International Mathematical Forum, Vol. 15, 2020, no. 2, 61 - 92
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2020.91249

The Logical Paradoxes

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Abstract

Basic notions which refer to the so called logical paradoxes are analyzed. For illustrating of their systematizing according to the kind of breaking of the requirements for comprised, consistent and reasonable enough system of the notions, some well-known such paradoxes are considered. The received results elucidate on the foundations of cognition.

Mathematics Subject Classification: Foundations of logics and of the sets

Keywords: Cognition, Logic, Truth, Paradox, Set

Introduction

As a primary nature, matter is an immediately existing actuality. Cognition arises as a means of realizing the secondary living nature and in general it consists in the mediating of an actuality. And as reflecting of the images in the subject's soul, the thinking represents a base for the self-mediating consciousizing nature.

As soon as cognition in general is mediating of an actuality, then logic in general is the way of mediating and we call human knowledge the result of the thinking. Since in the thinking are reflected as its not own objective and subjective realities so and its own ideal reality, by this it creates generality of knowledge, which represents the truth in general.

In the human cognition generally there are three fundamental scientific disciplines:

a) physics – directed to the empirical, the external, the primary –
seeking for the factual truth
b) formalics (in a more narrow sense called mathematics) –
directed to the regular, the internal, the secondary –
seeking for the formal truth
c) philosophy – directed to the whole, the united, the synthesisary –

seeking for the total truth.

To the three fundamental scientific disciplines correspond three principally different ways of thinking at:

a) penetrating in the objective reality – factual logic,

where thought finds itself as a primary actuality or element

b) extracting the subjective reality – formal logic,

where thought finds itself as a secondary actuality or law

c) catching the ideal reality – total logic,

where thought finds itself as a synthesisary actuality or symbol.

Conceived in this way, physics is a universal discipline, which comprises all human knowledge from its objective side, whereas the universal discipline formalics comprises all human knowledge from its subjective side, and the universal discipline philosophy comprises the same knowledge from its ideal side. Unlike the rest of scientific disciplines, mutually complementing fundamental scientific disciplines physics, formalics and philosophy are determined not by some topic but according to the utmost wide directedness of their principally different investigations. There is a more detailed presentation of the so specified notions in the treatment "System of Cognition" in the site cfi.atwebpages.com .

Under the notion logic in a more narrow sense one usually understands only the formal logics as a scientific discipline for drawing knowledge from preliminary established knowledge according to preliminary established rules. However, in its figurative sense with the word logic is designated also a consistency of thinking as well as regularity in some change in general. Besides, by the term informal logic one designates investigation in general of every inference from the viewpoint of total logic. The contemporary standard formal logic in its turn is based on the judgment as stated establishment of correspondence, called in it truth, or of discrepancy, called in it untruth, of determinate thing with determinate actuality. The propositional calculus and its extension first-order predicate calculus (shortly – predicate calculus) build one idealized form of commonly used way of reasoning on the base of this primary judgment relation. Based on preliminary established different from correspondence or discrepancy relations, an increasing multitude of non-classical formal logics is developed together with it, at that and the rules for drawing of conclusions are different. For solving some more special problems the very predicate calculus is also transformed to a higher level by suitable for the aim changes.

Every man represents a local triunity of material, living and consciousizing nature. Because of that we comprehend actuality in a guise of finite images, which are objectively determined in the perceptive sphere, subjectively determined in the conceptive sphere and ideally determined in the sphere of thought. To these three kinds of images correspond objects, topics and ideas respectively, jointly called things. The natural human language is the device by which topics are formed in the sphere of thought.

As a generality of knowledge, the truth in general is established by means of correlation of a determined thing to a determined actuality. At the factual logic this correlation becomes immediately in the objective reality as a determination of relation between objects, what the measurement represents. At the formal logic the correlation is mediated in the subjective reality as drawing of knowledge from preliminary established knowledge according to preliminary established rules, what the formal proving of an assertion represents. In the ideal reality the total logic requires the generality of knowledge to consider widely range, consistently and reasonably enough all circumstances which are essential for the unity of the system of knowledge.

The notion is a conditional sign for the caught into it essence of a phenomenon, and the definition represents a brief and rich in content determination of a notion. At determining of every notion, it is transformed into a topic of investigation for which we must make use of well-established notions in themselves, which are independent of it and compatible with each other, if we do not want to reach: a) to a not actuality at self-determining; b) to an unsoundness at inconsistency; c) to a wandering at negligence; or d) to delusions at non-conforming. Because of that, at determining of the sense of every notion, we must use such other notions that can enrich it in one comprised, consistent and reasonable enough system of the notions.

The Greek word παράδοξος means: 1. Unusual, improbable, strange 2. Contradictory to an established opinion, while the word παραδόξως means: Incredibly, astonishingly, strangely. The contemporary notion paradox is used for signifying of most different situations in all spheres of life, which for one reason or another, are perceived as unusual, unexpected or astonishing. The so called logical paradoxes are due to not understood breaking of the requirements for comprised, consistent and reasonable enough system of the notions**.**

According to the kind of the logical fault, which consists of breaking the requirements for comprised, consistent and reasonable enough system of the notions, the logical paradoxes may be subdivided into engendered by:

a) Self-determining, which cannot be actual because of at self-denying leads to a vicious logical circle and at self-affirming leads to impossibility to be delimited the correspondence from the discrepancy of a determined thing with a determined actuality.

b) Inconsistency, which cannot be substantial because it leads to self-contradiction.

c) Negligence, which cannot be accepted because it leads to wandering among indeterminatenesses.

d) Non-conformity, which cannot be accepted because it leads to delusions.

1 Paradoxes due to self-determining

The self-determining in the predicate calculus cannot be other than self-affirming or self-denying. Because of that in it are possible both infinitely many identically true formulae, as for example $x \vee \neg x$, and infinitely many identically untrue formulae, as for example $x \wedge \neg x$, whose truth and respectively untruth is independent of the truth value of the variables that participate in them, as is x. At self-denying of a thing, breaking logical requirements is postulating the law of excluded middle by the first formula $(x \vee \neg x)$ and the allowing after that of contradiction $(x \wedge \neg x)$ by self-denying.

The multitude different interpretations of the ancient logical paradox “I am lying” are an indicator about the problems that the revealing of logical faults at self-denying determination poses. The differences stem from not clarifying the notions in this multifold concealing of the self-denying behind no explicitness, inexactness, mixing and indeterminateness. The original formulation of the so compound judgment doesn’t expose explicitly the support of the assertion that the subject is lying, because this support is inferred from the affirmative announcing of the very same expressed assertion (by the same subject). The explicit generalized developed kind of the judgment is “This assertion is not assertion”, at which besides the contradiction $(x \wedge \neg x)$ also the indeterminateness of the judgment topic becomes clear, i.e. to which of the two assertions it is referred. The second concealing consists of this, that here the veracity of assertion is determined without using the exact words truth and untruth, but instead of them only the word lie that hints about the negation of the word truth is used wittingly. The exact concrete developed kind of the judgment is “This truth is untruth” $(x \wedge \neg x)$, at which the leading to a vicious logical circle self-denying of truth is immediately seen. The third concealing consists of mixing the implicit support of assertion with the open denying of support of assertion from only one declarer of them. This mixing of the support with the lie allows both to be examined from the side of their veracity. The forth concealing is in the above mentioned insufficient determinateness of the judgment topic in the original formulation. This permits to refer the assertion once to the declarer’s support for the very same expressed assertion, at that we are based on its validity, and second time to refer it to the denying of the support of assertion of the declarer, at that we are based on the denying validity. Another variant with indeterminate topic of self-denying assertion of the same declarer is “This determinateness is indeterminateness” $(x \wedge \neg x)$. In this way it is arrived at an illimitably repeating vicious logical circle.

