International Mathematical Forum, Vol. 15, 2020, no. 6, 277 - 281
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2020.91283

Why 3D Fragmentation Usually Leads to Cuboids: A Simple Geometric Explanation

Laxman Bokati, Olga Kosheleva, and Vladik Kreinovich

University of Texas at El Paso
500 W. University
El Paso, TX 79968, USA

Abstract

It has been empirically observed that the average shape of natural fragmentation results – such as natural rock fragments – is a distorted cube (known as cuboid). Recently, a complex explanation was provides for this empirical fact. In this paper, we propose a simple geometry-based physical explanation for the ubiquity of cuboid fragments.

Mathematics Subject Classification: 51M20 52B15 86A60

Keywords: fragmentation results, natural rock fragments, cuboid, regular polyhedra, symmetry

1 Formulation of the Problem

Empirical fact. The average shape of natural rock fragments – and other natural fragmentation results – is *cuboid*, a distorted cube; see, e.g., [4] and references therein. How can we explain this empirical fact?

A recent explanation. Recently, in [4], this empirical fact was explained: complex theoretical analysis of the dynamics of fragmentation, together with computer simulations of the corresponding random fragmentation process, indeed show that the majority of fragments have a cuboid form.

It is always desirable to have a simple explanation based on first principles. In physics, even when a phenomenon is obtained as a result of complex mathematical analysis and/or complex computer simulations, it is usually possible to have a simple physical explanation of this phenomenon – at least on the qualitative level – an explanation that should be ideally based on first principles; see, e.g., [5, 15].

It is therefore desirable to come up with a simple first-principle explanation for the cuboid phenomenon.

What we do in this paper. In this paper, we provide a geometric explanation for the ubiquity of cuboid fragments.

2 Our Explanation

Importance of symmetries. In this explanation, we will utilize the ideas of symmetry – one of the main ideas of modern physics; see, e.g., [5, 15].

The importance of symmetries is easy to explain. How do we know that any physical law is valid? For example, how do we know that when we drop an object, it will start falling with the acceleration of 9.81 m/sec^2? Well, we observed it at one location. Then we change orientation, repeat the experiment, and the result is the same. We shift ourselves to a different location – and we again observe the same phenomenon. After several such experiments, we conclude that this phenomenon is invariant (= does not change) under all possible rotations and shifts. Since every two location on the Earth surface can be obtained from each other by an appropriate shift, we conclude that this phenomenon is indeed universal.

This was a simple example, but this is a general idea of why we conclude that certain laws of physics are universal. The corresponding transformations can be more complex – they may involve re-scaling, changing the sign of all the electric charges, replacing each particle with its anti-particle, etc., but the main idea of symmetry remains the same. Symmetry is so ubiquitous that usually, new physical theories are formulated not in terms of differential equations – as it was in Newton's times – but in terms of corresponding symmetries. Moreover, theories that were originally formulated in terms of differential equations – such as Maxwell's equations of electrodynamics or Einstein's equations of gravity – can be reformulated in terms of symmetries; see, e.g., [8, 9, 12, 13].

In view of the importance of symmetries, we will use symmetries in our analysis.

What are the symmetries of the original pre-fragmentation state? In the first approximation, the yet-unfragmented rock is homogenous and isotropic, it has the same properties at each location and in each direction.

In other words, the original state of the rock is, in this approximation, invariant with respect to arbitrary shifts and rotations in 3-D space.

Fragmentation breaks some of these symmetries. Once cracks appear, some of the symmetries are violated. Indeed, the mechanical properties are different at the crack's location and at locations where the rock has not yet cracked. Thus, we no longer have invariance with respect to all possible shifts: namely, we do not have invariance with respect to shifts that move a location on a crack to a location where the rock is not cracked.

What are the symmetries of the transformed state? Such symmetry violations are ubiquitous in nature. According to statistical physics (see, e.g., [5, 15]), the most frequent transformations from a symmetric state are to a state in which the largest number of symmetries are preserved. For example, from a highly symmetric solid state, usually, a substance moves into a liquid state in which some symmetries are preserved and only then to a gas state. There exist direct transitions from solid to gas – dry ice is a good example – but they are rare.

This general fact is not just a summary of simple easily observed phenomenon: this general fact can explain, e.g., all geometrical forms that we observe in celestial bodies, how these forms evolve and which are more frequent ones, from the usual spiral shape of galaxies to the (approximate) geometric progression formed by distances from planets to the Sun; see, e.g., [6, 7, 14].

Let us therefore apply this general principle to fragmentation. According to this principle, fragments should have the most symmetric shapes.

Which shapes are the most symmetric? Every shape can be approximated, with any desired accuracy, by a polyhedron – just like every curve can be approximated by a piece-wise linear ones. Thus, without losing generality, we can assume that fragments have polyhedral shapes.

