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On Fuzzy b^*-Closed Sets in Fuzzy Topological Space

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Abstract

Fuzzy set was defined by Zadeh[15]in 1965 as a mapping from a set into a unit interval on the real line. It is characterized by a membership function which assigns to each object a grade of membership ranging from zero and one, in which the closer the image of an object to one, the higher the membership of it to the set. In 1968, C.L. Chang[11] then introduced fuzzy topological spaces.

In this paper, a new class of fuzzy sets in fuzzy topological spaces called fuzzy b^*-closed sets, fuzzy b^*-open sets, fuzzy b^*-closure and fuzzy b^*-interior are introduced and some of their properties are investigated. Also, we further studied some structures of fuzzy b^*-continuous mapping, fuzzy b^*-open mapping and fuzzy b^*-closed mapping which were introduced by S.S. Benchalli and Jenifer Karnel in [3]. We also defined another class of functions called the relative fuzzy b^*-continuous and fuzzy generalized b^*-continuous functions.

Keywords: fuzzy set, fuzzy topology, fuzzy b^*-open set, fuzzy b^*-closed set, fuzzy gb^*-continuous

1 Introduction

Fuzzy set was introduced by L.A. Zadeh[15] in his classical paper in 1965 as a class of objects with a continuum of grades of membership. It is characterized by a membership function which assigns to each object a grade of membership ranging from zero and one, in which the closer the image of an object to one, the higher the membership of it to the set. In 1968, the study of fuzzy topology was introduced by C.L. Chang[11]. Several authors then subsequently developed various concepts in fuzzy topological spaces.

In 2010, S.S. Benchalli and Jenifer Karnel[3] introduced the concept of fuzzy b-open sets and fuzzy b-closed sets in fuzzy topological spaces. They discussed the relationship of fuzzy b-open(closed) set with fuzzy pre-open set, fuzzy semi-open set, fuzzy semi-pre-open sets and fuzzy regular open set. Some properties of arbitrary union and arbitrary intersection of fuzzy b-open sets using the definition of fuzzy b-closure and fuzzy b-interior were also investigated. Moreover, the concept of fuzzy generalized b-closed and fuzzy generalized b-open set were also introduced and its interrelationship with fuzzy b-closed sets were also investigated.

S.S. Benchalli and Jenifer Karnel also introduced fuzzy b-continuous mapping, fuzzy b^*-continuous mapping, fuzzy b-open map and fuzzy b-closed map in fuzzy topological spaces in [4]. They also used the concept of fuzzy b-open sets in defining fuzzy b-continuous map and fuzzy b^*-continuous map. Interrelationship of fuzzy b-continuous map and fuzzy b^*-continuous map with fuzzy pre-continuous, fuzzy semi-continuous and fuzzy semi-pre-continuous were also established. Furthermore, composition of fuzzy $b-$continuous mapping and fuzzy $b-$open(closed) mapping were also presented. However, restriction of domain and range of a fuzzy b^*-continuous mapping were not considered. With these, the researcher is interested in extending their studies to investigate such concepts.

It is then interesting to introduce the new concept of fuzzy sets called fuzzy b^*-closed sets, fuzzy b^*-open sets, fuzzy b^*-closure and fuzzy b^*-interior. We investigate its behavior and various relationship with fuzzy closed set, fuzzy generalized open set and other fuzzy sets and some properties in relative fuzzy topology.

2 Preliminaries

Let X be a nonempty set and I be the unit interval $[0, 1]$. A fuzzy set A of X is characterized by the membership function, $\mu_A : X \to I$ where $\mu_A(x)$ is interpreted as the degree of membership of an element x in a fuzzy set A for each $x \in X$. For convenience, we do not distinguish between a fuzzy set A and its membership function μ_A, and if X is countable, we shall denote a fuzzy set A by the notation $A = \{(x_1, a_1), (x_2, a_2), \ldots, (x_n, a_n), \ldots\}$. The complement of a fuzzy set A, denoted by A^C, is $A^C(x) = 1 - A(x)$ for each $x \in X$. A family τ of fuzzy sets of X is called a fuzzy topology on X if $\mathcal{O}$ and $\mathcal{I}$ belongs to τ and τ is closed with respect to arbitrary union and finite intersection. Members of τ are called fuzzy open sets and their complement are fuzzy closed sets. A fuzzy set A is a subset of a fuzzy set B, denoted by $A \subseteq_f B$ if $A(x) \leq B(x)$ for all $x \in X$. The closure of a fuzzy set A of X is defined by $cl(A) = \bigwedge \Big\{F : F \text{ is fuzzy closed and } A \subseteq_f F\Big\}$ and the interior of A is defined as $int(A) = \bigvee \Big\{O : O \text{ is fuzzy open and } O \subseteq_f A\Big\}$.