The illimitably repeating vicious logical circle protrudes more clearly when the self-denying assertion is announced by two different declarers as the example is:

“Plato: The next assertion of Socrates is a lie.

Socrates: The saying by Plato is truth.”

If Plato is right that Socrates lies, from what Socrates is saying follows that Plato lies that Socrates lies. However if Plato lies that Socrates lies, according the predicate calculus $(\neg\neg x \leftrightarrow x)$, Socrates is right that Plato is right. And so on to infinity.

Another example for implicit permitting of contradiction by self-denying is the well-known Russell’s paradox. Usually the notion set is determined as “A set is a well-defined collection of distinct objects.” at which it is not outspoken marked that the collection contains these objects. However, this containing is inferred, because the primitive notion "collection" being undetermined is comprehended also as a unity of the objects participating in it, which unity contains them. The self-denying "this containing is not a containing" $(x \wedge \neg x)$ is concealed behind this not explicitly expressed containing. Russell's paradox about "the set of all sets which do not contain themselves" is built on these two logical faults. In open kind it reads "the containing of all containing which do not contain themselves". By the inferring notion "all such sets" here is put the question for applying the chosen criterion also to the set which according to the determination must contain as well as this what must not contain, i.e. to not contain oneself. Then it is seen that if it does not contain oneself it must contain oneself and if it contains oneself it does not contain oneself, and so on to infinity. With analogical concealing other analogical paradoxes may be also composed as to say "this loving is not a loving" $(x \wedge \neg x)$ for a man who loves only these people that do not love themselves. In this way the barber who shaves only those men who do not shave themselves will have for company the man who loves only these people that do not love themselves. Try to imagine the tragedy of Gottlob Frege, for whom the grounds of his two-volume work "The Foundations of Arithmetic" collapse after his learning about the paradox. Or else the efforts of Russell and Whitehead to avoid the same paradox in the three-volume work "Principia Mathematica".

At self-affirming of a thing the impossibility to be delimited the correspondence from discrepancy of a determined thing with a determined actuality protrudes most clearly at proving of the formal truth. Because this proving bases oneself on deduction, i.e. always has conditional correlation of one affirmation A to other affirmation B: "if A, than B", presented with the expression $\mathrm{A} \rightarrow \mathrm{B}$. In the formal logic is accepted, that the conditional affirmation $\mathrm{A} \rightarrow \mathrm{B}$ as a whole is untruth only when the affirmation A is truth and the affirmation B is untruth. The famous paradox of Haskell Curry manifests oneself at the formula $(\mathrm{A} \rightarrow \mathrm{B}) \rightarrow \mathrm{C}$, which says "if the affirmation $(\mathrm{A} \rightarrow \mathrm{B})$ is truth, than the affirmation C also is truth". If in the case is accepted the truth of the formula as a whole, with which it is self-affirmed, then the affirmation C may be as truth so and untruth, because with the last acceptance the affirmation $(\mathrm{A} \rightarrow \mathrm{B})$ to be untruth is impossible. Therefore in the formal logic the self-affirming is inadmissible.

2 Paradoxes due to inconsistency

The using of incompatible reasons for the determining of one and the same notion makes a work of the lawyers. Classical is the controversy of the famous sophist Protagoras with his gifted pupil Euathlus about payment of debt according to the entered agreement or according to the court's decision that are incompatible. The court is the one that must appraise the worth of the different reasons if it does not want to contradict itself because of the extreme inconsistency in their using. In the case, if the court orders that Euathlus should pay his debt to Protagoras, at that Euathlus loses the case, and then is neglected the agreement clause that states his debt to be paid after the first case is won by him. And if the court orders that Euathlus should not pay his debt, but he nevertheless must pay it according to the agreement clause, then the court decision is neglected. However, on the base of conversely using such extreme inconsistency, pointed by Euathlus, in both cases he must not pay his debt. Because, if the court orders that the debt should be paid, this decision may be neglected according to the agreement the debt to be paid, when Euathlus wins his first case. And if the court orders that the debt should not be paid, then the agreement clause stating that it be paid after the first won by Euathlus case, may be neglected. Ultimately however Protagoras will obtain his reward from Euathlus. Because of, if Euathlus wins the first case, at secondly initiated proceedings by Protagoras against Euathlus about the same debt, Euathlus surely will lose the secondary case. That's why their one-sided arguments and counter arguments demonstrate brightly only the inevitably limited human nature – everyone looks after number one.

The inconsistency of the crocodile that promises to return the child to his mother if she guesses whether he really will return it to her is much more arbitrary. After the mother's answer that her child will not be returned the crocodile concludes that it will really not be returned. Because, if she was right in her guess that it will not be returned, the saying to not return it will be not truth. Whereby, on base of another truth is neglected the condition that she had guessed. And if she was not right in her guess that her child will be returned it won't be returned, according to the yet neglected condition she should guess. The mother naturally opposes this inconsistency to the reverse inconsistency, in using the grounds at which she concludes that her child must be returned. Because, if she guessed correctly that it would not be returned, it must be returned according to the condition that she should have guessed. Whereby is neglected the truth that the child should not to be returned according to the saying that it won't be returned. And if she was not right in the guess that the child would be returned, it must be returned since otherwise the saying it to be returned won't be truth.

The arbitrariness in the crocodile's inconsistency is surpassed by the author of Pinocchio paradox. As it is well-known when Pinocchio lies his nose grows. On the question what will happen if he says "Now my nose will grow", the author comes to an impasse because of the self-contradiction that if his nose really grows it ought not to be grown, because Pinocchio will not tell a lie, but if his nose is not really grown it ought to be grown since Pinocchio will be telling a lie. The cause for the impasse consists of this that with the so put question it is wanted from Pinocchio to do something that he can't do. Namely, he cannot tell the truth by telling a lie openly, nor can he openly tell a lie by telling truth just for the qualities attributed to him. According to the same "logic" some people want from all-mighty God to make a stone that He also cannot lift.

3 Paradoxes due to negligence

The clear breakings of logical requirements in the presence of self-determining or inconsistency are got lost in the indeterminateness of turbid breakings which are connected with using of negligently determinate notions. The article “Richard’s paradox” on the Internet, for example, begins by the fact that “certain expressions of natural language define real numbers unambiguously” which is illustrated by the number $17.101010... = \frac{1693}{99}$. After that its author simply does not pay attention (or frankly forgets!) that the number presented in two different ways with conditional signs established in advance, can be defined unambiguously by at least two different expressions of natural language that describe its presentation by so the established signs. That is why he is confident that the infinite set of such different expressions as determinants of different real numbers can be arranged in unambiguous correspondence with the unambiguously ordered set of real numbers, which is impossible to be done after each one of the so determined numbers may always be defined unambiguously by more than one different expressions of the natural language. The striking indeterminateness of the so compound sequence of expressions probably is not sufficient. That is why an expression of natural language is composed, by which, according to Cantor's diagonal method another real number is determined. That real number as an expression of natural language must be some of the yet ordered numbers, but it is supposed that it cannot be any of them because of its difference with every one of them (See below the analysis of this method.). In agreement with contemporary mathematicians the author of the article explains himself the so arisen paradox with the impossibility of determining which expressions of the natural language actually define a real number and which do not define it. However the expression “the real number the integer part of which is 17 and the nth decimal place of which is 0 if n is even and 1 if n is odd” and the expression “one thousand six hundreds ninety three divided into ninety nine” undoubtedly define unambiguously the same real number. In contrast to them the author of the paradox Richard explains himself it with the fact that the expression for real number composed by Cantor's diagonal method cannot actually determine it unambiguously. Because this expression is referred to the construction of an infinite set of real numbers of which the same is a part. According to Richard this number cannot be included as some number of the row, since its definition does not meet the criteria to be included in the sequence of the definitions used for its construction. However, the expression for the number composed by Cantor's diagonal method squarely meets to the criteria to be included anywhere in the ambiguously composed sequence of expressions for unambiguously defined by them numbers, because it also represents such an expression of natural language.

