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Locally Most Powerful Group-Sequential Tests with Groups of Random Size

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Abstract

We consider sequential hypothesis testing based on observations which are received in groups of random size. The observations are assumed independent both within and between the groups, with a distribution depending on a real-valued parameter θ. We suppose that the group sizes are independent and their distributions are known, and that the groups are formed independently from the observations.

We are concerned with a problem of testing a simple hypothesis $H_0 : \theta = \theta_0$ against a composite alternative $H_1 : \theta > \theta_0$. For any (group-)sequential test, we take into account the following three characteristics: its error probability of the first type, the derivative of its power function at $\theta = \theta_0$, and the average cost of observations, under some natural assumptions about the cost structure. Under some mild regularity conditions, we characterize the structure of sequential tests maximizing the derivative of the power function among all sequential tests whose error probability of the first type and the average cost of observations are under some prescribed levels.

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Keywords: Sequential analysis, hypothesis testing, locally most powerful test, optimal stopping, optimal sequential tests

1 Introduction

We consider sequential hypothesis testing when the observations are received in groups of a random size, a model originally proposed by [2] for the case of two simple hypoteses.

In this article, we are concerned with testing a simple hypothesis against a one-sided alternative, supposing the same group-wise sampling scheme.

To treat the composite hypotheses case, we adhere to the approach of [1] based on local characteristics of sequential tests.

This is a “local” approach based on maximizing the derivative of the power function, at the point of the null hypothesis. [1] calls locally most powerful the tests that maximize the derivative of the power function among all the tests whose error probability of the first type and the average sample number do not exceed some prescribed levels.

In the case of random groups of observations, there are various ways to quantify the volume of observations taken for the analysis, for example, a special interest can be put on the total number of observations, or on the number of groups taken (see [2]). To unify the different approaches we introduce in this article a natural concept of cost of observations accounting for the number of groups and/or for the number of observations within the groups, and use the average cost as one of the characteristics to be taken into account. More specifically, we employ this, in the group-sequential context, exactly in the way the average sample number was employed in [1]. Respectively, the main goal we pursue in this article is to maximize the derivative of the power function among all group-sequential tests whose error probability of the first type and the average cost do not exceed some given levels. The tests that do the maximization are called locally most powerful (group-sequential) tests in this article.

This article heavily relies on our previous work (cf. [6] and [7]). In the former, we proved some useful inequalities for characteristics of group-sequential tests and obtained conditions of differentiability of their power functions. In the latter, we obtained the form of locally most powerful tests in the case of finite horizon. The present article completes the study providing the form of all locally most powerful group-sequential tests in the case of infinite horizon.

Section 2 contains the necessary notation and assumptions and formulates the problem to be resolved.

In Section 3, we characterize the optimal group-sequential tests.
Some lengthy proofs are placed in Section 4.

2 Notation, Assumptions, Problem Setting

We use in this article the notation and assumptions of [7].

Here is a list of the concepts defined in [7] we need in this work, the formal definitions can be found in [7].

- X_{kj}, $j = 1, 2, \ldots, \eta_k$ i.i.d. observations arriving in groups, $k = 1, 2, \ldots$
- η_k size of group k, obtained as a value of a random variable ν_k
- P_θ distribution of X_{kj}
- $H_0 : \theta = \theta_0$, $H_1 : \theta > \theta_0$ hypotheses to be distinguished
- $X_k^{(\eta_k)}$ a vector of observations in k-th group, $X_k^{(\eta_k)} = (X_{k1}, \ldots, X_{k\eta_k})$
- $(\mathfrak{X}, \mathscr{X})$ sample space of the observations
- f_θ Radon–Nikodym derivative of P_θ with respect to a σ-finite measure μ on $\mathscr{X}$
- $f_\theta^\eta(x^{(\eta)})$ joint density of $X^{(\eta)} = (X_1^{(\eta_1)}, \ldots, X_n^{(\eta_n)})$ observed in n groups
- $\mathscr{G} \subseteq \{0, 1, 2, \ldots\}$ the set of admissible group size
- $p_k(m) = P(\nu_k = m)$ distribution of k-th group size
- $\psi = \{\psi_\eta : \mathfrak{X}^\eta \mapsto [0, 1],\, \eta \in \mathscr{G}^n,\, n = 1, 2, \ldots\}$ stopping rule
- $\phi = \{\phi_\eta : \mathfrak{X}^\eta \mapsto [0, 1],\, \eta \in \mathscr{G}^n,\, n = 1, 2, \ldots,\}$ (terminal) decision rule
- (ψ, ϕ) group-sequential test
- $t_\eta^\psi = (1 - \psi_{(\eta_1)}) \ldots (1 - \psi_{(\eta_1, \ldots, \eta_{n-1})})$, $s_\eta^\psi = t_\eta^\psi \psi_\eta$
- $S_\eta^\psi = \{x \in \mathfrak{X}^\eta :\, s_\eta^\psi(x) > 0\}$, $T_\eta^\psi = \{x \in \mathfrak{X}^\eta : t_\eta^\psi(x) > 0\}$
- $\beta_\theta(\psi, \phi) = \sum_{n=1}^\infty \sum_{\eta \in \mathscr{G}^n} p(\eta) E_\theta s_\eta^\psi \phi_\eta$ power function
- $\dot{\beta}_\theta(\psi, \phi)$ its derivative
- $\alpha(\psi, \phi) = \beta_{\theta_0}(\psi, \phi)$ type-I error probability
- τ_ψ stopping time generated by stopping rule ψ