Definition 2.1. [12] Let A be a fuzzy set of X and τ be a fuzzy topology on X. Then the relative fuzzy topology on A, denoted by τ_A, is the family of fuzzy sets of X which are the intersection with A to τ-open fuzzy sets in X, that is, $\tau_A = \{A \wedge G : G \in \tau\}$. The pair (A, τ_A) is called a fuzzy subspace of (X, τ) and each fuzzy set $C \in \tau_A$ are then called fuzzy open sets in A.

Remark 2.2. It is important to note that in the relative fuzzy topology τ_A, the fuzzy sets in A are fuzzy sets B such that $B \subseteq_f A$ and the fuzzy open sets in A are understood to be $A \wedge G$ and not as a function whose domain is A for A is not a set but a fuzzy set.

Definition 2.3. [3, 8, 4] A fuzzy set A of a fuzzy topological space (X, τ) is called:

(a) fuzzy b-open (fuzzy b-closed) if $A \subseteq_f cl(intA) \vee int(clA)$ $\Big(int(clA) \wedge cl(intA) \subseteq_f A\Big)$.

(b) fuzzy pre-open (fuzzy pre-closed) if $A \subseteq_f int(clA)$ $\Big(cl(intA) \subseteq_f A\Big)$.

(c) fuzzy α-open (fuzzy α-closed) set if $A \subseteq_f int(cl(int\, A))$ $\Big(cl(int(clA)) \subseteq_f A\Big)$.

(d) fuzzy regular open (fuzzy regular closed) set if $A = int(cl\ A)$ $(A = cl(int\ A))$.

(e) fuzzy semi-open (fuzzy semi-closed) set if $A \subseteq_f cl(int\, A)$ $\Big(int(clA) \subseteq_f A\Big)$.

Remark 2.4. *Every fuzzy open (resp. closed) set is fuzzy b-open (resp. b-closed) set, fuzzy α-open (resp. α-closed), and fuzzy semi-open (resp. semi-closed) set.*

Theorem 2.5. [3] *Let (X, τ) be a fuzzy topological space.*

i. Arbitrary union of fuzzy $b-$open sets is fuzzy $b-$open.

ii. Arbitrary intersection of fuzzy $b-$closed sets is fuzzy $b-$closed.

Theorem 2.6. [11] *Let A and B be fuzzy sets of X such that $B \subseteq_f A$. Then the following holds:*

(i) $A \wedge int\ B \subseteq_f int_A B$. *If A is fuzzy open, equality holds.*

(ii) $A \wedge cl\ B = cl_A B$

Lemma 2.7. *Let (X, τ) be a fuzzy topological space and A be a fuzzy open set of X. If $G \subseteq_f A \subseteq_f X$ then the following holds:*

(i) $int_A\,(cl_A G) \subseteq_f int(cl\ G)$

(ii) $cl_A(int_A G) \subseteq_f cl(int\ G)$

3 Fuzzy b^*-Open Sets, Fuzzy b^*-Closed Sets

In this section, we will present some properties of fuzzy b-open sets which are not studied by S.S. Benchalli and Jenifer Karnel in [3] and [4]. We will also define another class of fuzzy set, the fuzzy b^*-closed and fuzzy b^*-open sets.

Theorem 3.1. *Let A, B be fuzzy sets in X and A be fuzzy open such that $B \subseteq_f A$. Then B is fuzzy b-open in X if and only if B is fuzzy b-open in A.*

Proof. Suppose B is fuzzy b-open in X. Then by Definition 2.3, $B \subseteq_f int(cl\ B) \vee cl(int\ B)$. Hence, we have $B = A \wedge B \subseteq_f A \wedge [int(cl\ B) \vee cl(int\ B)] = [A \wedge int(cl\ B)] \vee [A \wedge cl(int\ B)]$. Since A is fuzzy open, $A = int(A)$. Hence by Theorem 2.6(ii), we have

$$\begin{aligned} B &\subseteq_f [int(A) \wedge int(cl\ B)] \vee [A \wedge cl(int\ B)] \\ &= int(A \wedge cl\ B) \vee [A \wedge cl(int\ B)] = int(cl_A B) \vee cl_A(int\ B) \end{aligned}$$

Since $int(cl_A B) \subseteq_f A$ and $int(B) \subseteq_f B \subseteq_f A$, $int(cl_A B) = A \wedge int(cl_A B)$ and $cl_A(int\ B) = cl_A(A \wedge int\ B)$. Hence $B \subseteq_f [A \wedge int\,(cl_A B)] \vee cl_A(A \wedge int\ B)$. By Theorem 2.6(i), $B \subseteq_f [A \wedge int(cl_A B)] \vee cl_A(A \wedge int\ B) \subseteq_f int_A(cl_A B) \vee cl_A(int_A B)$. Therefore, B is fuzzy b-open relative to A.

