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Mannheim B-Curves in Weyl Space

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Abstract

We have defined Mannheim B-curves and Mannheim B-pair in three dimensional Weyl space W_3. Under the condition that the pair $(C, C^{\star})$ is a Mannheim B-pair in W_3, we have given some theorems such as the relation between Bishop vector fields of C and $C^{\star}$; the relation between Frenet vector fields of C and $C^{\star}$; the relation between Bishop curvatures of C and $C^{\star}$; the relation between Frenet curvatures of C and $C^{\star}$.

Mathematics Subject Classification: 53B25, 53A25

Keywords: Weyl space, Bishop frame, Mannheim B-curves, Mannheim B-pair

1 Introduction

Mannheim curves have been presented by Mannheim in 1878 and by Blum [3] in 1966. After that, Mannheim partner curves have been studied in Euclidean 3-space by Liu and Wang [8] in 2008 and by Orbay and Kasap [12] in 2009. Besides, Mannheim partner curves in dual space have been examined by Özkaldı et al. [15] in 2009. Generalized Mannheim curves have been given in Minkowski space-time E_1^4 by Akyiḡit et al. [1] in 2011 and in Euclidean 4-space by Matsuda and Yorozu [10] in 2009. Later, Mannheim offsets of ruled surfaces have been defined by Orbay et al. [13] in 2009 and dual Mannheim partner curves have been expressed bu Güngör and Tosun [4] in 2010. Furthermore, the quaternionic Mannheim curves in Euclidean 4-space, weakened Mannheim curves and quaternionic Mannheim curves of Aw(k)-type in Euclidean 3-space have been characterized by Okuyucu [11] in 2013, by Karacan [5] in 2011 and by Kızıltuḡ and Yaylı [6] in 2015, respectively.

Recently, Mannheim curves have been defined according to different frame such as Mannheim partner D-curves (Önder and Kızıltuḡ, 2012) [14] and Mannheim B-curves (Masal and Azak, 2017) [9].

2 Preliminaries

Let C be a curve in three dimensional Weyl space W_3. Let $\{\underset{1}{v}^i, \underset{2}{v}^i, \underset{3}{v}^i, \kappa_1, \kappa_2\}$ $\{\underset{1}{v}^i, \underset{1}{n}^i, \underset{2}{n}^i, k_1, k_2\}$ be the Frenet and Bishop apparatus [2] of C, respectively. Then, Frenet and Bishop formulas of C are expressed in the following form:

$$\begin{aligned} \underset{1}{v}^k \dot{\nabla}_k \underset{1}{v}^i &= \kappa_1 \underset{2}{v}^i \\ \underset{1}{v}^k \dot{\nabla}_k \underset{2}{v}^i &= -\kappa_1 \underset{1}{v}^i + \kappa_2 \underset{3}{v}^i \\ \underset{1}{v}^k \dot{\nabla}_k \underset{3}{v}^i &= -\kappa_2 \underset{2}{v}^i \end{aligned} \tag{1}$$

and

$$\begin{aligned} \underset{1}{v}^k \dot{\nabla}_k \underset{1}{v}^i &= k_1 \underset{1}{n}^i + k_2 \underset{2}{n}^i \\ \underset{1}{v}^k \dot{\nabla}_k \underset{1}{n}^i &= -k_1 \underset{1}{v}^i \\ \underset{1}{v}^k \dot{\nabla}_k \underset{2}{n}^i &= -k_2 \underset{1}{v}^i. \end{aligned} \tag{2}$$

Besides, the relation between Frenet and Bishop vector fields [7] and the curvatures of Frenet and Bishop [7] can be written as follows:

$$\begin{pmatrix} \underset{1}{v}^i \\ \underset{2}{v}^i \\ \underset{3}{v}^i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \underset{1}{v}^i \\ \underset{1}{n}^i \\ \underset{2}{n}^i \end{pmatrix} \tag{3}$$