- $d(k)$ cost of k observations in a group
- $\bar{d}_k = Ed(\nu_k)$ average cost of observations in k-th group
- $D(\psi) = \sum_{n=1}^{\infty} \sum_{\eta \in \mathscr{G}^n} p(\eta) d(\eta) E_{\theta_0} s_\eta^\psi$ average total cost due to the stopping rule ψ
- $\mathscr{F} = \{\psi : E_{\theta_0} \tau_\psi < \infty\}$
- $\mathscr{F}^N = \{\psi : t_\eta^\psi \equiv 0, \text{ for all } \eta \in \mathscr{G}^{N+1}\}$ the set of truncated (at time N) stopping rules

We assume that Assumptions 1–4 of [6] are satisfied. Then, by Theorem 2 of [6], the power function of any test (ψ, ϕ) with $\psi \in \mathscr{F}$ is differentiable at $\theta = \theta_0$, and its derivative at $\theta = \theta_0$ is equal to

$$\dot{\beta}_{\theta_0}(\psi, \phi) = \sum_{n=1}^{\infty} \sum_{\eta \in \mathscr{G}^n} p(\eta) E_{\theta_0} s_\eta^\psi \phi_\eta z_\eta,$$

where, for any $\eta \in \mathscr{G}^n$,

$$z_\eta = z_\eta(x_1^{(\eta_1)}, \ldots, x_n^{(\eta_n)}) = \sum_{i=1}^{n} z_{\eta_i}(x_{i1}, \ldots, x_{i\eta_i})$$

and

$$z_{\eta_i} = z_{\eta_i}(x_{i1}, \ldots, x_{i\eta_i}) = \sum_{j=1}^{\eta_i} \dot{f}_{\theta_0}(x_{ij}) / f_{\theta_0}(x_{ij}).$$

Our goal is to characterize all the tests (ψ, ϕ) with $\psi \in \mathscr{F}$ that maximize the derivative of the power function,

$$\dot{\beta}_{\theta_0}(\psi, \phi) \to \max_{\psi, \phi}, \tag{1}$$

in the class of tests (ψ, ϕ) with $\psi \in \mathscr{F}_0 \subset \mathscr{F}$, such that

$$\alpha(\psi, \phi) \leq \alpha \quad \text{and} \quad D(\psi) \leq D, \tag{2}$$

where $\alpha \in [0, 1)$ and $D > 0$ are some fixed constants. If such test exists, it is called *locally most powerful* (see [1], [4], [5]). In [9] and [8], only the tests with the same α-level are allowed ($\alpha(\psi, \phi) = \alpha$ in (2)).

In [7] we solved the maximization problem (1)–(2), for truncated sequential tests ($\psi \in \mathscr{F}^N$). In this article we will solve this maximization problem for non-truncated sequential tests ($\psi \in \mathscr{F}_0$).

3 Reduction to optimal stopping problem and Structure of optimal stopping rules

The problem of maximization (1) under conditions (2) is routinely reduced to minimizing the Lagrange function

$$L(\psi,\phi)=L(\psi,\phi;b,c)=cD(\psi)+b\alpha(\psi,\phi)-\dot\beta_{\theta_0}(\psi,\phi), \tag{3}$$

where $b\in\mathbb{R}$ and $c>0$ are some constant multipliers (to be quite formal, one can apply Theorem 1 in [7] with $\Delta=\{(\psi,\phi):\psi\in\mathscr{F})\}$.