Conversely, suppose B is fuzzy b-open relative to A. Then $B \subseteq_f int_A(cl_A B) \vee cl_A(int_A B)$. And by Theorem 2.7, $B \subseteq_f int(cl\ B) \vee cl(int\ B)$ Therefore, B is fuzzy b-open in X □

We will now introduce the notion of fuzzy b^*-closed sets.

Definition 3.2. A fuzzy set A of a fuzzy topological space (X, τ) is called a **fuzzy b^*-closed set** if $int(cl\ A) \subseteq_f U$, whenever $A \subseteq_f U$ and U is fuzzy b-open.

Remark 3.3. Note that $\mathcal{O}$ is fuzzy b^*-closed since $int(cl\ \mathcal{O}) = int(\mathcal{O}) = \mathcal{O}$ which is always a subset of any fuzzy b-open set U. Similarly, $\mathcal{I}$ is fuzzy b^*-closed since the only fuzzy b-open set U that contains $\mathcal{I}$ is $\mathcal{I}$, i.e. $U = \mathcal{I}$. Thus $int(cl\ \mathcal{I}) = int(\mathcal{I}) = \mathcal{I}$ which is a subset of U.

Example 3.4. Consider the fuzzy topology $\tau = \{\mathcal{O}, \mathcal{I}, A, B\}$ on X where $X = \{a, b, c\}$, $A = \{(a, 0.7), (b, 0.3), (c, 1)\}$, $B = \{(a, 0.7), (b, 0), (c, 0)\}$ and $\tau = \{\mathcal{O}, \mathcal{I}, A, B\}$. Then fuzzy b^*-closed sets are $\mathcal{O}$, $\mathcal{I}$ and D, where D is defined as $D(a) \leq 0.3$, for any $D(b)$ and $D(c)$.

To see this, we have by Remark 3.3, $\mathcal{O}$ and $\mathcal{I}$ are fuzzy b^*-closed sets. Hence, we only have to consider the following cases:

case 1: If $D(b) \leq 0.3$ and $D(c) > 0$

Note that the fuzzy b-open set that contains D are $A, \mathcal{I}$ and a fuzzy set $C = \{(a, C(a)), (b, C(b)), (c, C(c))\}$ where $C(a) > 0.3$, $C(b) > 0.3$ and $C(c) > D(c)$. Now, $cl(D) = B^C \wedge \mathcal{I} = B^C$. Thus, $int(cl\ D) = int(B^C) = \mathcal{O}$. Let U be equal to A, $\mathcal{I}$ and C. Then $int(cl\ D) = \mathcal{O} \subseteq_f U$ for each fuzzy b-open set U containing D.

case 2: If $D(b) \leq 0.3$ and $D(c) = 0$

Note that the fuzzy b-open set that contains D are $A, \mathcal{I}$ and a fuzzy set $C = \{(a, C(a)), (b, C(b)), (c, C(c))\}$ where $C(a) > 0.3$, $C(b) > 0.3$ and for any $C(c)$. Now, $cl(D) = A^C \wedge B^C \wedge \mathcal{I} = A^C$. Thus, $int(cl\ D) = int(A^C) = \mathcal{O}$. Let U be equal to A, $\mathcal{I}$ and C. Then $int(cl\ D) = \mathcal{O} \subseteq_f U$ for each fuzzy b-open set U containing D.

case 3: If $D(b) > 0.3$ and $D(c) > 0$

Note that the fuzzy b-open set that contains D are $\mathcal{I}$ and a fuzzy set C where

$C(a) > 0.3$, $C(b) > D(b)$ and $C(c) > D(c)$. Now, $cl(D) = B^C \wedge \mathcal{I} = B^C$. Thus, $int(cl\ D) = int(B^C) = \mathcal{O}$. Let U be equal to $\mathcal{I}$ and C. Then $int(cl\ D) = \mathcal{O} \subseteq_f U$ for each fuzzy b-open set U containing D.

case 4: If $0.3 < D(b) \leq 0.7$ and $D(c) = 0$

Note that the fuzzy b-open set that contains D are $\mathcal{I}$ and a fuzzy set C where