and $\kappa_1 = \overset{2}{\underset{1}{\gamma}}$, $\kappa_2 = -\overset{2}{\underset{31}{\tau}}$, $\kappa_2 = \underset{1}{v}^k \dot{\nabla}_k \theta$, $k_1 = \kappa_1 \cos\theta$, $k_2 = \kappa_1 \sin\theta$, $k_1^2 + k_2^2 = \kappa_1^2$, $k_1 = \overset{2}{\underset{1}{\gamma}} \cos\theta - \overset{3}{\underset{1}{\gamma}} \sin\theta$, $k_2 = \overset{2}{\underset{1}{\gamma}} \sin\theta + \overset{3}{\underset{1}{\gamma}} \cos\theta$, where $\overset{r}{\underset{s}{\gamma}} = \overset{r}{\underset{s}{\mathrm{T}}}_k \underset{s}{v}^k$ and $\overset{q}{\underset{rs}{\tau}} = \overset{q}{\underset{r}{\mathrm{T}}}_k \underset{s}{v}^k (r \neq s; \quad r, s, q = 1, 2, 3)$ are called geodesic curvature and Chebyshev curvature of the first kind of the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ [16].

3 Mannheim B-Curves in Weyl Space

Let C and $C^\star$ be curves in three dimensional Weyl space W_3. Let us denote arc-length of C and $C^\star$ by s and $s^\star$, respectively. Then, we can express the curves C and $C^\star$ in the form $C : x^i = x^i(s)$ and $C^\star : \overset{\star}{x}{}^i = \overset{\star}{x}{}^i(s^\star)(i = 1, 2, 3)$, respectively. Let us denote Bishop apparatus of C and $C^\star$ by $\{\underset{1}{v}{}^i, \underset{1}{n}{}^i, \underset{2}{n}{}^i, k_1, k_2\}$ $\{\overset{\star}{\underset{1}{v}}{}^i, \overset{\star}{\underset{1}{n}}{}^i, \overset{\star}{\underset{2}{n}}{}^i, \overset{\star}{k}_1, \overset{\star}{k}_2\}$, respectively.

Definition 3.1. *If the Bishop vector field $\underset{1}{n}{}^i$ coincides with the Bishop vector field $\overset{\star}{\underset{2}{n}}{}^i$ at the corresponding points of curves C and $C^\star$, then the curve C is called a Mannheim partner B-curve of $C^\star$ and $(C, C^\star)$ is also called Mannheim B-pair.*

This can be formalized as

$$C(s) = C^\star(s^\star) + \lambda(s^\star)\overset{\star}{\underset{2}{n}}(s^\star) \tag{4}$$

or

$$x^i(s) = \overset{\star}{x}{}^i(s^\star) + \lambda(s^\star)\overset{\star}{\underset{2}{n}}{}^i(s^\star) \tag{5}$$

where λ is a function of $s^\star$, see Figure 1.

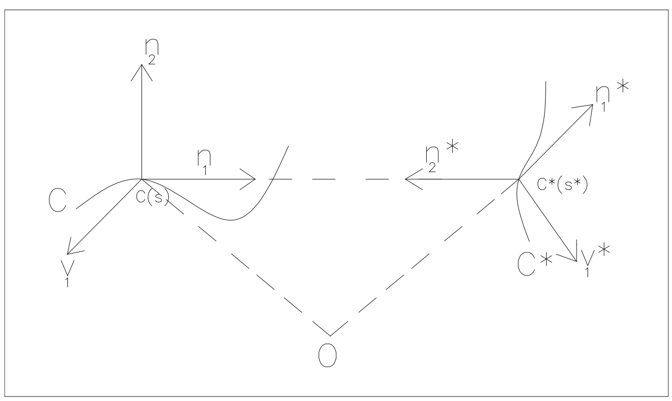

Figure 1: Mannheim B-Curves

Let us obtain the relation between the Bishop vector fields of C and $C^\star$ such that the pair $(C, C^\star)$ is a Mannheim B-pair:

Since $\underset{1}{v}{}^i$ is orthogonal to $\underset{1}{n}{}^i$ and $\underset{1}{n}{}^i = \overset{\star}{\underset{2}{n}}{}^i$, $\underset{1}{v}{}^i$ is orthogonal to $\overset{\star}{\underset{2}{n}}{}^i$. Therefore, $\underset{1}{v}{}^i$ can be written as

$$\underset{1}{v}{}^i = \gamma_1 \overset{\star}{\underset{1}{v}}{}^i + \gamma_2 \overset{\star}{\underset{1}{n}}{}^i \tag{6}$$

or

$$\underset{1}{v}{}^i = \cos\alpha \overset{\star}{\underset{1}{v}}{}^i + \sin\alpha \overset{\star}{\underset{1}{n}}{}^i \tag{7}$$

where $\alpha = \sphericalangle(\underset{1}{v}^i, \overset{\star}{\underset{1}{v}}^i)$, $\gamma_1 = g_{ij}\underset{1}{v}^i\overset{\star}{\underset{1}{v}}^j = \cos\alpha$ and $\gamma_2 = g_{ij}\underset{1}{v}^i\overset{\star}{\underset{1}{n}}^j = \cos(\frac{\pi}{2}-\alpha) = \sin\alpha$.

Since $\underset{2}{n}^i$ is orthogonal to $\underset{1}{n}^i$ and $\underset{1}{n}^i = \overset{\star}{\underset{2}{n}}^i$, $\underset{2}{n}^i$ is orthogonal to $\overset{\star}{\underset{2}{n}}^i$. Then, $\underset{2}{n}^i$ can be written as

$$\underset{2}{n}^i = \eta_1\overset{\star}{\underset{1}{v}}^i + \eta_2\overset{\star}{\underset{1}{n}}^i \tag{8}$$

or

$$\underset{2}{n}^i = -\sin\alpha\overset{\star}{\underset{1}{v}}^i + \cos\alpha\overset{\star}{\underset{1}{n}}^i \tag{9}$$

where $\eta_1 = g_{ij}\underset{2}{n}^i\overset{\star}{\underset{1}{v}}^j = \cos(\frac{\pi}{2}+\alpha) = -\sin\alpha$ and $\eta_2 = g_{ij}\underset{2}{n}^i\overset{\star}{\underset{1}{n}}^j = \cos\alpha$.

By means of (7) and (9), we can express this relation among Bishop vector fields in the following matrix form:

$$\begin{pmatrix} \underset{1}{v}^i \\ \underset{1}{n}^i \\ \underset{2}{n}^i \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 1 \\ -\sin\alpha & \cos\alpha & 0 \end{pmatrix} \begin{pmatrix} \overset{\star}{\underset{1}{v}}^i \\ \overset{\star}{\underset{1}{n}}^i \\ \overset{\star}{\underset{2}{n}}^i \end{pmatrix}. \tag{10}$$

Theorem 3.2. *If the pair $(C, C^\star)$ is a Mannheim B-pair then λ is a non-zero constant.*

Proof. Let $(C, C^\star)$ be a Mannheim B-pair. Then (5) is satisfied.

If we take the prolonged covariant derivative of (5) in the direction of $\overset{\star}{\underset{1}{v}}^k$, we get

$$\begin{aligned} \overset{\star}{\underset{1}{v}}^k\overset{\star}{\dot{\nabla}}_k x^i &= \overset{\star}{\underset{1}{v}}^k\overset{\star}{\dot{\nabla}}_k \overset{\star}{x}^i + (\overset{\star}{\underset{1}{v}}^k\overset{\star}{\dot{\nabla}}_k\lambda)\overset{\star}{\underset{2}{n}}^i + \lambda\overset{\star}{\underset{1}{v}}^k\overset{\star}{\dot{\nabla}}_k\overset{\star}{\underset{2}{n}}^i \\ &= (1-\lambda\overset{\star}{k}_2)\overset{\star}{\underset{1}{v}}^i + (\overset{\star}{\underset{1}{v}}^k\overset{\star}{\dot{\nabla}}_k\lambda)\overset{\star}{\underset{2}{n}}^i. \end{aligned} \tag{11}$$

Let us denote left hand side of (11) by

$$\overset{\star}{\underset{1}{v}}^k\overset{\star}{\dot{\nabla}}_k x^i = (\underset{1}{v}^k\dot{\nabla}_k x^i).A. \tag{12}$$

By using (12) in (11), we have

$$\underset{1}{v}^i.A = (1-\lambda\overset{\star}{k}_2)\overset{\star}{\underset{1}{v}}^i + (\overset{\star}{\underset{1}{v}}^k\overset{\star}{\dot{\nabla}}_k\lambda)\overset{\star}{\underset{2}{n}}^i. \tag{13}$$

Multiplying (13) by $g_{ij}\underset{1}{v}^j$ and summing on i and j, we obtain

$$A = (1-\lambda\overset{\star}{k}_2)\cos\alpha \tag{14}$$

where $g_{ij}\overset{\star}{\underset{2}{n}}{}^{i}\underset{1}{v}^{j} = g_{ij}\underset{1}{n}^{i}\underset{1}{v}^{j} = 0$.