Let us describe the decision rules that minimize $L(\psi,\phi)$ over ϕ, given any ψ.

Let us write $\phi\simeq I_{\{F_1\preccurlyeq F_2\}}$ (say) when $I_{\{F_1<F_2\}}\leq\phi\leq I_{\{F_1\leq F_2\}}$.

For example, it follows from Theorem 2 in [7] that the minimum of (3) is attained by ϕ defined as

$$\phi_\eta\simeq I_{\{b\preccurlyeq z_\eta\}} \tag{4}$$

for all $\eta\in\mathscr{G}^n$, $n=1,2,\dots,N$.

By virtue of Theorem 2 in [7]

$$L(\psi;b,c)=\inf_\phi L(\psi,\phi;b,c)=\sum_{n=1}^\infty\sum_{\eta\in\mathscr{G}^n}p(\eta)E_{\theta_0}s_\eta^\psi(cd(\eta)+g(b-z_\eta)) \tag{5}$$

for any $\psi\in\mathscr{F}$.

To get to a minimum of (5) over all truncated stopping rules $\psi\in\mathscr{F}^N$, in [7] the following construction is used.

Let $V_i^N=V_i^N(z)=V_i^N(z;c)$, $V_i^N:\mathbb{R}\mapsto\mathbb{R}^-$, $i=N,N-1,\dots,1$, defined in the following backward-recursive way: $V_N^N(z;c)=g(z)\equiv\min\{0,z\}$, and, recursively for $i=N,N-1,\dots,2,1$,

$$V_{i-1}^N(z;c)=\min\{g(z),c\sum_{\eta\in\mathscr{G}}p_i(\eta)d(\eta)+R_{i-1}^N(z;c)\} \tag{6}$$

where

$$R_{i-1}^N(z;c)=\sum_{\eta\in\mathscr{G}}p_i(\eta)E_{\theta_0}V_i^N(z-z_\eta;c). \tag{7}$$

For any stopping rule ψ let us define its truncation $\psi^N\in\mathscr{F}^N$ by $\psi_i^N\equiv\psi_i$ for $1\leq i\leq N-1$, and $\psi_N^N\equiv 1$.

Let also

$$L_N(\psi)=L_N(\psi;b,c)=L(\psi^N;b,c). \tag{8}$$

As $\psi^N\in\mathscr{F}^N$, it follows from Theorem 3 [7] and (8) that

$$L_N(\psi;b,c)\geq c\bar d_1+R_0^N(b;c). \tag{9}$$

Our plan is to pass to the limit in (9) as $N\to\infty$.

Let us prove that V_i^N and R_i^N defined in (6)-(7) have a limit as $N\to\infty$.

Lemma 3.1 *For all $N \geq 1$, $i \leq N+1$, for all $z \in \mathbb{R}$*
1) $V_i^N(z;c) \geq V_i^{N+1}(z;c)$,
2) $R_i^N(z;c) \geq R_i^{N+1}(z;c)$.

Proof of 1) is by induction over $i = N, N-1, \dots 1$. 2) follows from 1) by the Lebesgue monotone convergence theorem if we take (7) into account. □

Because by Lemma 3.1 V_i^N and R_i^N are non-decreasing with respect to N for each $z \in \mathbb{R}$, there exist limits $V_i(z) = V_i(z;c) = \lim_{N\to\infty} V_i^N(z;c)$, $R_i(z) = R_i(z;c) = \lim_{N\to\infty} R_i^N(z;c)$, for every $i \geq 1$. Passing to the limit, as $N \to \infty$, in (6) and (7) for any fixed $i = 0, 1, 2, \dots$, we get

$$V_{i-1}(z;c) = \min\{g(z), c\sum_{\eta\in\mathscr{G}} p_i(\eta)d(\eta) + R_{i-1}(z;c)\}, \tag{10}$$

$$R_{i-1}(z;c) = \sum_{\eta\in\mathscr{G}} p_i(\eta) E_{\theta_0} V_i(z - z_\eta; c). \tag{11}$$

The following lemma is a variation of Lemma 9 in [5] whose proof is the same, except that Theorem 1 of [6] is used instead of Theorem 1 of [5].