$C(a) > 0.3$, $C(b) > D(b)$ and $C(c) > D(c)$. Now, $cl(D) = A^C \wedge B^C \wedge \mathcal{I} = A^C$. Thus, $int(cl\ D) = int(A^C) = \mathcal{O}$. Let U be equal to I and C. Then $int(cl\ D) = \mathcal{O} \subseteq_f U$ for each fuzzy b-open set U containing D.

case 5: If $D(b) > 0.7$ and $D(c) = 0$

Note that the fuzzy b-open set that contains D are $\mathcal{I}$ and a fuzzy set C where

$C(a) > 0.3$, $C(b) > D(b)$ and $C(c) > D(c)$. Now, $cl(D) = B^C \wedge \mathcal{I} = B^C$.

Thus, $int(cl\ D) = int(B^C) = \mathcal{O}$. Let U be equal to I and C. Then $int(cl\ D) = \mathcal{O} \subseteq_f U$ for each fuzzy b-open set U containing D.

Hence, $D = \{(a, D(a)), (b, D(b)), (c, D(c))\}$ is fuzzy b^*-closed sets where $D(a) \leq 0.3$, for any $D(b)$ and $D(c)$.

Definition 3.5. Let A be a fuzzy set of X. A is said to be **fuzzy b^*-open** if A^C is fuzzy b^*-closed.

Remark 3.6. A is fuzzy b^*-open if and only if $F \subseteq_f cl\,(int\ A)$ where F is fuzzy $b-$closed and $F \subseteq_f A$.

Proof. Let A be a fuzzy b^*-open and F be fuzzy b-closed such that $F \subseteq_f A$. Then A^c is fuzzy b^*-closed and $A^c \subseteq_f F^c$ where F^c is fuzzy b-open. Hence, $int\,(cl\,(A^c)) \subseteq_f U$ where $A^c \subseteq_f U$ and U is fuzzy b-open. Since F^c is one of the fuzzy b-open sets that contains A^c, we have $int\,(cl\,(A^c)) \subseteq_f F^c$. Taking the complements both sides, $F \subseteq_f [int\,(cl\,(A^c))]^c \subseteq_f cl\,(cl\,(A^c))^c = cl\,(int\,(A^c)^c) = cl\,(int\,(A))$.

Conversely, let U be fuzzy b-open and $A^C \subseteq_f U$. Then $U^C \subseteq_f A$ where U^C is fuzzy b-closed. By the assumptions, $U^C \subseteq_f cl\,(int\,(A))$ implying $[cl\,(int\,(A))]^C \subseteq_f U$. By Theorem 2.6, we have $int\left(cl\left(A^C\right)\right) = int\,(int\,(A))^C = [cl\,(int\,(A))]^C \subseteq_f U$. That is, $int(cl\ A^C) \subseteq_f U$. Therefore A^C is fuzzy b^*-closed and that A is fuzzy b^*-open. □

The next result shows that the image of any fuzzy closed set contained in the intersection of the interior of the closure of a fuzzy b^*-closed set and its complement is always less than or equal to 0.5.

Theorem 3.7. *If A is a fuzzy b^*-closed set and F is fuzzy closed with $F \subseteq_f int(clA) \wedge A^C$, then $F(x) \leq 0.5$ for all $x \in X$.*

Proof. Let A be a fuzzy b^*-closed set in X. Suppose there exist a fuzzy closed set F with $F \subseteq_f int(clA) \wedge A^C$ and $F(x) > 0.5$ for some $x \in X$. Then $F \subseteq_f int(cl\ A)$ and $F \subseteq_f A^C$. Since $F \subseteq_f A^C$, we have $A \subseteq_f F^C$. But F is a fuzzy closed set hence, F^C is fuzzy open. By Remark 2.4, F^C is fuzzy b-open. Now, since A is fuzzy b^*-closed set, by Definition 3.2, $int(cl\ A) \subseteq_f U$ where U is fuzzy b-open and $A \subseteq_f U$. Thus, F^C is one of the $U's$. That is, $int(cl\ A) \subseteq_f F^C$. But $F \subseteq_f int(cl\ A)$. Hence $F \subseteq_f F^C$. However, note that $F(x) > 0.5$ for some $x \in X$ so that $F^C(x) < 0.5$ for some $x \in X$. Hence F cannot be subset of F^C for $F \subseteq_f F^C$ is possible only if $F(x) \leq 0.5$ for all $x \in X$. Thus, for each $x \in X$, $F(x) \leq 0.5$. □

Theorem 3.8. *Let A be a fuzzy set of X. If A is fuzzy closed, then A is fuzzy b^*-closed.*