From (13) and (14), we have

$$\underset{1}{v}^{i}(1-\lambda\overset{\star}{k}_2)\cos\alpha = (1-\lambda\overset{\star}{k}_2)\overset{\star}{\underset{1}{v}}{}^{i} + (\overset{\star}{\underset{1}{v}}{}^{k}\overset{\star}{\dot{\nabla}}_k\lambda)\overset{\star}{\underset{2}{n}}{}^{i}. \tag{15}$$

Multiplying (15) by $g_{ij}\overset{\star}{\underset{2}{n}}{}^{j}$ and summing on i and j, we obtain

$$\overset{\star}{\underset{1}{v}}{}^{k}\overset{\star}{\dot{\nabla}}_k\lambda = 0 \tag{16}$$

where $g_{ij}\underset{1}{v}^{i}\overset{\star}{\underset{2}{n}}{}^{j} = g_{ij}\underset{1}{v}^{i}\underset{1}{n}^{j} = 0$, $g_{ij}\overset{\star}{\underset{1}{v}}{}^{i}\overset{\star}{\underset{2}{n}}{}^{j} = 0$ and $g_{ij}\overset{\star}{\underset{2}{n}}{}^{i}\overset{\star}{\underset{2}{n}}{}^{j} = 1$.

Thus, λ is a non-zero constant. □

Theorem 3.3. *If $(C, C^\star)$ is a Mannheim B-pair in W_3, then there are the following relations between Bishop vector fields of C and $C^\star$:*

$$\underset{1}{v}^{i} = \varepsilon\overset{\star}{\underset{1}{v}}{}^{i}, \quad \underset{1}{n}^{i} = \varepsilon\overset{\star}{\underset{2}{n}}{}^{i}, \quad \underset{2}{n}^{i} = \varepsilon\overset{\star}{\underset{1}{n}}{}^{i}$$

where $\varepsilon = \pm 1$ for $\alpha = 0$ and $\alpha = \pi$.

Proof. Let $(C, C^\star)$ be a Mannheim B-pair in W_3. Then λ is a non-zero constant. So, we get from (13)

$$\underset{1}{v}^{i}A = (1-\lambda\overset{\star}{k}_2)\overset{\star}{\underset{1}{v}}{}^{i}. \tag{17}$$

Multiplying (17) by $g_{ij}\overset{\star}{\underset{1}{v}}{}^{i}$ and summing on i and j, we have

$$A\cos\alpha = 1-\lambda\overset{\star}{k}_2 \tag{18}$$

where $g_{ij}\underset{1}{v}^{i}\overset{\star}{\underset{1}{v}}{}^{j} = \cos\alpha$ and $g_{ij}\overset{\star}{\underset{1}{v}}{}^{i}\overset{\star}{\underset{1}{v}}{}^{j} = 1$.

Using (14) in (18), we get $\cos^2\alpha = 1$. That is, $\cos\alpha = 1$ for $\alpha = 0$ and $\cos\alpha = -1$ for $\alpha = \pi$. So, $\sin\alpha = 0$ is obtained.