Lemma 3.2 *For all $b > 0$, $c > 0$, for all $\psi \in \mathscr{F}$*

$$L(\psi; b, c) \geq -\frac{\gamma_1\delta_1}{8c\delta_2}.$$

Remark 3.3 *It follows from Lemma 3.2 that*

$$\inf_{\psi\in\mathscr{F}} L(\psi; b, c) = c\bar{d}_1 + R_0(b;c) \geq -\frac{\gamma_1\delta_1}{8c\delta_2} > -\infty$$

for all $b > 0$ and $c > 0$.

This also implies that

$$c\bar{d}_{k+1} + R_k(b;c) > -\frac{\gamma_1\delta_1}{8c\delta_2}$$

for all $b > 0$, $c > 0$ and all $n \in \mathscr{G}^k$ $k \geq 0$. Indeed, by construction R_k is "the R_0 function" for the problem of testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$ about the parameter of the distribution of the process $X_1, X_2, \dots$, for which $X_1, X_2, \dots \sim f_\theta$. We have now that the right-hand side of (9) goes to $c\bar{d}_1 + R_0(b;c)$, as $N \to \infty$.

To pass to the limit in the left-hand side of (9), we need now that

$$L_N(\psi; b, c) \to L(\psi; b, c), \text{ as } N \to \infty, \text{ for any } b \in \mathbb{R} \text{ and } c > 0. \tag{12}$$

Let $\mathscr{F}_0$ be the set of all stopping rules in $\mathscr{F}$ satisfying (12).

In fact, one might think that (12) is fulfilled for any $\psi \in \mathscr{F}$, and it is very likely that this is true. Unfortunately we could not find a simple way to prove this fact.

On the other hand, we are pretty sure that any optimal sequential test should satisfy (12). If it does not, there will exist $N_i \to \infty$, as $i \to \infty$, such that $L_{N_i}(\psi) - L(\psi) > \varepsilon > 0$ for any i. Therefore, no truncation of the optimal ψ at N_i will permit getting closer than by $\varepsilon > 0$ to the minimum value $L(\psi)$. Because truncations may be thought of as some type of intervention of *force majeure* able to instantly interrupt the testing procedure, we think the behavior like this is not appropriate in statistical applications.

Lemma 3.4 *It holds that* $\inf_{\psi \in \mathscr{F}_0} L(\psi; b, c) = c\bar{d}_1 + R_0(b; c)$ *for any* $b \in \mathbb{R}$ *and* $c > 0$.

The proof of Lemma 3.4 is laid down in the Appendix.

The following Theorem characterizes the structure of optimal stopping rules in case such exist.

Theorem 3.5 *If there exists* $\psi \in \mathscr{F}_0$, *such that*

$$L(\psi; b, c) = \inf_{\psi' \in \mathscr{F}_0} L(\psi'; b, c), \tag{13}$$

then

$$\psi_\eta \simeq I_{\{g(b - z_\eta) \preccurlyeq c\bar{d}_{n+1} + R_n(b - z_\eta; c)\}} \tag{14}$$

P_{θ_0}*-almost surely on* T_η^ψ *for all* $\eta \in \mathscr{G}^n$, *and* $n = 1, 2, \ldots$.

Conversely, if ψ *satisfies* (14) P_{θ_0}*-almost surely on* T_η^ψ *for all* $\eta \in \mathscr{G}^n$, *for all* $n = 1, 2, \ldots$, *and* $\psi \in \mathscr{F}_0$, *then it satisfies* (13) *as well.*

The proof of Theorem 3.5 is also placed in the Appendix.

It is easy to see from Lemma 4 of [7] that if

$$c\bar{d}_{n+1} + R_n(0; c) \le 0,$$

then in each of the regions $\{z \le 0\}$ and $\{z \ge 0\}$ there exists a unique solution to the equation

$$c\bar{d}_{n+1} + R_n(z; c) = g(z), \tag{15}$$

$A_n = A_n(c) \le 0$ and $B_n = B_n(c) \ge 0$ (see the proof of Lemma 4 in [7]). In such case denote

$$\Delta_n = \Delta_n(c) = (A_n(c), B_n(c)) \quad \text{and} \quad \bar{\Delta}_n = \bar{\Delta}_n(c) = [A_n(c), B_n(c)].$$

In case

$$c\bar{d}_{n+1} + R_n(0) > 0,$$

let

$$\Delta_n(c) = \bar{\Delta}_n(c) = \emptyset.$$

Then it is easy to see that (14) is equivalent to

$$I\{b - z_\eta \in \Delta_n(c)\} \leq 1 - \psi_\eta \leq I\{b - z_\eta \in \bar{\Delta}_n(c)\}. \tag{16}$$

Corollary 3.6 *The statement of Theorem* 3.5 *holds true when* (14) *is substituted with* (16).