The words in the natural language are usually polysemantic in meaning that is determined additionally by the context. In addition, they are characterized also by a number of other determinatenesses such as the grammatical, the literal, the syllabical, etc. As they do not pay attention to all this (or frankly forget it!) some people try to divide all adjectives into two essentially different classes:

a) autological – that describe themselves giving as examples the English word *English*, *unhyphenated* and *pentasyllabic*

b) heterological – that do not describe themselves giving as examples *long* (because it is not a long word and is shorter than *short*), *hyphenated*, and *monosyllabic*.

The multitude of qualitatively different one from the other kinds of characteristics of one word shows the groundlessness of the comprehension that it describes itself by marking only one of them. It is more exactly to be said that the word is referred to oneself in some way. However, because of the difference of the characteristics, one may find words, which according to one of them, can belong to the one of the two classes, while according to other of them, can belong to the other of the two classes. Appropriate for the aim is proven to be the concealed self-denying “this self-referring is not self-referring” $(x \wedge \neg x)$ in the form of the so determined notion heterological. Because if the word heterological is autological, i.e. if it is referring to itself, its meaning is that it does not refer to itself, hence it ought to be heterological. But if it is heterological, i.e. if it doesn’t refer to itself, it ought not to concern anyone of its characteristics. However, by self-marking of the one’s own name that contradicts its meaning, it concerns one of them on account of that it ought to be autological. Unlike it, the self-marking of the one’s own name at the autological word is in harmony with its meaning. Some more indeterminateness is introduced also by words such as the word loud that becomes autological when pronounced with big intensity of the voice and is transmuted into heterological, when it is not pronounced with big intensity of the voice.

The Berry’s paradox is formulated by the phrase in the form "The smallest positive integer not definable in under sixty letters.". Naturally, at the different languages of which the words in writing are presented by letters, the number of letters in the respective phrase is different, however the announced in it restricted number is chosen to be bigger than is their factual number in it, in order to seem that it is turned out the self-denying “This not defining is defining” $(\neg x \wedge x)$. In a given natural language the most exact definition of a positive integer with words usually is made by a consecutively announcement of all its determinable digits of their respective places at used place-value notation. At the determining of such a number by means of such kind of defining and at a restricted number of letters, inevitably is reached up to “the smallest positive integer” for which determining phrase are needed a bigger number of letters than their restricted number. The seemingly determination of namely this number by the quoted phrase with less number than the restricted number of letters is achieved thanks to the no announced replacing of its most exact definition with a much more inexact determination, which is trustingly accepted as of equal value of its most exact definition. However, with fairly smaller letters than the used for composing the quoted phrase much more exactly determined can be defined, for example, the irrational number π as “the ratio of a circle's circumference to its diameter”, by using a phrase with such an extending of the degree of indetermination of the notion defining a number. This indicates that within the so broadly comprehended determinateness of the notion defining a number all infinitely many positive integers can be defined on principle. Because they have only finite quantity predictably distributed determinable digits which are regularly alternated at adding of a unit to every one of them that precedes it, in contrast to the irrational numbers that have infinite quantity unlimited from regularity not predictably distributed determinable digits. One way of such an extended defining of the infinitely many positive integers is, for example: after defining a positive integer with smaller letters than the used for composing of the quoted phrase, for defining of every next bigger and bigger number we can use the expression “number bigger by a unit than the preceding”, because at the so chosen indeterminateness of defining such a number it is enough to indicate only the preceding number.

In this case, the problem to be determined the minimum number of letters or words for a certain natural language, by which a phrase can be composed that should have the sense that it cannot be composed by the so determined way, is more interesting. The result obtained will be used as a criterion for the power of expressiveness of the respective language.

4 Paradoxes due to non-conformity

4.0 The present set theory

The present set theory does not conform to the circumstances that there are three basic kinds of numbers sets: a) finite sets – as are the enumerable finite countal subsets of the countal natural numbers N; b) chronal, which are countable infinite countal sets – as are the countable infinite set of the countal natural numbers N and its countable infinite proper subsets, as well as the countable infinite set of the rational numbers Q and its countable infinite proper subsets; c) spatial, which are uncountable infinite numbers sets – as are the one-dimensional uncountable infinite set of the real numbers R and its uncountable infinite proper subsets, as well as the next two-dimensional, three-dimensional and in general with a finite number of dimensions uncountable infinite numbers sets and their respective uncountable infinite proper subsets. (To avoid the irrelevant term “cardinal cardinality” here, the term countal is used instead of the term “cardinal” to mean the quality of the kind of natural numbers – the respective other kind of natural numbers are the ordinal numbers. See the use below of the notions countal cardinality and ordinal cardinality.) Because of the essential difference among the properties of the three kinds of numbers sets, the quantity of the elements belonging to them, called their cardinality, is compared in three qualitatively different ways. Bertrand Bolzano showed that the unambiguous and reversible correspondence between the elements of two finite sets is enough for ensuring of equality between their cardinalities. At the spatial sets however such a correspondence is not enough. For the relation between the cardinalities at the spatial sets is determined by means of the relation between the sizes of the covered of them extents in space with their dimension.

In the next three subdivisions of this research an attempt for conformation with the indicated circumstances is made, at that essential results are obtained, which concern the logical paradoxes and the foundations of mathematics.

4.1 History

Long before Galileo Galilei, it was known that on the one hand not all integers are exact squares, and on the other hand, the exact squares of all integers are integers. However, Galilei is the man who explained this peculiarity with an unlikeness of the properties of the finite sets and the properties of the infinite sets. At that stage, he found that for the finite sets, it was meaningful to distinguish smaller, equal or bigger, whereas for the infinite sets, we can only say that they are infinite [1].

The next deep insight into the difference between the properties of the finite sets and the properties of the spatial sets is in the posthumously published book of Bernard Bolzano [2]. It is clearly shown in (§20, §21, §22, and §24) of this book that the unambiguous and reversible correspondence between the elements of two finite sets is enough to ensure equality between the enumerable finite countal quantities of their elements. At the spatial sets, however, the availability of such a correspondence is not enough for determining equality between the uncountable infinite numerical quantities of their elements. For the relation between the uncountable infinite numerical quantities of the elements of the spatial sets is measured by means of the relation between the sizes of the covered of them extents in space with their dimension. In the case scrutinized by Bolzano such sizes are the lengths that they cover on the number line. An illustrative example for this is presented with the equation $5y = 12x$ and its corresponding Fig. 1 in §20, where it is seen, that the infinite set of the real numbers in the interval from 0 to 5 is in unambiguous and reversible correspondence with the infinite set of the real numbers in the interval from 0 to 12. However, the first set undoubtedly represents a proper subset of the second set because the length covered by it is only part of the length covered by the second set. At the end of §28, Bolzano summarizes his discovery in such a way: "… the correct calculation of the infinite aims at not … calculating the infinite set in itself …, but determining of *a relation* between one infinite and another…". Moreover, in §48 it was shown, with examples, the incomparability of the set of points on the number line with the set of points on an infinite surface, and of the set of points on an infinite surface with the set of points in the infinite three-dimensional space. With this in mind, the quantity of their elements is measured with qualitatively different units of measurement respectively for length, for area, and for volume.

At the end of the 19^{th} century, Georg Cantor used a combination of the elements of two sets for determining the presence or the absence of an unambiguous and reversible correspondence between them as a criterion for the relation "smaller than", "equal to" or "bigger than" between the cardinalities of these sets. In this way he delimited the chronal countable infinite countal set of the natural numbers N as incomparably small in relation to the spatial uncountable infinite number set of the real numbers R. Unfortunately up to now it is not realized that in this way the chronal sets became a border between the finite sets and the spatial sets.