If obtained results are used in equation (10), we have

$$\underset{1}{v}^{i} = \varepsilon\overset{\star}{\underset{1}{v}}{}^{i}, \quad \underset{1}{n}^{i} = \varepsilon\overset{\star}{\underset{2}{n}}{}^{i}, \quad \underset{2}{n}^{i} = \varepsilon\overset{\star}{\underset{1}{n}}{}^{i}$$

where $\cos\alpha = \varepsilon = \pm 1$ for $\alpha = 0$ and $\alpha = \pi$. □

Theorem 3.4. *Let $(C, C^\star)$ be a Mannheim B-pair in W_3. Let us denote their Frenet apparatus by $\{\underset{1}{v}^{i}, \underset{2}{v}^{i}, \underset{3}{v}^{i}, \kappa_1, \kappa_2\}$ and $\{\overset{\star}{\underset{1}{v}}{}^{i}, \overset{\star}{\underset{2}{v}}{}^{i}, \overset{\star}{\underset{3}{v}}{}^{i}, \overset{\star}{\kappa}_1, \overset{\star}{\kappa}_2\}$ in W_3, respectively. The relations between Frenet vector fields of C and $C^\star$ are given by*

$$\begin{aligned}
\overset{\star}{\underset{1}{v}}{}^{i} &= \varepsilon\underset{1}{v}^{i}\\
\overset{\star}{\underset{2}{v}}{}^{i} &= \sin(\theta^\star-\varepsilon\theta)\underset{2}{v}^{i} - \varepsilon\cos(\theta^\star-\varepsilon\theta)\underset{3}{v}^{i}\\
\overset{\star}{\underset{3}{v}}{}^{i} &= \cos(\theta^\star-\varepsilon\theta)\underset{2}{v}^{i} - \varepsilon\sin(\theta^\star-\varepsilon\theta)\underset{3}{v}^{i}
\end{aligned}$$

where $\varepsilon = \pm 1$ for $\alpha = 0$ and $\alpha = \pi$.

Proof. Let $(C, C^\star)$ be a Mannheim B-pair in W_3. Let us express (3) for Frenet and Bishop apparatus of $C^\star$:

$$\begin{pmatrix} \overset{\star}{\underset{1}{v}}{}^i \\ \overset{\star}{\underset{2}{v}}{}^i \\ \overset{\star}{\underset{3}{v}}{}^i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta^\star & \sin\theta^\star \\ 0 & -\sin\theta^\star & \cos\theta^\star \end{pmatrix} \begin{pmatrix} \overset{\star}{\underset{1}{v}}{}^i \\ \overset{\star}{\underset{1}{n}}{}^i \\ \overset{\star}{\underset{2}{n}}{}^i \end{pmatrix} \tag{19}$$

where $\theta^\star = \sphericalangle(\overset{\star}{\underset{2}{v}}{}^i, \overset{\star}{\underset{1}{n}}{}^i)$.

Using (3), (4) and Theorem 3.3, we get

$$\begin{pmatrix} \overset{\star}{\underset{1}{v}}{}^i \\ \overset{\star}{\underset{2}{v}}{}^i \\ \overset{\star}{\underset{3}{v}}{}^i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta^\star & \sin\theta^\star \\ 0 & -\sin\theta^\star & \cos\theta^\star \end{pmatrix} \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & 0 & -\cos\alpha \\ 0 & 1 & 0 \end{pmatrix}.$$

$$.\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \underset{1}{v}{}^i \\ \underset{2}{v}{}^i \\ \underset{3}{v}{}^i \end{pmatrix} \tag{20}$$

or

$$\begin{aligned} \overset{\star}{\underset{1}{v}}{}^i &= \cos\alpha \underset{1}{v}{}^i \\ \overset{\star}{\underset{2}{v}}{}^i &= \{\sin\theta^\star\cos\theta - \sin\theta\cos\alpha\cos\theta^\star\}\underset{2}{v}{}^i + \{-\sin\theta^\star\sin\theta - \cos\theta\cos\alpha\cos\theta^\star\}\underset{3}{v}{}^i \\ \overset{\star}{\underset{3}{v}}{}^i &= \{\cos\theta^\star\cos\theta + \sin\theta\sin\theta^\star\cos\alpha\}\underset{2}{v}{}^i + \{-\cos\theta^\star\sin\theta + \cos\theta\sin\theta^\star\cos\alpha\}\underset{3}{v}{}^i \end{aligned} \tag{21}$$

or

$$\begin{aligned} \overset{\star}{\underset{1}{v}}{}^i &= \varepsilon \underset{1}{v}{}^i \\ \overset{\star}{\underset{2}{v}}{}^i &= \sin(\theta^\star - \varepsilon\theta)\underset{2}{v}{}^i - \varepsilon\cos(\theta^\star - \varepsilon\theta)\underset{3}{v}{}^i \\ \overset{\star}{\underset{3}{v}}{}^i &= \cos(\theta^\star - \varepsilon\theta)\underset{2}{v}{}^i + \varepsilon\sin(\theta^\star - \varepsilon\theta)\underset{3}{v}{}^i \end{aligned} \tag{22}$$

where $\varepsilon = \cos\alpha$; $\varepsilon = 1$ for $\alpha = 0$, $\varepsilon = -1$ for $\alpha = \pi$. □