As an immediate consequences of Theorems 1 and 2 of [7], Theorem 3.5 and Corollary 3.6, we get

Theorem 3.7 *Let $b > 0$, $c > 0$ be arbitrary constants. Let $\psi \in \mathscr{F}_0$ be any stopping rule satisfying* (14) *P_{θ_0}-almost everywhere on T_η^ψ for all $\eta \in \mathscr{G}^n$, $n = 1, 2, \ldots$, and let the decision rule ϕ satisfy (4) P_{θ_0}-almost surely on S_η^ψ for all $\eta \in \mathscr{G}^n$, $n = 1, 2, \ldots$.*

Then the test (ψ, ϕ) is locally most powerful for testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$ in the following sense: for any (ψ', ϕ') with $\psi' \in \mathscr{F}$, such that

$$D(\psi') \leq D(\psi) \quad \text{and} \quad \alpha(\psi', \phi') \leq \alpha(\psi, \phi), \tag{17}$$

it holds

$$\dot{\beta}_{\theta_0}(\psi, \phi) \geq \dot{\beta}_{\theta_0}(\psi', \phi'). \tag{18}$$

The inequality in (18) *is strict if at least one of the inequalities in* (17) *is strict.*

If there are equalities in all of the inequalities in (17) *and* (18)*, then ψ' satisfies* (4) *P_{θ_0}-almost surely on $T_\eta^{\psi'}$ for all $\eta \in \mathscr{G}^n$, $n = 1, 2, \ldots$ (with ψ_η replaced by ψ'_n) and ϕ' satisfies* (4) *(with ϕ_η replaced by ϕ'_n) P_{θ_0}-almost surely on S_η^ψ for all $\eta \in \mathscr{G}^n$, $n = 1, 2, \ldots$.*

Remark 3.8 *Theorem 3.7 remains valid for the "lower-tail" formulation: one just has to substitute, simultaneosly, $b < 0$ for $b > 0$, $\bar{\phi}_\eta = 1 - \phi_\eta$ for ϕ_η, and $H_1 : \theta < \theta_0$ for $H_1 : \theta > \theta_0$. The argument for this is the same as in the proof of Theorem 5.3 of [4].*

To conclude, let us apply the results above to the particular case of i.i.d. group sizes. Let $p_i(k) = p_1(k)$, for all $i = 1, 2, \ldots$, for all $k \in \mathscr{G}$.

In this case the subindices k in (10) and (11) may be dropped, resulting in $V(z; c) = \min\{g(z), c\bar{d}_1 + R(z; c)\}$, $R(z; c) = \sum_{\eta_1 \in \mathscr{G}} p(\eta_1) E_{\theta_0}(z - z_{\eta_1})$, respectively. This turns (14) into

$$I_{\{g(b - z_\eta) < c\bar{d}_1 + R(b - z_\eta; c)\}} \leq \psi_\eta \leq I_{\{g(b - z_\eta) \leq c\bar{d}_1 + R(b - z_\eta; c)\}}.$$

Now the roots of the equation (15), $A_i = A = A(c) \leq 0$ and $B_i = B = B(c) \geq 0$, are independent of i, $\Delta_i = \Delta = (A, B)$, $\bar{\Delta}_i = \bar{\Delta} = [A, B]$, which yields

$$I\{b - z_\eta \in \Delta(c)\} \leq 1 - \psi_\eta \leq I\{b - z_\eta \in \bar{\Delta}(c)\}. \tag{19}$$

Thus, a test (ψ, ϕ) with ψ satisfying (19) and ϕ satisfying (4) is locally most powerful for testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$ in case of i.i.d. group sizes.

In particlular, if each group contains only one observation, then $p_i(1) = 1$, $i = 1, 2, \ldots$, then the group-sequential tests become "pure" locally most powerful sequential tests studied in [1], [4], [5].