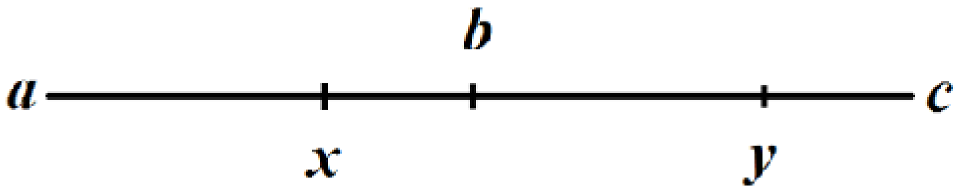

Fig. 1. A graph corresponding to the equation 5y = 12x.

The next step with consequences which are unrealized is consisted of marking of the quantity of elements of the set N with one sign, as is the letter $\aleph_0$ (aleph-null). With this marking the countable infinite countal quantity of elements $\aleph_0$ is transmuted into a border unambiguously determined quantity of elements, which by definition is incomparably big toward the enumerable finite countal quantity of elements *n* of a finite set and incomparably small in relation to the uncountable infinite numerical quantity of elements of the spatial set R. (Cause for the not realizing is the difference in the kind determinateness of the relations between comparable enumerable finite countal quantities, between incomparable enumerable finite countal quantities and countable infinite countal quantities, and between incomparable countable infinite countal quantities and uncountable infinite numerical quantities.) Along with this it is not understood that the so determined border quantity of elements $\aleph_0$ represents the natural unit of measurement for cardinality of the chronal sets. Presumably Cantor did not know about Bolzano's quoted work, and obviously he did not realize that as the border quantity of elements $\aleph_0$ was incomparably small in relation to the uncountable infinite numerical quantity of elements of the uncountable infinite number set R, similarly, it was incomparably big toward the enumerable finite countal quantity of elements *n* of any finite set. The simultaneous arising of so many basic and qualitatively different in kind transformations did not enable Cantor to realize sufficiently good as the essential difference among the properties of the three kinds of sets, so and $\aleph_0$ as natural unit of measurement for the cardinality of the chronal sets. With this in mind, he hurriedly accepted the sufficiency of the unambiguous and reversible correspondence between the elements of the finite sets for equality of their cardinalities, as also being valid for the chronal sets and as well as for the spatial sets. Because of that, he did not succeed in seeing the grandiosity of the world of the chronal sets discovered by him, as well as the still more complicated world of the spatial sets. That is why the presentation of these two worlds engenders many delusions. One of them is the delusion that the set of the points in the interval from 0 to 1 has the same quantity of elements as every space with a finite number of dimensions.

The discovered by Bernard Bolzano insufficiency of the unambiguous and reversible correspondence between the elements of the spatial sets for equality of their cardinalities is displayed by specific way and at the chronal sets.

4.2 The chronal sets

A set may be defined as a mentally created unity that contains well determined and distinct one from other things, called elements of the set. Since every kind of infinity represents a concrete determinateness which is idealized as unlimited, and the set N is a combination of all of the kinds of the simplest idealized quantitative regularities, therefore the basic qualities of the standard defined sets are revealed most easily when their elements are countal natural numbers. Utmost general characteristic of such number set X is the quantity of elements belonging to it, called cardinality of the set and marked with $|X|$. The set X is countable and infinite when at its juxtaposition with the countable infinite set N each element from the quantity of elements of the set X can be paired with exactly one element from the determined by a regularity infinite quantity of elements of the set N. The set X is enumerable and finite when at pairing of its quantity of elements through the countal unit 1 with the determined by a regularity infinite quantity of elements of the set N, the quantity of elements of the set X is exhausted up to a finite countal number n from the infinite quantity of elements of the set N. We will call the cardinality of such an enumerable finite countal set an enumerable finite countal cardinality n.

The juxtaposition between two finite sets by a harmonious combination of their elements represents bijection when, between these elements, there is an arbitrarily realizable unambiguous and reversible correspondence, marked also as one-to-one (1–1) correspondence. At such a correspondence, each element of the one set is paired with exactly one element of the other set, and each element of the other set is paired with exactly one element of the first set. Thus through the availability of bijection between two finite sets is established equality between their enumerable finite countal cardinality n.

At the juxtaposition between two chronal sets, the harmonious combination of their elements can be realized through two essentially different ways: a) by a not regularity-conforming unambiguous and reversible (non-conformal mutually countable) correspondence between the elements of their infinite quantities – see Fig. 6; and b) by a regularity-conforming unambiguous and reversible (conformal mutually countable) correspondence between the elements of their infinite quantities – see Fig. 2, Fig. 3, and Fig. 4. On the one hand, the infinite set N of the countal natural numbers is in non-conformal mutually countable correspondence with every one of its infinite proper subsets, whatever they also are among each other, because of their regularity and infinity. Because of that, the non-conformal mutually countable correspondence between their elements cannot be a criterion for the quantity of their elements. On the other hand, their quantities of elements most often are essentially different on account of the differences among the kinds of regularities, which determine the different kinds of chronal sets. For example, without doubt, the infinite quantity of the even numbers represents exactly half from the entire infinite quantity of the elements $\aleph_0$ of the set N, whereas the other half of its infinite quantity of elements is exactly represented by the infinite quantity of its odd numbers. Note that at the so done estimating for the relation between the cardinalities of these chronal sets, one uses not the countal unit 1 for enumerable finite countal cardinality *n*, which unit of measurement is inapplicable in the case, but $\aleph_0$ as a unit of measurement for chronal cardinality. That is why, at the juxtaposition between two chronal sets, their cardinalities must be determined not by the (same for all of them) non-conformal mutually countable correspondence between their elements, but by a conformal mutually countable correspondence. In the general case, the greater of them does not take part in the correspondence with all of its elements.

When the cardinalities of two chronal sets are essentially different, as, for example, the cardinality $\aleph_0$ of the set N and the cardinality a_s of the set of the even numbers, then the harmonious combination between the quantities of their elements in the case can be done conformably in three different ways, as described below:

a) Injection first at $a_s < \aleph_0$ – see Fig. 2. When the determined by the regularity evenness smaller infinite quantity of elements of the set of the even numbers are put in conformal mutually countable one-to-one (1–1) correspondence with the determined by the same regularity infinite quantity of elements of the proper subset of the even numbers of the set N, at which in the case, the odd numbers from the set with bigger quantity of elements remain without correspondence to the elements of the set with the smaller quantity of elements.

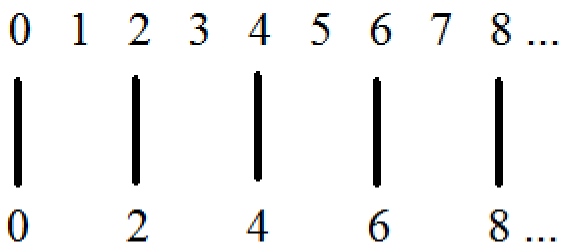

Fig. 2. Illustration of the conformal mutually countable injection first.

b) Injection second at $a_s < \aleph_0$ – see Fig. 3. When the determined by the regularity evenness smaller infinite quantity of elements of the set of the even numbers are put in conformal mutually countable one-to-one (1–1) correspondence with the determined by the analogical regularity infinite quantity of elements of the proper subset of the odd numbers of the set N, at which in the case, the even numbers from the set with bigger quantity of elements remain without correspondence to the elements of the set with the smaller quantity of elements.

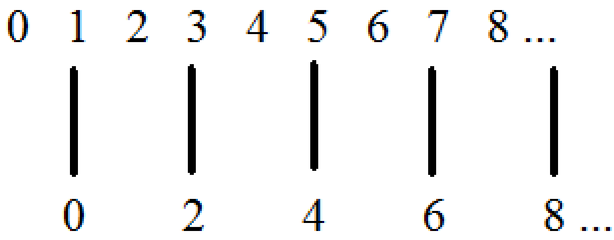

Fig. 3. Illustration of the conformal mutually countable injection second.

c) Surjection at $\aleph_0 > a_s$ – see Fig. 4. When more than one elements of the set with the bigger quantity of elements are put in a conformal, in this case, in a two-to-one (2–1) (responding to the kind of regularities mutually countable) correspondence with every one of the elements of the set with the smaller quantity of elements, at which in the case, there are no remaining without correspondence elements of the set with the bigger quantity of them.