Corollary 3.9. *Every fuzzy open set is fuzzy b^*-open set.*

Theorem 3.10. *Let A be a fuzzy set of X. If A is fuzzy α-closed, then A is fuzzy b^*-closed.*

Corollary 3.11. *Every fuzzy α-open set is fuzzy b^*-open.*

Theorem 3.12. *Let A be a fuzzy set of X. If A is fuzzy regular open, then A is fuzzy b^*-closed.*

Theorem 3.13. *Let A be a fuzzy set of X. If A is fuzzy semi-closed in X, then A is fuzzy b^*-closed.*

Corollary 3.14. *Every fuzzy semi-open set is fuzzy b^*-open.*

Theorem 3.15. *Let A be a fuzzy set of X. If A is both fuzzy b-open and fuzzy b^*-closed then A is fuzzy b-closed.*

Proof. Let A be both fuzzy b^*-closed and fuzzy b-open set. Then $int(clA) \subseteq_f O$, where O is fuzzy b-open and $A \subseteq_f O$. Since A is fuzzy b-open, A is one of the $O's$. Hence $int(clA) \subseteq_f A$. Now, $cl(intA) \wedge int(clA) \subseteq_f int(clA) \subseteq_f A$. Therefore, A is fuzzy b-closed. □

Theorem 3.16. *Let A be fuzzy b^*-closed set and $A \subseteq_f B \subseteq_f cl(A)$. Then B is fuzzy b^*-closed in X.*

Proof. Let U be fuzzy b-open set such that $B \subseteq_f U$. Then $A \subseteq_f U$. Since A is fuzzy b^*-closed, $int(cl\ A) \subseteq_f U$. Now, $B \subseteq_f cl(A)$ implying $cl(B) \subseteq_f cl(cl\ A) = cl(A)$. Consequently, $int(cl\ B) \subseteq_f int(cl\ A) \subseteq_f U$. Therefore, B is fuzzy b^*-closed. □

Theorem 3.17. *Let A, B be fuzzy sets of X such that $B \subseteq_f A$ and A is fuzzy open. If B is fuzzy b^*-closed set in A and A is fuzzy b^*-closed in X, then B is fuzzy b^*-closed set in X.*

Proof. Let U be fuzzy b-open in X such that $B \subseteq_f U$. Since A is fuzzy b^*-closed set and A is fuzzy open, A is itself fuzzy b-open, hence, $int(clA) \subseteq_f A$. With $B \subseteq_f A$, we have $int(cl\ B) \subseteq_f int(cl\ A) \subseteq_f A$. Thus $int\,(cl\ B) = A \wedge int\,(cl\ B)$. But A is fuzzy open, hence $A = int(A)$. Thus, $int(cl\ B) = A \wedge int(A) \wedge int(cl\ B)$. Therefore, Theorem 2.6(i) and (ii), we have $int(cl\ B) = A \wedge int(A) \wedge int(cl\ B) = A \wedge int\,(A \wedge cl\ B) = A \wedge int\,(cl_A B) = int_A\,(cl_A B)$. Similarly, $A \wedge U$ is fuzzy b-open in X, by Theorem 2.5(ii). Since $A \wedge U \subseteq_f A$, by Theorem 3.1, $A \wedge U$ is fuzzy b-open in A. Hence, $int(cl\ B) = int_A\,(cl_A B) \subseteq_f A \wedge U \subseteq_f U$ where $B \subseteq_f A \wedge U$ and $A \wedge U$ is fuzzy b-open. Therefore, B is fuzzy b^*-closed in X. □

4 Fuzzy b^*-closure & Fuzzy b^*-interior of a Fuzzy Set

We now introduce notion of fuzzy b^*-closure and fuzzy b^*-interior of a fuzzy set.

Definition 4.1. Let A be a fuzzy set in a fuzzy topological space X. Then **fuzzy b^*-closure of** A (or simply $fb^*cl(A)$) and **fuzzy b^*-interior of** A (or simply $fb^*int(A)$) are defined respectively as:
$fb^*cl(A) = \bigwedge \left\{ F : A \subseteq_f F \text{ and } F \text{ is fuzzy } b^*\text{-closed set of } X \right\}$ and $fb^*int(A) = \bigvee \left\{ O : O \subseteq_f A \text{ and } O \text{ is fuzzy } b^*\text{-open set of } X \right\}$.

Remark 4.2. *Let A be a fuzzy set in a fuzzy topological space (X, τ). Then $A \subseteq_f fb^*cl(A)$ and $fb^*int(A) \subseteq_f A$.*

Theorem 4.3. *Let (X, τ) be a fuzzy topological space. If A is fuzzy b^*-closed then*
$A = fb^*cl(A)$.