Which polyhedra are the most symmetric? This is a well-studied geometrical question. Polyhedra have *vertices*, these vertices are connected by *edges*, and edges form planar *faces*. In the original non-fragmented state, every two locations could be transformed into one another by an appropriate shift, every two lines could be transformed into one another by an appropriate combination of shift and rotation, and every two planes could also be transformed into one another by an appropriate combination of shift and rotation.

In a fragmented states, we can not longer transform each spatial location into another one while preserving the physical properties: In each shift or rotation of a polyhedron, vertices are transformed into vertices, edges into edges, and faces into faces. But at least we can require that every two edges of the polyhedron can be transformed into each other by an appropriate combination of rotation and shift, and likewise every two edges can be transformed into each other, and every two faces can be transformed into each other. Such polyhedra are known as *regular*; see, e.g., [1, 2, 3].

All regular polyhedra are known – they include five convex ones (known as *Platonic solids*): tetrahedron, cube, octahedron, dodecahedron, and icosahedron, as well as four non-convex one. Which of these none forms should we observe?

Let us take into account that we are talking about fragmentation. In this paper, we are not interested in abstract geometric forms, we are interested in shapes obtained by fragmentation. Thus, we are interested in shapes that, when placed back together, would fill the original 3D space. Polyhedra that have this property are known as *space-filling polyhedra*; see, e.g., [11].

This explains the cuboids. Interestingly, the only space-filling regular polyhedron is the cube; see, e.g., [1, 10] and references therein.

This explains the ubiqity of cuboids among rock fragments.

Acknowledgments. This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), HRD-1834620 (CAHSI Includes), and HRD-1242122 (Cyber-ShARE Center of Excellence).

References

[1] H. S. M. Coxeter, *Regular Polytopes*, Dover, New York, 1973.

[2] H. S. M. Coxeter, *The Beauty of Geometry: Twelve Essays*, Dover, New York, 1999.

[3] P. R. Cromwell, *Polyhedra*, Cambridge University Press, Cambridge, UK, 2008.

[4] G. Domokos, D. J. Jerolmak, F. Kun, and J. Török, Plato's cube and the natural geometry of fragmentation', *Proceedings of the US National Academy of Sciences*, 2020, paper 2001037117. https://doi.org/10.1073/pnas.2001037117

[5] R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Addison Wesley, Boston, Massachusetts, 2005.

[6] A. Finkelstein, O. Kosheleva, and V. Kreinovich, Astrogeometry: towards mathematical foundations, *International Journal of Theoretical Physics*, **36** (4) (1997), 1009–1020. https://doi.org/10.1007/bf02435798

[7] A. Finkelstein, O. Kosheleva, and V. Kreinovich, Astrogeometry: geometry explains shapes of celestial bodies, *Geombinatorics*, **VI** (4) (1997), 125–139.

[8] A. M. Finkelstein and V. Kreinovich, Derivation of Einstein's, Brans-Dicke and other equations from group considerations, in: Y. Choque-Bruhat and T.-M. Karade (eds.), *On Relativity Theory. Proceedings of the Sir Arthur Eddington Centenary Symposium, Nagpur India 1984*, Vol. 2, World Scientific, Singapore, 1985, 138–146.

[9] A. M. Finkelstein, V. Kreinovich, and R. R. Zapatrin. Fundamental physical equations uniquely determined by their symmetry groups, *Lecture Notes in Mathematics*, Springer- Verlag, Berlin-Heidelberg-N.Y., Vol. 1214, 1986, 159–170. https://doi.org/10.1007/bfb0075964

[10] M. Gardner, *The Sixth Book of Mathematical Games from Scientific American*, University of Chicago Press, Chicago, Illinois, 1984.

[11] B. Grünbaum and G. C. Shephard, Tilings with congruent tiles, *Bulletin of American Mathematical Society*, **3** (1980), 951–973. https://doi.org/10.1090/s0273-0979-1980-14827-2

[12] V. Kreinovich, Derivation of the Schroedinger equations from scale invariance, *Theoretical and Mathematical Physics*, **8** (3) (1976), 282–285. https://doi.org/10.1007/bf01032102

[13] V. Kreinovich and G. Liu, We live in the best of possible worlds: Leibniz's insight helps to derive equations of modern physics, in: R. Pisano, M. Fichant, P. Bussotti, and A. R. E. Oliveira (eds.), *The Dialogue between Sciences, Philosophy and Engineering. New Historical and Epistemological Insights, Homage to Gottfried W. Leibnitz 1646–1716*, College Publications, London, 2017, 207–226.

[14] S. Li, Y. Ogura, and V. Kreinovich, *Limit Theorems and Applications of Set Valued and Fuzzy Valued Random Variables*, Kluwer Academic Publishers, Dordrecht, 2002. https://doi.org/10.1007/978-94-015-9932-0

[15] K. S. Thorne and R. D. Blandford, *Modern Classical Physics: Optics, Fluids, Plasmas, Elasticity, Relativity, and Statistical Physics*, Princeton University Press, Princeton, New Jersey, 2017.

Received: August 3, 2020; Published: August 17, 2020