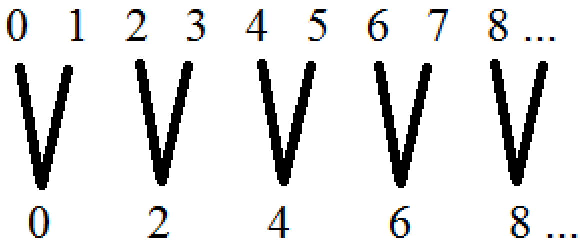

Fig. 4. Illustration of conformal mutually countable surjection.

When the cardinalities of the two chronal sets are the same, as in the case of the cardinality a_{s1} of the set of the even numbers and the cardinality a_{s2} of the set of the odd numbers, then the harmonious combination between their elements can be done only in one way, analogically to call it:

d) Bijection at $a_{s1} = a_{s2}$ – see Fig. 5. When the non-conformal and the conformal mutually countable one-to-one (1–1) correspondence coincide, with which the equivalence of the cardinalities of two chronal sets is established, similar to the equivalence of the cardinalities of two finite sets at bijection.

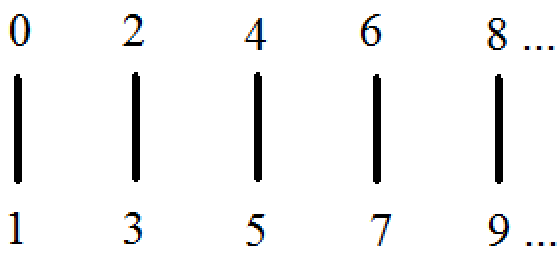

Fig. 5. Illustration of a bijection at chronal sets.

For a more exact determination of the relation between the cardinality $\aleph_0$ of the set N and the cardinality of any of its infinite proper subsets, as well as the relation between the cardinalities of any two of its infinite proper subsets, we should use the foreseen by Bolzano relation between the respective sums of all of the terms, which are in the scope of a determined distance r from the top of the sequence of the natural numbers. For example, the relation between the cardinality $\aleph_0$ of the set N and the cardinality of the set of squares of countal natural numbers is determined correctly and more and more exactly by the relations of the successively determined respective sums: 1 to 1; $1 + 2 + 3 + 4 = 10$ to $5 = 1 + 4$; $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$ to $14 = 1 + 4 + 9$; and so on. In the case these sums are determined for the distances of which are located several consecutive initial terms of the sequence of the squares of the countal natural numbers. Thus, on the one hand, we obtain the sum presented only by the terms of the sequence of the squares of the countal natural numbers for a determined distance from the top of the sequence of the countal natural numbers. Whereas, on the other hand, to the same sum is added the sum presented by the countal natural numbers, which are not exact squares of such numbers. The infinite set of the numbers, which are not exact squares, increases in progressing arithmetic progression, which after the first step ($2^2 = 4$), consists of the numbers 2 and 3, and with every next step of the sequence of the squares of the countal natural numbers, the number of the added such numbers is increased by two, while the numbers in themselves become bigger and bigger.

Bolzano's quoted work shows that the cardinalities of the spatial sets are determined by the relation between the sizes of lengths, areas, and volumes, which they cover on their corresponding extent: line, surface, and the three-dimensional space. At that each of them is comparable according to cardinality only with a set of its dimension. Therefore, the Archimedean property of comparability between two quantities is possible only at availability of unit of measurement, which is uniform with both of the quantities. Besides, we must distinguish the unambiguous and reversible correspondence between the different spatial sets, such as: the set of points in the interval from 0 to 1 and the set of points on the whole number line or the set of points in some other finite interval of it; the set of points on the whole number line and the set of points on an infinite surface; the set of points on an infinite surface and the set of points in the infinite three-dimensional space; and as well as at the remaining cases for every space with more finite dimensions.

From the considered examples with chronal sets, it follows that their initial comparing by cardinality is based on the kind of regularity, which determines them in relation to the chosen for unit of measurement cardinality $\aleph_0$. The chronal cardinality $\aleph_0$ is incomparable with an enumerable finite countal cardinality *n* of a finite set, similar to the incomparable with a finite sum $1 + 1 + 1 + \ldots < \omega$ hyperreal numbers in contemporary non-standard analysis. Generally, because the essential difference among the properties of the finite sets, of the chronal sets and of the spatial sets, their cardinalities are measured respectively by incomparable with each other units of measurement, namely: with the countal unit 1, with the countable infinite countable cardinality $\aleph_0$, and, with the uncountable infinite numerical cardinality of the uncountable infinite number set of a pleriminary chosen for unit of measurement part of spacial extent, which has equal number of dimentions with the number of dimentions of the spatial extent covered by the measured kind of set.

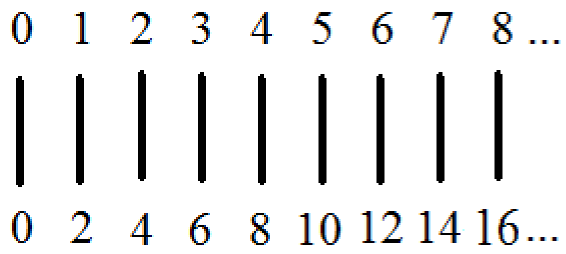

Fig. 6. Illustration of non-conformal mutually countable correspondence.

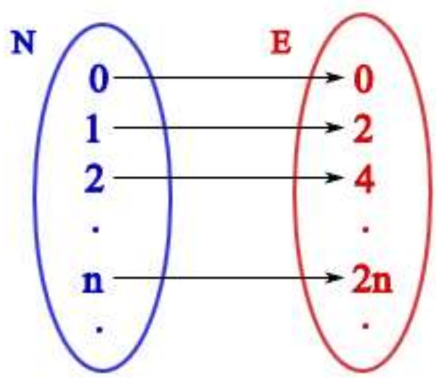

Fig. 7. Illustration of mistakenly understood bijection.

On Fig. 6 is shown the non-conformal mutually countable correspondence between the elements of the whole set of the countal natural numbers N with the elements of its proper subset of the even numbers. On Fig. 7, the same correspondence is illustrated as a unilateral pairing of the elements of the whole set N with the elements of the proper subset *E*. On Fig. 7, the presented non-conformal mutually countable correspondence between the elements of the sets N and *E* is now given as an example for bijection, that is, as a criterion for equivalence between infinite quantities of their elements. However, the definition for proper subset states that all the elements of the proper subset *X* should belong to the whole set *Y* and the proper subset *X* should be different from the whole set *Y*, i.e. $(X \subset Y) \Leftrightarrow (\forall x(x \in X \rightarrow x \in Y)) \wedge (X \neq Y)$. From this, it follows, that the proper subset *X* must have smaller quantity of elements than the quantity of elements of the whole set *Y*. In this instance, the subset *E* of the even numbers should have a smaller quantity of elements than the whole quantity of elements of the set N. In this way, one reaches to the obvious contradiction $(x \wedge \neg x)$ that the sets N and *E* now have an equal quantity of elements now that one of them has smaller quantity of elements than the other. That is why it is incorrectly to accept for bijection the depicted in Fig. 7 non-conformal mutually countable correspondence between the sets N and *E*. Besides, above already is shown a definition for bijection at the chronal sets which is similar to the definition for bijection at the finite sets. Because of the indicated existing absurdity, it is incorrectly considered that each one of the countable infinite proper subsets of the set N has the same cardinality as N. From here also follows the incorrect definition: one countal set is infinite *if it has a proper subset with the same cardinality*, instead of the correct definition: if it is in non-conformal mutually countable correspondence with a proper subset.