Proof. Suppose A is fuzzy b^*-closed set. Since $A \subseteq_f A$ and $A \in \left\{ F : F \text{ is fuzzy } b^*\text{-closed set of } X \text{ and } A \subseteq_f F \right\}$, A is the smallest of such F, we have $A = \bigwedge \left\{ F : F \text{ is fuzzy } b^*\text{-closed set of } X \text{ and } A \subseteq_f F \right\} = fb^*cl(A)$. □

Theorem 4.4. *Let (X, τ) ba a fuzzy topological space. If A is fuzzy b^*-open, then*
$A = fb^*int(A)$.

Proof. Suppose A is a fuzzy b^*-open set. Since $A \subseteq_f A$, $A \in \left\{ O : O \text{ is fuzzy } b^*\text{-open set of } X \text{ and } O \subseteq_f A \right\}$. Since A is the largest of such O, we have $A = \bigvee \left\{ O : O \subseteq_f A \text{ and } O \text{ is fuzzy } b^*\text{-open set of } X \right\} = fb^*int(A)$. □

Corollary 4.5. *Let (X, τ) ba a fuzzy topological space. If A is fuzzy set in X, then*
$cl(A) = fb^*cl(cl(A))$.

Theorem 4.6. *Let A be a fuzzy set in X. Then the following are true:*

(i) $[fb^*int(A)]^C = fb^*cl\left(A^C\right)$

(ii) $[fb^*cl(A)]^C = fb^*int\left(A^C\right)$

Proof.

(i) By Definition 4.1, $fb^*int(A) = \bigvee \left\{ O : O \subseteq_f A \text{ and } O \text{ is fuzzy } b^*\text{-open set of } X \right\}$. Taking the complement, we have

$$\begin{aligned}
[fb^*int(A)]^C &= \left[\bigvee \left\{ O : O \subseteq_f A \text{ and } O \text{ is fuzzy } b^*\text{-open set of } X \right\} \right]^C \\
&= \bigwedge \left\{ O^C : O \subseteq_f A \text{ and } O \text{ is fuzzy } b^*\text{-open set of } X \right\} \\
&= \bigwedge \left\{ O^C : A^C \subseteq_f O^C \text{ and } O^C \text{ is fuzzy } b^*\text{-closed set of } X \right\} \\
&= fb^*cl\left(A^C\right)
\end{aligned}$$

Therefore, $[fb^*int(A)]^C = fb^*cl\left(A^C\right)$

(ii) By Definition 4.1, $fb^*cl(A) = \bigwedge \left\{ F : A \subseteq_f F \text{ and } F \text{ is fuzzy } b^*\text{-closed set of } X \right\}$. Taking the complement, we have

$$\begin{aligned}
[fb^*cl(A)]^C &= \left[\bigwedge \left\{ F : A \subseteq_f F \text{ and } F \text{ is fuzzy } b^*\text{-closed set of } X \right\} \right]^C \\
&= \bigvee \left\{ F^C : A \subseteq_f F \text{ and } F \text{ is fuzzy } b^*\text{-closed set of } X \right\} \\
&= \bigvee \left\{ F^C : F^C \subseteq_f A^C \text{ and } F^C \text{ is fuzzy } b^*\text{-open set of } X \right\} \\
&= fb^*int\left(A^C\right)
\end{aligned}$$

Therefore, $[fb^*cl(A)]^C = fb^*int\left(A^C\right)$

□

5 Fuzzy b^*-continuous and Fuzzy gb^*-continuous

We will now discuss some properties of fuzzy b^*-continuous, fuzzy b^*-open, and fuzzy b^*-closed functions which are not introduced by S.S. Benchalli and Jenifer Karnel in [4]. We will also define another class of functions called the relative fuzzy b^*-continuous and fuzzy generalized b^*-continuous functions.

Theorem 5.1. *Let $f : X \to Y$ and $g : Y \to Z$ be two functions. If f is fuzzy b^*-continuous, onto and $g \circ f$ is fuzzy b^*-closed, then g is fuzzy b^*-closed function.*

Proof. Let A be a fuzzy b-closed set in Y. Since f is fuzzy b^*-continuous, $f^{-1}(A)$ is fuzzy $b-$closed set in X. Now, since $g \circ f$ is fuzzy b^*-closed function and f is onto, $(g \circ f)(f^{-1}(A)) = g(f(f^{-1}(A))) = g(A)$ is fuzzy $b-$closed set in Z. Therefore, $g : Y \to Z$ is fuzzy b^*-closed function. □