4 Proofs

We need this additional lemma.

Lemma 4.1 *For each $i = 0, 1, \ldots, N$, $N = 1, 2, \ldots$:*
1) $R_i(z) \leq V_n(z) \leq g(z)$, $z \in \mathbb{R}$;
2) $R_i(z)$ *is concave and continuous on* $\mathbb{R}$;
3) $R_i(z)$ *is non-decreasing on* $\mathbb{R}$;
4) $z - R_i(z)$ *is non-decreasing on* $\mathbb{R}$;
5) $g(z) - R_i(z) \to 0$, *as* $z \to \pm\infty$.

Proof. Statements 1) – 4) follow from respective statements of Lemma 3 of [7] by passing to the limit, as $N \to \infty$ (the continuity in 2) follows from concavity). To prove statement 5), it is sufficient to show that $z - R_i(z) \to 0$ as $z \to -\infty$ and $R_i(z) \to 0$, as $z \to +\infty$.

To prove $R_i(z) \to 0$, $z \to +\infty$, it is sufficient to show, in view of (11) and the monotone convergence theorem, that $V_i(z) \to 0$, $z \to +\infty$. By statement 3), the limit $\lambda_i = \lambda_i(c) = \lim_{z\to+\infty} V_i(z; c)$ exists for $i = 1, 2, \ldots$. By (11), $\lim_{z\to\infty} R_{i-1}(z, c) = \lambda_i(c)$ for $i = 1, 2, \ldots$. Passing to the limit as $z \to +\infty$ in (10), we get $\lambda_i(c) = \min\{g(z), c\bar{d}_{i+1} + \lambda_{i+1}(c)\}$ for $i = 1, 2, \ldots$. Now it is obvious that if for some $i \geq 1$ $\lambda_i \leq 0$, then $\lambda_i(c) = c\bar{d}_{i+1} + \lambda_{i+1}(c) < 0$, therefore, $\lambda_{i+1}(c) = c\bar{d}_{i+2} + \lambda_{i+2}(c) < 0$ and so on for all other i's. Thus, $\lambda_{i+1}(c) = \lambda_i(c) - c\bar{d}_{i+1}$, $\lambda_{i+2}(c) = \lambda_{i+1}(c) - c\bar{d}_{i+2} = \lambda_i(c) - c(\bar{d}_{i+1} + \bar{d}_{i+2})$, $\ldots$ $\lambda_{i+k}(c) = \lambda_i(c) - c\sum_{i=1}^{k} \bar{d}_{i+1}$. As a consequence, $R_{i+k-1}(0; c) \leq \lambda_i(c) - kc\min_j\{\bar{d}_j\}$, for all $k \geq 1$. Since $\min_j\{\bar{d}_j\} > 0$, we get a contradiction with the fact that

$$R_{i+k-1}(0; c) > -\frac{\gamma_1\delta_1}{8c\delta_2} - c\bar{d}_{i+k} \tag{20}$$

(see Remark 3.3). Thus, $\lambda_i(c) = \lim_{z\to+\infty} R_{i-1}(z; c) = 0$ for all $i \geq 1$.

Now consider the case $z \to -\infty$. Since $E_{\theta_0} z_{\eta_i} = 0$, $\sum_{\eta_i \in \mathscr{G}} p_i(\eta_i) E_{\theta_0} z = z$ for all $\eta_i \in \mathscr{G}$, we have

$$V_{i-1}^N(z;c) - z = \min\{\min\{0,-z\}, \\ c\bar{d}_i + \sum_{\eta_i \in \mathscr{G}} p_i(\eta_i) E_{\theta_0} V_i^N(z - z_{\eta_i};c) - \sum_{\eta_i \in \mathscr{G}} p_i(\eta_i) E_{\theta_0}(z - z_{\eta_i})\};$$

passing to the limit as $N \to \infty$, we get

$$V_{i-1}(z;c) - z = \min\{\min\{0,-z\}, \\ c\bar{d}_i + \sum_{\eta_i \in \mathscr{G}} p_i(\eta_i) E_{\theta_0} V_i(z - z_{\eta_i};c) - \sum_{\eta_i \in \mathscr{G}} p_i(\eta_i) E_{\theta_0}(z - z_{\eta_i})\}. \quad (21)$$