Paradoxically in this case is how it is possible for so much time to be neglected the conformal mutually countable correspondence when determining the cardinality of the chronal sets, even after unambiguously determined cardinality $\aleph_0$ of the set of the countal natural numbers N. For the set N consists of an infinite quantity of countable infinite countal proper subsets, which because of the determining them different kinds of regularities may contain as many as we want of more and more smaller parts of the countable infinite countal cardinality $\aleph_0$ of the set N. An example for such countable infinite proper subsets of the set N, that contain gradually decreasing parts of its countable infinite countal cardinality $\aleph_0$, is the next infinite succession:

a_1) 1, 2, 3, …, n, … $\aleph_0$

a_2) 2, 4, 6, …, 2n, … $\aleph_0/2$

a_3) 3, 6, 9, …, 3n, … $\aleph_0/3$

----

a_k) k, k2, k3, …, kn, … $\aleph_0/k$

----

where k is a countal natural number bigger than unit.

With using of $\aleph_0$ as a unit of measurement for the countable infinite countal cardinality of the chronal sets now we can without a problem determine as the faster decreasing countable infinite countal cardinality of the sets, being presented by the terms of the sequences with equal powers of the countal natural numbers, at the infinite succession of their increasing powers:

a) $1^2, 2^2, 3^2, \dots, n^2, \dots$

b) $1^3, 2^3, 3^3, \dots, n^3, \dots$

c) $1^4, 2^4, 3^4, \dots, n^4, \dots$

----

so and the more faster decreasing countable infinite countal cardinality of the sets, being presented by the terms of the sequences with increasing powers of the prime numbers, at the infinite succession of these numbers:

d) $2^1, 2^2, 2^3, \dots, 2^n, \dots$

e) $3^1, 3^2, 3^3, \dots, 3^n, \dots$

f) $5^1, 5^2, 5^3, \dots, 5^n, \dots$

----

which are used in Hilbert’s so-called “paradox” of the Grand Hotel.

Somehow or another, the essential difference between the non-conformal mutually countable correspondence and the conformal mutually countable correspondence at the determining of the relation between the cardinality $\aleph_0$ of the set N and the respective cardinalities of its infinite proper subsets remained unnoticed for more than three centuries. The unambiguously determined cardinality $\aleph_0$ added to Galileo's paradox also the Cantor's paradox for "the set of all enumerable finite countal sets", due to neglecting the difference between the properties of the finite sets and the properties of the chronal set of the countal natural numbers N. The difference between these properties is visually demonstrated with the aforementioned Hilbert's "paradox."

The Hilbert's "paradox" of the Grand Hotel is based on the unity of the infinitely many kinds of countable infinite proper subsets of the set N of the countal natural numbers (without the zero). Despite the fact that all of its infinitely many rooms are already occupied, after appropriate displacement of the tenants in them the Grand Hotel can always accommodates not only another guest but also infinitely many new guests at once, as well as all guests that arrive simultaneously by infinitely many coaches with infinitely many guests each. To accommodate another new guest, one simultaneously moves the guest currently in room 1 to room 2, the guest currently in room 2 to room 3, and so on, moving every guest from his current room n to room $n + 1$. After this, room 1 is vacant and the new guest can be accommodated into this room. To accommodate an infinite number of new guests at once, every guest currently in room n is moved into room $2n$, and the infinite new guests are accommodated in the rooms with odd numbers. The third problem is solved most elegantly by moving every of the available guests from room n in room 2^n, while the guests from the first coach are accommodated in the rooms 3^n, the guests from the second coach are accommodated in the rooms 5^n, the guests from the third coach are accommodated in the rooms 7^n, and so on to the infinity of the prime numbers, at that, the rooms 1, 6, 10, 12, 14, 15, 18, 20, 21, 22 and so on to the infinity of the numbers that are not powers of the prime numbers remain vacant.

In objective reality the measurement represents a determination of relation between objects. The process of measurement is always connected with a conception for the objects which are correlated. This process becomes a terminate entity, called measuring, only when the obtained result is interpreted. Every object is described by some essential indications, called its qualities. The physical magnitudes are characterizations of the objective reality that correspond to determined qualities. At measurement, some physical magnitudes exhibit comparable according to degree properties, called uniform quantities y of the measured magnitude Y. The magnitude Y is a primary insignia by which we can in general distinguish one thing from other different things, whereas the quantity y is a secondary insignia by which we are able to distinguish one from other different things of the one and the same magnitude when they exhibit themselves as comparable according to degree properties of this magnitude. In general, quantity is comparable according to the degree property of some magnitude.

The initial notion of a number x arises from the universal designation of the relation between a measured uniform quantity *y* and its part *u* chosen in advance, which serves as a unit of measurement

$$\mathrm{x} = \frac{y}{u}. \qquad (1)$$

At that a size of the quantity *y* is the expression x*u*, where the measure number x is called the value of the quantity *y*, which is presented by unit of measurement *u*. The remaining two letters in Equation (1) are also idealized as numbers in the theory of the numbers.

Contemporary factual logic indisputably establishes that at every determining of relation between uniform quantities according to Equation (1), the very determinateness of the measure number x always has finite size. In physics, this size is expressed by the number *m* of its reliably determined digits, which begin with its first most reliably determined different from zero digit and end with its last reliably-enough determined digit, called the significant digits of the measure number. Idealizing as unlimited the processes with numbers increases the number of their determinable digits. The process of division, for example, finishes in some cases without excess after a finite number of steps, at which the obtained number is presented with a finite quantity of determinable digits. In other cases, however, this process cannot be finished due to obtaining an excess which is periodically reiterating. Because of that, from the place of appearance of such of excess and farther, the obtained number is presented with an infinite quantity of predictably distributed determinable digits composed by periodically reiterating of the same group of combinations, without repetition of a finite quantity of digits. We can call completed the first kind of rational numbers as distinct from the second kind of uncompleted rational numbers. At the unlimited quantity of other kinds of processing with numbers, as for example, at some root extractions are obtained the so-called irrational numbers with infinite quantity of unpredictably distributed determinable digits. The first big shock in formal logic happens at the discovery of the incommensurability of the diagonal of the ideal square with its side, which is an example of an irrational number.

The infinite set Q of the rational numbers is presented in a positional countal system (positional numeral system) by two qualitatively different infinite proper subsets: a) by a proper subset q_c of the completed numbers, which are countable and with predictably distributed digits due to the regularity of successive alternation of a finite quantity of digits when adding a unit to every such preceding number, analogously to the infinite set N; and b) by a proper subset q_u of the uncompleted numbers, which are countable and with predictably distributed digits due to the regularity of a periodically reiterating group of a combination without repetition of a finite quantity of digits. For that reason the infinite set Q of the rational numbers as a whole is also countable. However, between the infinite quantity of elements of the set N and the infinite quantity of elements of the proper subset q_c of the completed rational numbers of the set Q, there is a determined by analogical regularity conformal mutually countable one-to-one (1–1) correspondence, whereas the elements of the proper subset q_u of the uncompleted rational numbers of the set Q remain without such a correspondence with the elements of the set N. For that reason, the infinite countal cardinality $\aleph_q$ of the set Q, as a sum of the infinite countal cardinalities of the two kinds of its countable infinite proper subsets, is bigger than the infinite countal cardinality $\aleph_0$ of the set N. The very proper subset q_u of the uncompleted numbers of the set Q is an infinite set composed by the all sorts of terms of the infinite set of its proper subset q_c of completed numbers, every one of which is consecutively reproduced with the all sorts of prolongations of all sorts of the periodically reiterating groups of combinations without repetition of a finite quantity of digits. Because of that the set Q represents a countable infinite countal set with the possibly greatest countable infinite countal cardinality $\aleph_q$. Cantor mistakenly accepts the non-conformal mutual countability as a criterion for the same cardinality at the chronal sets and presents uniformly the two kinds subsets of the rational numbers through the relation $\frac{a}{b}$, in contrast to their evident presentation as different kinds in a positional countal system. Thus, by the non-conformal mutual countability of the set Q of the rational numbers with the set N of the countal natural numbers proved by Cantor, is concealed the bigger countable infinite countal cardinality $\aleph_q$ of the set Q than the countable infinite countal cardinality $\aleph_0$ of the set N.