Theorem 5.2. *If $f : X \to Y$ is fuzzy $b-$closed and $g : Y \to Z$ is fuzzy b^*-closed, then $g \circ f : X \to Z$ is a fuzzy b-closed.*

Proof. Let A be a fuzzy closed set in X. Since f is fuzzy $b-$closed, $f(A)$ is fuzzy $b-$closed in Y. Also, g is fuzzy b^*-closed map so, $g(f(A))$ is fuzzy $b-$closed in Z. But $(g \circ f)(A) = g(f(A))$. Hence, $(g \circ f)(A)$ is fuzzy $b-$closed in Z for every fuzzy closed set A in X. Therefore, $g \circ f$ is fuzzy b^*-closed. □

Lemma 5.3. *If $f : X \to Y$ is onto, then $f(\mathcal{I})$ is fuzzy open.*

Proof. Let $f : X \to Y$ be onto. Then $f(\mathcal{I})(y) = \sup\left\{\mathcal{I}(z), z \in \left\{f^{-1}(y)\right\}\right\} = 1$. Since this is true for all $y \in Y$, $f(\mathcal{I})$ is the $\mathcal{I}$ function on Y. Hence $f(\mathcal{I}) = \mathcal{I}$ is fuzzy open in Y. □

Theorem 5.4. (Restriction of Range)
If $f : X \to Y$ is fuzzy b^*-continuous and onto, then $f : X \to f(\mathcal{I})$ is fuzzy b^*-continuous.

Proof. Let A be fuzzy b-open in $f(\mathcal{I})$. Then by Theorem 3.1, A is fuzzy b-open in Y for $f(\mathcal{I})$ is fuzzy open. Since $f : X \to Y$ is fuzzy b^*-continuous, $f^{-1}(A)$ is fuzzy b-open in X. Therefore, $f : X \to f(\mathcal{I})$ is fuzzy b^*-continuous. □

We will now introduce the notion of relative fuzzy b^*-continuity.

Definition 5.5. Let $f : (X, \tau) \to (Y, \mathcal{U})$ be a mapping of fuzzy topological spaces, (A, τ_A) and $(B, \mathcal{U}_B)$ be relative fuzzy topologies of (X, τ) and $(Y, \mathcal{U})$, respectively with $f(A) \subseteq_f B$. Then $f : (A, \tau_A) \to (B, \mathcal{U}_B)$ is said to be **relative fuzzy b^*-continuous** if $f^{-1}(O) \wedge A$ is fuzzy b-open in A for each fuzzy b-open set O in B.

Theorem 5.6. *Let $f : (X, \tau) \to (Y, \mathcal{U})$ be a bijective function of fuzzy topological spaces. Then the following statements are equivalent:*

(i) $f : (X, \tau) \to (Y, \mathcal{U})$ *is fuzzy b^*-continuous.*

(ii) $f : (A, \tau_A) \to (B, U_B)$ *is relative fuzzy b^*-continuous for every fuzzy open set A in X and fuzzy open set B in Y where $f(A) = B$.*

Proof. Let O be fuzzy b-open in B. By Theorem 3.1, O is fuzzy b-open in Y. Since $f:(X,\tau)\to(Y,\mathcal{U})$ is fuzzy b^*-continuous, $f^{-1}(O)$ is fuzzy b-open in X. With $O\subseteq_f B$ and $B=f(A)$, $f^{-1}(O)\subseteq_f f^{-1}(B)=A\subseteq_f X$. Since $f^{-1}(O)$ is fuzzy b-open in X, $f^{-1}(O)$ is fuzzy b-open in A, by Theorem 3.1. Also, since $f^{-1}(O)\subseteq_f A$, $f^{-1}(O)\wedge A=f^{-1}(O)$. Therefore, $f:(A,\tau_A)\to(B,U_B)$ is relative fuzzy b^*-continuous.

Suppose (ii) holds and that G be fuzzy b-open in Y. Since $G\subseteq_f \mathcal{I}\subseteq_f Y$ and $\mathcal{I}$ is fuzzy open, by Theorem 3.1, G is fuzzy b-open in $\mathcal{I}$. Since $f^{-1}(\mathcal{I})(x)=\mathcal{I}(f(x))=1$ for all $x\in X$, $f^{-1}(\mathcal{I})(x)=\mathcal{I}$. Hence, by (ii) with $f:(\mathcal{I},\tau_{\mathcal{I}})\to(\mathcal{I},\mathcal{U}_{\mathcal{I}})$ and G being fuzzy b-open in $\mathcal{I}$, $f^{-1}(G)\wedge\mathcal{I}=f^{-1}(G)$ is fuzzy b-open in $\mathcal{I}$. Implying further that $f^{-1}(G)$ is fuzzy b-open in X by Theorem 3.1, for $f^{-1}(G)\subseteq_f \mathcal{I}\subseteq_f X$. Therefore, $f:(X,\tau)\to(Y,\mathcal{U})$ is fuzzy b^*-continuous. □