By Lemma 1 of [7], as $N \to \infty$, it follows that functions $V_i^N(z) - z$ are non-increasing for $i = 1, 2, \ldots$, so there exists a limit $\lambda_i(c) = \lim_{z \to -\infty} V_i(z;c) - z \leq 0$ for each $i = 1, 2, \ldots$. In the same manner as above, passing to the limit as $z \to -\infty$ in (21), we get $\lambda_i = \min\{0, c\bar{d}_{i+1} + \lambda_{i+1}\}$, $i = 1, 2, \ldots$. As we suppose $\lambda_i < 0$, we obtain $\lambda_{i+k}(c) \leq \lambda_i(c) - kc \min_j\{\bar{d}_j\}$. as $k \to \infty$. Therefore, for all $z \leq 0$ by Lemma 4) (4.1) $R_{i+k-1}(z;c) - z \leq \lambda_i(c) - kc \min_j\{\bar{d}_j\}$. For $z = 0$ we get $R_{i+k-1}(0;c) \leq \lambda_i(c) - kc \min_j\{\bar{d}_j\}$, which is in contradiction with (20) for $i = 1, 2, \ldots$, so we get $\lambda_i(c) = \lim_{z \to -\infty} R_{i-1}(z;c) - z = 0$ for $i = 1, 2, \ldots$.

4.1 Proof of Lemma 3.4

Denote $U = \inf_{\psi \in \mathscr{F}_0} L(\psi)$, $U_N = \inf_{\psi \in \mathscr{F}^N} L(\psi)$. Hence, $U_N = \bar{c}\bar{d}_1 + R_0^N(b;c)$ for any $N = 1, 2, \ldots$. It is obvious that $U_N \geq U$ for any $N = 1, 2, \ldots$, so, $\lim_{N \to \infty} U_N \geq U$. Let us show that in fact $\lim_{N \to \infty} U_N = U$.

Suppose the contrary: $\lim_{N \to \infty} U_N = U + 4\varepsilon$ with some $\varepsilon > 0$; this would imply $U_N \geq U + 3\varepsilon$ for all large enough N. By definition of U, there exists a stopping rule ψ, such that $U \leq L(\psi) \leq U + \varepsilon$. Since, by definition of $\mathscr{F}_0$, $L_N(\psi) \to L(\psi)$, as $N \to \infty$, we have $L_N(\psi) \leq U + 2\varepsilon$ for all large enough N. Because, by definition, $U_N \leq L_N(\psi)$, we have that $U_N \leq U + 2\varepsilon$ for all large enough N, and we get a contradiction.

Hence, $U = \lim_{N \to \infty} U_N = c\bar{d}_1 + \lim_{N \to \infty} R_0^N(b;c) = c\bar{d}_1 + R_0(b;c)$. □

4.2 Proof of Theorem 3.5

In the proof of Theorem 3 of [7] we defined the function

$$Q_i^N(\psi) = \sum_{j=1}^{i-1} \sum_{\eta \in \mathscr{G}^j} p(\eta) E_{\theta_0} s_\eta^\psi (cd(\eta) + g(b - z_\eta)) + \sum_{\eta \in \mathscr{G}^i} p(\eta) E_{\theta_0} t_\eta^\psi (cd(\eta) + V_i^N(b - z_\eta))$$

and showed that for all $\psi \in \mathscr{F}$, $i = 1, 2, \dots$

$$Q_{i+1}^N(\psi) \geq Q_i^N(\psi). \tag{22}$$

For any $\psi \in \mathscr{F}$, $i = 1, 2, \dots$ by Lemma 3.1 and Lebesgue convergence theorem $Q_i^N(\psi)$ converges to

$$Q_i(\psi) = \sum_{j=1}^{i-1} \sum_{\eta \in \mathscr{G}^j} p(\eta) E_{\theta_0} s_\eta^\psi (cd(\eta) + g(b - z_\eta)) \\ + \sum_{\eta \in \mathscr{G}^i} p(\eta) E_{\theta_0} t_\eta^\psi (cd(\eta) + V_i(b - z_\eta)).$$

Passing to the limit in (22), as $N \to \infty$, for $i = 1, 2, \dots,$ we get

$$Q_{i+1}(\psi) \geq Q_i(\psi), \tag{23}$$

where both sides are finite, thus, for $i = 1, 2, \dots,$

$$L(\psi; b, c) \geq Q_{i+1}(\psi) \geq Q_i(\psi) \geq c\bar{d}_1 + R_0(b; c). \tag{24}$$

Now let ψ satisfy (13). By Lemma 3.4, $L(\psi; b, c) = c\bar{d}_1 + R_0(b; c)$; hence there are equalities in all the inequalities in (24).