Cantor's paradox for "the set of all enumerable finite countal sets" is due to not taking into account the impossibility for enumerable finite countal cardinality *n* to be compared with by definition the incomparably with it countable infinite countal cardinality $\aleph_0$. When attempting to compare them, a procedure for increasing of the enumerable finite countal cardinality *n* of finite set by counting up its subsets is used. For this aim it is begun with the empty set of the finite set, continued one by one by the sets of all of its elements, passed on through the successively generated sets of their various combinations (without repetition) by two, by three and so on, coming to the set of the last combination of all its elements. In this way, the number *n* of the elements of the initial finite set turns out less than the number 2^n of the so counted up its subsets. Here, it is important to note that this procedure for increasing of the cardinality of every finite set inevitably transmutes the new set into an enumerable finite countal set of countal natural numbers. The validity of the relation

$$2^n > n \qquad (2)$$

between the enumerable finite countal cardinalities of two consecutive terms of the respective infinite sequence, as in the case represent the sets, which are consecutively created by counting up of the subsets of every preceding enumerable finite countal set, is proved by mathematical induction which consists of tautological confirmation of an established property for the terms of the respective sequence.

On the one hand Cantor accepts that the established Inequality (2) for the property of the terms of the infinite sequence of enumerable finite countal sets is valid and for "the set of all enumerable finite countal sets", which represents determined in other words the set N. For all numbers of the set N are namely enumerable finite countal sets of indistinguishable one from other units. However, one thing are the properties of the terms of this infinite sequence of enumerable finite countal sets and quite different thing are the properties of the infinite set, which is presented by the unity of its terms, so as one thing are the properties of the particular grains and quite different thing are the properties of the heap of such grains, noticed yet by the ancient Greeks. Especially in the case, when such a heap is infinite. Cantor did not consider this circumstance, due to that he accepted the validity of the Inequality (2) also for the N itself. On the other hand, by choosing one sign for marking its countable infinite countal cardinality, as is the letter $\aleph_0$, he fixes it as unambiguously determined countable infinite countal cardinality, which is incomparably big with respect to a determined enumerable finite countal cardinality *n*. Because of that, the countable infinite countal cardinality $\aleph_0$ cannot be reached and still less it can be surpassed by albeit infinitely increasing itself in this finite way an enumerable finite countal cardinality *n*. In this case, the contradiction $(x \wedge \neg x)$ consists of Cantor's attempt to compare two incomparable by definition cardinalities *n* and $\aleph_0$.

Cantor supposes that at an infinite consecutive increasing of the enumerable finite countal cardinality *n* of a finite set by counting up its subsets, the continuously increasing enumerable finite countal cardinality *n* of the so created finite sets can not only reach, but may also surpass the incomparable with it countable infinite countal cardinality $\aleph_0$ of the set N. What is more, Cantor does not stop here. He accepts as proven, that there is only one countable infinite countal cardinality $\aleph_0$ of the countable infinite countal set N and that there can also be another, qualitatively different from it, the already uncountable infinite number set of the real numbers R with uncountable infinite numerical cardinality *c*. He proposes for dependence between the two essentially different kinds of infinite cardinalities based on the Inequality (2) equation:

$$2^{\aleph_0} = c \tag{3}$$

– the so-called continuum hypothesis at which it is supposed that between $\aleph_0$ and *c* there are no other cardinalities. After that groundlessly is proceeded with an infinite series of alephs $\aleph_0 < \aleph_1 < \aleph_2 < \ldots$, founded on similar dependences among them.

In the binary positional countal system the countal natural numbers, the rational numbers and the irrational numbers may be presented. However, in the framework of the scheme at Cantor's diagonal method for proving the existence of an uncountable infinite set, the horizontally situated different sequences of zeros and units can present unambiguously only the infinite quantity of elements $\aleph_0$ of the set of the countal natural numbers N, as soon as the enumeration of these sequences is with the infinite sequence of the same natural numbers N – see Fig.8. For the cardinality of the set of the rational numbers $\aleph_q > \aleph_0$, and the cardinality |I| of the set I of the irrational numbers obviously is incomparably bigger than $\aleph_0$.

The first digit of the sequence *s* under the scheme is opposite to the first digit of the sequence s_1 in the scheme, the second digit of *s* is opposite to the second digit of s_2, the third digit of *s* is opposite to the third digit of s_3 and so on to the infinity of the countal natural numbers. With this in mind, it is intuitively expected for the sequence *s* to be different from all of the sequences of the countable natural numbers within the scheme, which will be correct for a finite scheme.

$$
\begin{array}{ll}
s_1 & = 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \ldots \\
s_2 & = 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \ldots \\
s_3 & = 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0 \ldots \\
s_4 & = 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1 \ldots \\
s_5 & = 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1 \ldots \\
s_6 & = 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0 \ldots \\
s_7 & = 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0 \ldots \\
s_8 & = 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1 \ldots \\
s_9 & = 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0 \ldots \\
s_{10} & = 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1 \ldots \\
s_{11} & = 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0 \ldots \\
\vdots & \quad \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \ddots
\end{array}
$$

$$
s = 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1 \ldots
$$

Fig. 8. The scheme of the diagonal method.

However, the scheme on Fig. 8 is infinite and by definition it contains all possible such sequences. Because of that, in this way it is impossible to create a sequence different from them, as it is impossible also for the biggest countal natural number to be created as all of them can always be increased by a unit. On the same impossibility is based also the possibility of Hilbert's Grand Hotel to accommodate additionally so many new guests regardless of the initial occupation of all its infinitely many rooms.

Drawn by the idea of an uninterrupted and consequently uncountable set R of the real numbers, Cantor endeavored to prove the uncountability of the set R by the diagonal number, groundlessly accepted as "different" from the elements of the set N. At that, he did not manage to see the (less than $\aleph_0$) countable infinite countal cardinalities of the different kinds of infinite proper subsets of the set N as well as the (bigger than $\aleph_0$) infinite countal cardinality $\aleph_q$ of the countable infinite set Q of the rational numbers, because of the groundless acceptance of the non-conformal mutual countability between the elements of two different kinds of chronal sets for equality of their cardinalities. Because of that, he groundlessly presupposes that the chronal sets may have only one countable infinite countal cardinality $\aleph_0$, whereas the spatial sets may have at least the uncountable infinite numerical cardinality *c* of the set R. Hence, if T is a set composed of the set N and the diagonally created number, by assuming that the set T is countable "it is proven" that it cannot be countable and thus at the present time the uncountability of the set R of the real numbers is considered as proven.

It is interesting to note that, although the diagonal number is believed to be "different" from the elements of the set N, up to now, the possibility of forming the infinite set of diagonal numbers N_1, different from the set N, has not been used. The diagonal set N_1 should be countable, like N, due to the regular method of its creation. Then the first summary "upset" $T = N + N_1$ should be countable infinite but with bigger countable infinite countal cardinality than $\aleph_0$. At infinite serial applying of the diagonal method to every next summary "upset," one seemingly infinite sequence of such countable infinite countal "upsets" is formed with increasingly bigger countable infinite countal cardinality than the countable infinite countal cardinality $\aleph_0$. By this, as well as with the paradox for "the set of all enumerable finite countal sets," the doubts of Cantor's contemporaries in the theory created by him might be explained. Quite other is the question, that after the determining of the countable infinite countal cardinality $\aleph_0$ of the set of the countal natural numbers N as an unit of measurement for cardinality of the chronal sets, there remains for us to unravel the interrupted continuity of the number continuum.