Theorem 3.5. *Let $(C, C^\star)$ be a Mannheim B-pair in W_3. Then the relation between the second Bishop curvature $\overset{\star}{k}_2$ of the curve $C^\star$ and the first Bishop curvature k_1 of the curve C is as follows:*

$$\overset{\star}{k}_2 = \frac{k_1}{1 + \lambda k_1}.$$

Proof. Let $(C, C^\star)$ be a Mannheim B-pair in W_3. Then $\underset{1}{v}^i A = (1-\lambda \overset{\star}{k}_2)\underset{1}{\overset{\star}{v}}{}^i$ is satisfied.

Taking prolonged covariant derivative of this equality in the direction of $\underset{1}{\overset{\star}{v}}{}^k$, we have

$$\begin{aligned}(\underset{1}{v}^k \dot{\nabla}_k \underset{1}{v}^i)A^2 + \underset{1}{v}^i(\underset{1}{\overset{\star}{v}}{}^k \overset{\star}{\dot{\nabla}}_k A) = \\ -\lambda(\underset{1}{\overset{\star}{v}}{}^k \overset{\star}{\dot{\nabla}}_k \overset{\star}{k}_2)\underset{1}{\overset{\star}{v}}{}^i + (1-\lambda\overset{\star}{k}_2)(\underset{1}{\overset{\star}{v}}{}^k \overset{\star}{\dot{\nabla}}_k \underset{1}{\overset{\star}{v}}{}^i)\end{aligned} \tag{23}$$

or

$$\begin{aligned}(k_1 \underset{1}{n}^i + k_2 \underset{2}{n}^i)A^2 + \underset{1}{v}^i(\underset{1}{\overset{\star}{v}}{}^k \overset{\star}{\dot{\nabla}}_k A) = \\ -\lambda(\underset{1}{\overset{\star}{v}}{}^k \overset{\star}{\dot{\nabla}}_k \overset{\star}{k}_2)\underset{1}{\overset{\star}{v}}{}^i + (1-\lambda\overset{\star}{k}_2)\overset{\star}{k}_1 \underset{1}{\overset{\star}{n}}{}^i + (1-\lambda\overset{\star}{k}_2)\overset{\star}{k}_2 \underset{2}{\overset{\star}{n}}{}^i.\end{aligned} \tag{24}$$

Multiplying (24) by $g_{ij}\underset{1}{n}^j$ and summing on i and j, we get

$$k_1 A^2 = (1-\lambda\overset{\star}{k}_2)\overset{\star}{k}_2 \tag{25}$$

where $g_{ij}\underset{1}{n}^i\underset{1}{n}^j = 1$, $g_{ij}\underset{1}{v}^i\underset{1}{n}^j = 0$, $g_{ij}\underset{1}{\overset{\star}{v}}{}^i\underset{1}{n}^j = g_{ij}\underset{1}{\overset{\star}{v}}{}^i\underset{2}{\overset{\star}{n}}{}^j = 0$, $g_{ij}\underset{1}{\overset{\star}{n}}{}^i\underset{1}{n}^j = g_{ij}\underset{1}{\overset{\star}{n}}{}^i\underset{2}{\overset{\star}{n}}{}^j = 0$, and $g_{ij}\underset{2}{\overset{\star}{n}}{}^i\underset{1}{n}^j = g_{ij}\underset{2}{\overset{\star}{n}}{}^i\underset{2}{\overset{\star}{n}}{}^j = 1$.

