frac{\cosh(\alpha y)\operatorname{csch}(\beta y)}{y} dy = -\frac{1}{2}\log\left(1 + e^{-\frac{i\pi\alpha}{\beta}}\right) - \frac{1}{2}\log\left(1 + e^{\frac{i\pi\alpha}{\beta}}\right) \tag{15}$$

and setting $k = -1$ and $\alpha = 0$ for the second equation to get

$$\int_0^\infty \frac{\operatorname{csch}(\beta y)}{y} dy = -\log(2) \tag{16}$$

Now we subtract equation (16) from (15) to get

$$\begin{aligned}\int_0^\infty \frac{\cosh(\alpha y)\operatorname{csch}(\beta y) - \operatorname{csch}(\beta y)}{y} dy &= -\frac{1}{2}\log\left(1 + e^{-\frac{i\pi\alpha}{\beta}}\right) - \frac{1}{2}\log\left(1 + e^{\frac{i\pi\alpha}{\beta}}\right) + \log(2) \\ &= \frac{1}{2}\left(\log(2) - \log\left(\cos\left(\frac{\pi\alpha}{\beta}\right) + 1\right)\right) \\ &= -\log\left(\cos\left(\frac{\pi\alpha}{2\beta}\right)\right)\end{aligned} \tag{17}$$

where $\Re(\beta) > \Re(\alpha)$.

5.2 Derivation of entry 3.529.1 in [7]

Using equation (14) setting $k = -2$ and $\beta = 1$ and taking the first partial derivative with respect to α we get

$$\int_0^\infty \frac{\operatorname{csch}(y)\sinh(\alpha y)}{y^2} dy = \frac{i\operatorname{Li}_2\left(-e^{-i\pi\alpha}\right)}{2\pi} - \frac{i\operatorname{Li}_2\left(-e^{i\pi\alpha}\right)}{2\pi} \tag{18}$$

Next we will use (14) setting $k = -1$ and $\beta = 1$ to get

$$\int_0^\infty \frac{\operatorname{csch}(y)\cosh(\alpha y)}{y} dy = -\frac{1}{2}\log\left(1 + e^{-i\pi\alpha}\right) - \frac{1}{2}\log\left(1 + e^{i\pi\alpha}\right) \tag{19}$$

Here we subtract equation (18) at $\alpha = 1$ from (19) at $\alpha = 0$ to get

$$\int_0^\infty \frac{1}{y^2} - \frac{\operatorname{csch}(y)}{y} dy = \log(2) \tag{20}$$

which simplifies to

$$\int_0^\infty \frac{y\operatorname{csch}(y) - 1}{y^2} dy = -\log(2) \tag{21}$$

5.3 Derivation of entry 3.529.3 in [7]

Using equation (14) and getting two equations by multiply by both sides by β then by setting $k = -1$, $\alpha = 0$ and $\beta = a$ for the first and multiply by both sides by β and setting $k = -1$, $\alpha = 0$ and $\beta = b$ for the second and subtracting we get

$$\int_0^\infty \frac{a\operatorname{csch}(ay) - b\operatorname{csch}(by)}{y} dy = (b - a)\log(2) \tag{22}$$

where $\Re(a) > 0$ and $\Re(b) > 0$.

5.4 Derivation of entry 3.524.5 in [7]

Using equation (14) and simplifying the left-hand side we get

$$\begin{aligned}\int_0^\infty y^k \cosh(\alpha y)\operatorname{csch}(\beta y) dy &= \frac{i\pi^{k+1}\left(\frac{1}{\beta}\right)^{k+1}\left(\mathrm{Li}_{-k}\left(-e^{-\frac{i\pi\alpha}{\beta}}\right) + \mathrm{Li}_{-k}\left(-e^{\frac{i\pi\alpha}{\beta}}\right)\right)}{(-i)^k - i^k} \\ &= -\frac{1}{2}\pi^{k+1}\left(\frac{1}{\beta}\right)^{k+1}\csc\left(\frac{\pi k}{2}\right)\left(\mathrm{Li}_{-k}\left(-e^{-\frac{i\pi\alpha}{\beta}}\right)\right. \\ &\qquad \left. + \mathrm{Li}_{-k}\left(-e^{\frac{i\pi\alpha}{\beta}}\right)\right)\end{aligned} \tag{23}$$

where $\Re(\beta) > \Re(\alpha)$. The Hurwitz equivalent to the Polylogarithm function is derived using Joncquière's relation (1.11.16) in [5].

5.5 Derivation of entry 3.525.3 in [7]

Using equation (11) and setting $k = -1$, $a = e$, $\beta = \pi$ and simplifying the left-hand side we get

$$\begin{aligned}\int_0^\infty \frac{y\operatorname{csch}(\pi y)\cosh(\alpha y)}{y^2 + 1} dy &= \frac{1}{2}\left(-1 + e^{i\alpha}\log\left(1 + e^{-i\alpha}\right) + e^{-i\alpha}\log\left(1 + e^{i\alpha}\right)\right) \\ &= \frac{1}{2}(\alpha\sin(\alpha) + \cos(\alpha)\log(2(\cos(\alpha) + 1)) - 1)\end{aligned} \tag{24}$$

where $-\pi < \Re(\alpha) < \pi$.

5.6 Derivation of entry 3.525.4 in [7]

Using equation (11) and setting $k = -1$, $a = e$, $\beta = \pi/2$ and simplifying the left-hand side, expanding the complex exponentials using Euler's formula and using the identities for $\tan^{-1}(x)$ we get

$$\begin{aligned}\int_0^\infty \frac{y\operatorname{csch}\left(\frac{\pi y}{2}\right)\cosh(\alpha y)}{y^2+1}dy &= 1 + e^{i\alpha}\tan^{-1}\left(e^{-i\alpha}\right) + e^{-i\alpha}\tan^{-1}\left(e^{i\alpha}\right) \\ &= \frac{1}{2}\pi\cos(\alpha) + \sin(\alpha)\tanh^{-1}(\sin(\alpha)) - 1\end{aligned} \tag{25}$$

where $-\pi/2 < \Re(\alpha) < \pi/2$. In this evaluation we used the identities

$$\tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

and

$$\tan^{-1}(x) = \frac{1}{2i}\log\left(\frac{x-i}{x+i}\right)$$

where $x > 0$.

5.7 Derivation of entry 3.525.8 in [7]

Using equation (11) setting $k = -1$, and replacing a by e^a and simplifying we get

$$\begin{aligned}\int_0^\infty \frac{y\cosh(\alpha y)\operatorname{csch}(\beta y)}{a^2+y^2}dy = \frac{1}{2}\Bigg(&\frac{\pi}{a\beta} - e^{-\frac{i\pi\alpha}{\beta}}\Phi\left(-e^{-\frac{i\pi\alpha}{\beta}}, 1, \frac{a\beta}{\pi}+1\right) \\ &- e^{\frac{i\pi\alpha}{\beta}}\Phi\left(-e^{\frac{i\pi\alpha}{\beta}}, 1, \frac{a\beta}{\pi}+1\right)\Bigg)\end{aligned} \tag{26}$$

where $\Re(\beta) > \Re(\alpha)$. This solution represents the analytic continuation of the definite integral, whereas the listed solution in [7] is bounded by positive b values.

6 Conclusion

In this paper, we have presented a novel method for deriving some interesting definite integrals by Gröber and Bierens de Haan using contour integration. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

Acknowledgements. This paper is fully supported by the Natural Sciences and Engineering Research Council (NSERC) Grant No. 504070.

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Received: July 18, 2020; Published: August 8, 2020