Inequality (23) is equivalent to

$$\sum_{\eta \in \mathscr{G}^i} p(\eta) \int t_\eta^\psi (cd(\eta) + \psi_\eta g(b - z_\eta) + (1 - \psi_\eta)(c \sum_{\eta_{i+1} \in \mathscr{G}} p_{i+1}(\eta_{i+1}) d(\eta_{i+1}) \\ + \sum_{\eta_{i+1} \in \mathscr{G}} p_{i+1}(\eta_{i+1}) \int V_{i+1}(b - z_\eta - z_{\eta_{i+1}}) f_{\theta_0}^{\eta_{i+1}} d\mu^{\eta_{i+1}})) f_{\theta_0}^\eta d\mu^\eta \\ \geq \sum_{\eta \in \mathscr{G}^i} p(\eta) \int t_\eta^\psi (cd(\eta) + \min\{g(b - z_\eta), c \sum_{\eta_{i+1} \in \mathscr{G}} p_{i+1}(\eta_{i+1}) d(\eta_{i+1}) \\ + \sum_{\eta_{i+1} \in \mathscr{G}} p_{i+1}(\eta_{i+1}) \int V_{i+1}(b - z_\eta - z_{\eta_{i+1}}) f_{\theta_0}^{\eta_{i+1}} d\mu^{\eta_{i+1}}) f_{\theta_0}^\eta d\mu^\eta \}. \tag{25}$$

By Lemma A.1 of [4] the equality in (25) (and hence, in (23) and (24)) is attained if and only if (14) holds P_{θ_0}-almost surely on T_η^ψ, $\eta \in \mathscr{G}^i$. The “only if” part is proved.

Let now ψ satisfy (14) P_{θ_0}-almost surely on T_η^ψ, for all $\eta \in \mathscr{G}^n$, $n = 1, 2, \dots$. By Lemma A.1 of [4] an equality in (25), and hence in (24), is attained, resulting in $Q_i(\psi) = Q_{i-1}(\psi) = \dots = Q_1(\psi) = c\bar{d}_1 + R_0(b; c)$, $i = 1, 2, \dots$. Note

$$c\bar{d}_1 + R_0(b; c) = Q_i(\psi) = L_i(\psi) + \sum_{\eta \in \mathscr{G}^i} p_i(\eta) E_{\theta_0} t_\eta^\psi (V_i(b - z_\eta) - g(b - z_\eta)); \tag{26}$$

$i = 1, 2, \ldots$, let us show that $E_{\theta_0} t_\eta^\psi (g(b - z_\eta) - V_i(b - z_\eta; c)) \to 0$, $\eta \in \mathscr{G}^i$, as $i \to \infty$. By statement 1) of Lemma 4.1

$$E_{\theta_0} t_\eta^\psi (g(b - z_\eta) - V_i(b - z_\eta; c)) \leq E_{\theta_0} t_\eta^\psi (g(b - z_\eta) - R_i(b - z_\eta; c)). \qquad (27)$$

By statements 3), 4) of Lemma 4.1 and Remark 3.3, for $z \in \mathbb{R}$ we have

$$0 \leq g(z) - R_i(z; c) \leq -R_i(0; c) = -\frac{\gamma_1 \delta_1}{8c\delta_2} - c\bar{d}_{i+1} < \infty$$

Thus, from (27) it follows that

$$0 \leq E_{\theta_0} t_\eta^\psi (g(b - z_\eta) - V_i(b - z_\eta)) \leq \left(\frac{\gamma_1 \delta_1}{8c\delta_2} + c\bar{d}_{i+1} \right) P_{\theta_0}(\tau_\psi \geq i) \to 0,$$

as $i \to \infty$, because $E_{\theta_0} \tau_\psi < \infty$. Thus, $\lim_{i \to \infty} L_i(\psi) = c\bar{d}_1 + R_0(b; c)$. By (12) $\lim_{i \to \infty} L_i(\psi) = L(\psi)$, therefore $L(\psi) = c\bar{d}_1 + R_0(b; c) = \inf_{\psi' \in \mathscr{F}} L(\psi')$.

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