Using (18) in (24), we get

$$k_1 = \frac{\overset{\star}{k}_2}{1-\lambda\overset{\star}{k}_2} \quad \text{or} \quad \overset{\star}{k}_2 = \frac{k_1}{1+\lambda k_1}. \tag{26}$$

The proof is completed. □

Remark 3.6. *Using* [7] *in* (26), $\overset{\star}{k}_2$ *can also be expressed in the following form:*

$$\overset{\star}{k}_2 = \frac{\overset{2}{\underset{1}{Ƨ}}\cos\theta - \overset{3}{\underset{1}{Ƨ}}\sin\theta}{1+\lambda(\overset{2}{\underset{1}{Ƨ}}\cos\theta - \overset{3}{\underset{1}{Ƨ}}\sin\theta)}. \tag{27}$$

Theorem 3.7. *Let* $(C, C^\star)$ *be a Mannheim B-pair in* W_3. *Then the following equalities are satisfied:*

$$k_1 A = \cos\alpha\overset{\star}{k}_2$$

$$\overset{\star}{k}_1 + k_2 A = 0.$$

Proof. Let $(C, C^\star)$ be a Mannheim B-pair. Then $\underset{1}{v}^i = \cos\alpha \overset{\star}{\underset{1}{v}}{}^i$ is obtained from (10) and Theorem 3.3.

Taking prolonged covariant derivative of this equality in the direction of $\overset{\star}{\underset{1}{v}}{}^k$, we get

$$(\underset{1}{v}^k \dot{\nabla}_k \underset{1}{v}^i)A = \cos\alpha \overset{\star}{\underset{1}{v}}{}^k \overset{\star}{\dot{\nabla}}_k \overset{\star}{\underset{1}{v}}{}^i \tag{28}$$

or

$$(k_1 \underset{1}{n}^i + k_2 \underset{2}{n}^i)A = \cos\alpha(\overset{\star}{k}_1 \overset{\star}{\underset{1}{n}}{}^i + \overset{\star}{k}_2 \overset{\star}{\underset{2}{n}}{}^i). \tag{29}$$

Multiplying (29) by $g_{ij} \overset{\star}{\underset{1}{n}}{}^j$ and summing on i and j, we have

$$-k_2 A = \overset{\star}{k}_1 \tag{30}$$

or

$$\overset{\star}{k}_1 + k_2 A = 0 \tag{31}$$

where

$$\overset{\star}{\underset{1}{n}}{}^i = \varepsilon_{jik} \overset{\star}{\underset{2}{n}}{}^i \overset{\star}{\underset{1}{v}}{}^k = \varepsilon_{jik} \underset{1}{n}^j \cos\alpha \underset{1}{v}^k = -\cos\alpha \varepsilon_{jki} \underset{1}{v}^k \underset{1}{n}^i = -\cos\alpha \underset{2}{n}^j.$$

Multiplying (29) by $g_{ij} \overset{\star}{\underset{2}{n}}{}^j$ and summing on i and j, we have

$$k_1 A = \cos\alpha \overset{\star}{k}_2 \tag{32}$$

where $g_{ij} \underset{1}{n}^i \overset{\star}{\underset{2}{n}}{}^j = g_{ij} \underset{1}{n}^i \underset{1}{n}^j = 1$, $g_{ij} \underset{2}{n}^i \overset{\star}{\underset{2}{n}}{}^j = g_{ij} \underset{2}{n}^i \underset{1}{n}^j = 0$, $g_{ij} \overset{\star}{\underset{1}{n}}{}^i \overset{\star}{\underset{2}{n}}{}^j = 0$ and $g_{ij} \overset{\star}{\underset{2}{n}}{}^i \overset{\star}{\underset{2}{n}}{}^j = 1$. □

Remark 3.8. *Using* [7], (27) *and* (30), $\overset{\star}{k}_1$ *can also be expressed in the following form:*

$$\overset{\star}{k}_1 = -\frac{\overset{2}{\underset{1}{\mathfrak{S}}} \sin\theta + \overset{3}{\underset{1}{\mathfrak{S}}} \cos\theta}{1 + \lambda(\overset{2}{\underset{1}{\mathfrak{S}}} \cos\theta - \overset{3}{\underset{1}{\mathfrak{S}}}) \sin\theta} \cos\alpha. \tag{33}$$