Theorem 5.7. *Let $f:(X,\tau)\to(Y,\mathcal{U})$ be a function of fuzzy topological spaces. Then the following statements are equivalent:*

(i) $f:(X,\tau)\to(Y,\mathcal{U})$ is fuzzy b^*-open.

(ii) $f:(A,\tau_A)\to(B,U_B)$ is fuzzy b^*-open for every fuzzy open set A in X and fuzzy open set B in Y.

Proof. $(i)\Rightarrow(ii)$ Let O be fuzzy b-open in A. Then, O is fuzzy b-open in X. Since $f:(X,\tau)\to(Y,\mathcal{U})$ is fuzzy b^*-open, $f(O)$ is fuzzy b-open in Y. With $O\subseteq_f A\subseteq_f X$, $f(O)\subseteq_f f(A)\subseteq_f Y$. Since $f:(A,\tau_A)\to(B,U_B)$ is a mapping, by definition, $f(A)\subseteq_f B$. Hence $f(O)\subseteq_f B\subseteq_f Y$. Therefore, $f(O)$ is fuzzy b-open in B since B is fuzzy open in Y. Hence, $f:(A,\tau_A)\to(B,U_B)$ is fuzzy b^*-open.

$(ii)\Rightarrow(i)$ Let G be fuzzy b-open in X. Since $G\subseteq_f \mathcal{I}\subseteq_f X$ and $\mathcal{I}$ is fuzzy open, by Theorem 3.1, G is fuzzy b-open in $\mathcal{I}$. By (ii), with $f:(\mathcal{I},\tau_{\mathcal{I}})\to(\mathcal{I},\mathcal{U}_{\mathcal{I}})$ is fuzzy b^*-open, then $f(G)$ is fuzzy b-open in $\mathcal{I}$ and so it is fuzzy b-open in Y, by Theorem 3.1. Therefore, $f:(X,\tau)\to(Y,\mathcal{U})$ is fuzzy b^*-open. □

We will now present the notion of fuzzy generalized b^*-continuous function and its properties.

Definition 5.8. *A function $f:X\to Y$ is* **fuzzy generalized b^*-continuous** *or* **fuzzy gb^*-continuous** *if $f^{-1}(A)$ is fuzzy b^*-open in X for each fuzzy b^*-open set A in Y.*

Theorem 5.9. *Let $f:X\to Y$ be a bijective function. Then the following statements are equivalent:*

(i) f is fuzzy gb^*-continuous

(ii) If F is fuzzy b^*-closed in Y, then $f^{-1}(F)$ is fuzzy b^*-closed in X.

(iii) For each fuzzy b^*-open set V in Y, there exist a fuzzy b^*-open set U in X such that $f(U) = V$.

Proof. $(i) \Rightarrow (ii)$ Let F be fuzzy b^*-closed set in Y Then F^C is fuzzy b^*-open in Y. Since f is fuzzy gb^*-continuous, $f^{-1}\left(F^C\right)$ is fuzzy b^*-open in X. Note that $f^{-1}\left(F^C\right) = \left[f^{-1}(F)\right]^C$. Hence $\left[f^{-1}(F)\right]^C$ is fuzzy b^*-open in X. Therefore, $f^{-1}(F)$ is fuzzy b^*-closed in X.

$(ii) \Rightarrow (iii)$ Let V be fuzzy b^*-open in Y. Then V^C is fuzzy b^*-closed in Y. By (ii), $f^{-1}\left(V^C\right)$ is fuzzy b^*-closed in X. Hence $\left[f^{-1}(V)\right]^C$ is fuzzy b^*-closed in X. Thus $f^{-1}(V)$ is fuzzy b^*-open in X. Let $U = f^{-1}(V)$. Then $f(U) = f\left(f^{-1}\right)(V) = V$, since f is bijective.

$(iii) \Rightarrow (i)$ Let V be fuzzy b^*-open in Y. By (iii), there exist a fuzzy b^*-open set U in X such that $f(U) = V$. Hence $U = f^{-1}(f(U)) = f^{-1}(V)$. That is, $f^{-1}(V)$ is fuzzy b^*-open in X. Therefore, f is fuzzy b^*-continuous. □

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