Theorem 3.9. *Let* $(C, C^\star)$ *be a Mannheim B-pair in* W_3*. Then there is a relation between the first Frenet curvatures of* C *and* $C^\star$ *in the following form:*

$$\overset{\star}{\kappa}_1 = \kappa_1 |A|.$$

Proof. Let $(C, C^{\star})$ be a Mannheim B-pair in W_3. Using (30) and (32), we get

$$\begin{aligned} \overset{\star}{k}_1^{2} + \overset{\star}{k}_2^{2} &= ({k_1}^2 + {k_2}^2)A^2 \\ \overset{\star}{\kappa}_1^{2} &= \kappa_1^2 A^2 \\ \overset{\star}{\kappa}_1 &= \kappa_1 |A|. \end{aligned} \tag{34}$$

□

Remark 3.10. *Using* [7] *and* (34), *we obtain*

$$\underset{1}{\overset{2}{\overset{\star}{\text{Ƨ}}}} = \underset{1}{\overset{2}{\text{Ƨ}}}|A| \tag{35}$$

where $\overset{\star}{\kappa}_1 = \underset{1}{\overset{2}{\overset{\star}{\text{Ƨ}}}} = \underset{1}{\overset{2}{\overset{\star}{\mathrm{T}}}}_k \underset{1}{\overset{\star}{v}}^k$ *and* $\kappa_1 = \underset{1}{\overset{2}{\text{Ƨ}}} = \underset{1}{\overset{2}{\mathrm{T}}}_k \underset{1}{v}^k$.

Theorem 3.11. *Let* $(C, C^{\star})$ *be a Mannheim B-pair in* W_3. *Then*

$$\underset{1}{\overset{\star}{c}}^i = \underset{1}{c}^i (1 - \lambda \overset{\star}{k}_2)$$

is satisfied where $\underset{1}{\overset{\star}{c}}^i$ *are the geodesic vector fields of the net* $(\underset{1}{\overset{\star}{v}}, \underset{2}{\overset{\star}{v}}, \underset{3}{\overset{\star}{v}})$ *occurred by the vector fields of* $C^{\star}$ *and* $\underset{1}{c}^i$ *are the geodesic vector fields of the net* $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ *occurred by the vector fields of* C *in* W_3 *[16].*

Proof. Let $(C, C^{\star})$ be a Mannheim B-pair in W_3.

Taking prolonged covariant derivative of the first equation in (22) in the direction of $\underset{1}{\overset{\star}{v}}^k$, we have

$$\underset{1}{\overset{\star}{v}}^k \dot{\overset{\star}{\nabla}}_k \underset{1}{\overset{\star}{v}}^i = \varepsilon \underset{1}{\overset{\star}{v}}^k \dot{\overset{\star}{\nabla}}_k \underset{1}{v}^i = \varepsilon (\underset{1}{v}^k \dot{\nabla}_k \underset{1}{v}^i) A \tag{36}$$

$$\underset{1}{\overset{p}{\overset{\star}{\mathrm{T}}}}_k \underset{1}{\overset{\star}{v}}^k \underset{p}{\overset{\star}{v}}^i = \varepsilon \underset{1}{\overset{p}{\mathrm{T}}}_k \underset{1}{v}^k \underset{p}{v}^i A \quad (p = 1, 2, 3) \tag{37}$$

$$\underset{1}{\overset{p}{\overset{\star}{\text{Ƨ}}}} \underset{p}{\overset{\star}{v}}^i = \varepsilon \underset{1}{\overset{p}{\text{Ƨ}}} \underset{p}{v}^i A \tag{38}$$

$$\underset{1}{\overset{\star}{c}}^i = \varepsilon \underset{1}{c}^i A. \tag{39}$$

Using (14) in (39), we get

$$\underset{1}{\overset{\star}{c}}^i = \underset{1}{c}^i (1 - \lambda \overset{\star}{k}_2). \tag{40}$$

□

By means of (27) and Theorem 3.11, we can give the following theorem:

Theorem 3.12. *Let $(C, C^{\star})$ be a Mannheim B-pair in W_3. The net $(\overset{\star}{\underset{1}{v}}, \overset{\star}{\underset{2}{v}}, \overset{\star}{\underset{3}{v}})$ is a geodesic net in W_3 if and only if the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ is a geodesic net in W_3.*

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