Similar to the paradox for "the set of all enumerable finite countal sets" is the Burali-Forti paradox for "the set of all enumerable finite ordinal numbers", because every non-initial such number represents the set of its ordinal natural numbers that precede it. Similar to the unambiguously determined countable infinite countal cardinality $\aleph_0$ of the set of the countal natural numbers N is unambiguously determined and countable infinite ordinal cardinality ω of the set of the ordinal natural numbers N_α. Although the just-described, seemingly infinite sequence of countable infinite countal "upsets" with increasingly bigger countable infinite countal cardinality than the countable infinite countal cardinality $\aleph_0$ was not created, Cantor already created a seemingly infinite sequence of countable infinite ordinal "upsets" with increasingly bigger countable infinite ordinal cardinality than the countable infinite ordinal cardinality ω. Similar to the groundlessly used dependence $2^{\aleph_0} = c$ for relation between the countable infinite countal cardinality $\aleph_0$ of the set of the countal natural numbers N and the uncountable infinite numerical cardinality c of the countal number continuum R, for relation between the countable infinite ordinal cardinality ω of the set of the ordinal natural numbers N_α and the uncountable infinite numerical cardinality ω_1 of the ordinal number continuum R_ω, the cardinality of the set of the terms of the seemingly infinite sequence of countable infinite ordinal "upsets" of the ordinal natural numbers N_α has groundlessly been accepted. As after the uncountable infinite numerical cardinality $c = \aleph_1$ of the countal number continuum R is groundlessly continued by an infinite series of alephs with incomparably increasing uncountability at its every next term, so after the uncountable infinite numerical cardinality ω_1 of the ordinal number continuum R_ω is groundlessly continued by an infinite series of omegas with incomparably increasing uncountability at its every next term. Simply, the fact has been disregarded that the criterion for determining uncountability bigger than that of the countal number continuum and respectively than that of the ordinal number continuum ought to be the established by Bolzano increase of the incomparability of the cardinalities of such spatial sets with the increasing of the number of the dimensions of the extent covered by them.

4.3 The continuum hypothesis

From the scrutinized examples with the chronal sets it follows that their chronal cardinalities may be arranged according to their decreasing cardinalities in this way:

1. Possibly the biggest countable infinite countal cardinality $\aleph_q$ of the countable infinite countal set of the rational numbers Q.

2. The cardinality $\aleph_{qu} = \aleph_q - \aleph_0$ of the countable infinite countal proper subset q_u of the uncomplete numbers of the set Q.

3. The cardinality $\aleph_0$ of the countable infinite countal set of the countal natural numbers N.

4. The unlimitedly decreasing countable infinite countal cardinalities $\aleph_{0k} = \aleph_0/k$ of the countable infinite countal proper subsets a_k of the set N, at which their cardinality remains a finite part of the cardinality $\aleph_0$ with the unlimitedly increasing of the countal natural number k.

5. The countable infinite countal cardinalities p_i, which are more and more exactly determined by the relation $s_i = \frac{S_{ir}}{S_{nr}}$, where i is a countal natural number for marking of a determined of the infinitely many countable infinite countal proper subsets of the set N, which are different from the sets a_k, S_{ir} is the sum of all terms of such a subset in the scope of distance r from the top of the sequence of the natural numbers, and S_{nr} is the sum of all terms of the sequence of the set N, which are in the scope of distance r from its top. The cardinalities p_i are incomparably big toward the enumerable finite countal cardinalities *n*, however, they incline to zero in comparison with the countable infinite countal cardinality $\aleph_0$. After the most slowly inclining to zero cardinality of the set, being presented by the terms of the sequence of the squares of the countal natural numbers, the cardinalities of the remaining such countable infinite countal proper subsets of the set N may be arranged according to the rapidity with which their respective relation s_i inclines to zero.

Or shortly:

$$p_i < \aleph_{0k} < \aleph_0 < \aleph_{qu} < \aleph_q. \tag{4}$$

This arrangement of the chronal countable infinite countal cardinalities may be connected with an analogical arrangement of the quantitatively generalized spatial uncountable infinite numerical cardinalities of the respectively quantitatively generalized standardly determined spatial uncountable infinite number sets.

As a unity of its elements the set is something different from them. The different countal natural numbers, for example, represent sets of different quantities of indistinguishable from each other units. However, the standardly determined set of such numbers is already another kind of set that in other way is different as from its elements so and according to the kind and the quantity of its contents from other similar sets. In a more wide sense of quantitative determinateness we may distinguish: sets of the same elements on one and the same level (such is the level of the sets of indistinguishable from each other units of the countal natural numbers), from sets of different elements on one and the same level (such is the level of a standardly determined set of different numbers), and from quantitatively generalized sets of elements which are a sum of determined in a different way on different levels elements (such are the elements of the two differently determined levels of the quantities of a standardly determined set of numbers).

In a positional countal system the generalized cardinality C_t of quantitatively generalized standardly determined countable infinite countal number set is a sum of its numerical cardinality C_n and of its digital cardinality C_d, which is a sum of the digital cardinalities of all its numbers. The countable infinite countal numbers sets are composed of infinite quantity numbers, every one of which is with a finite quantity of predictably distributed determinable digits or with an infinite quantity of predictably distributed determinable digits. Because of that their generalized cardinality is quantitatively countable infinite. The very predictability of the distribution of the determinable digits is based on the two described kinds of periodical regularities at numerical marking of different quantities engendered by the primary operations with numbers: adding, subtracting, multiplying and dividing. However, numerically marking different quantities are obtained and by infinitely many other kinds of operations with numbers, at which the distribution of the determinable digits is unpredictable because it does not obey these periodical regularities and as such it is chaotic. For this chaos the digital cardinality of an irrational number is quantitatively uncountable infinite, because it is equal to the uncountable quantity unequal permutations of its infinitely many unpredictably distributed determinable digits, in contrast to the countable quantity unequal permutations at the finitely many or at the infinitely many predictably distributed determinable digits of a rational number. In its turn the digital cardinality of any digitally presented number is determined by the quantity of its determinable digits, by their kind and place in a positional countal system, and as well as by its quality predictability of their distribution according to some regularity or unpredictability of their distribution for lack of such regularity. In the set N of the countal natural numbers every one of them is with a finite quantity of predictably distributed determinable digits, due to that it has finite digital cardinality. In the case this digital cardinality represents a sum of the digital cardinality of every one of its predictably distributed determinable digits, equal to the number of the undistinguishable from each other units that marks every digit, times the power of which it is multiplied according to the place of the respective digit in the positional countal system. Because of that the generalized numerical and digital cardinality $C_t = C_n + C_d = \aleph_{0s}$ of the quantitatively generalized set N_S of the countal natural numbers is quantitatively countable infinite.

As the countable infinite countable numerical cardinality $\aleph_{qu}$ of the proper subset q_u of the set Q is bigger than the countable infinite countable numerical cardinality $\aleph_0$ of the set N, so and the generalized numerical and digital cardinality $\aleph_{qus}$ of the quantitatively generalized proper subset q_{us} of the quantitatively generalized set Q_S of the rational numbers is bigger than the generalized numerical and digital cardinality $\aleph_{0s}$ of the quantitatively generalized set N_S of the countal natural numbers. Analogically, $\aleph_{qus} < \aleph_{qs}$, where $\aleph_{qs}$ is quantitatively generalized numerical and digital cardinality of the quantitatively generalized set Q_s of the rational numbers. However, the countable generalized numerical and digital cardinality $\aleph_{qs}$ of the limited from regularities countable infinite quantitatively generalized set Q_S of the rational numbers should be less than the uncountable digital cardinality $|I_p|$ of the unlimited of regularities and because of that uncountable infinite set of digits of any one separately taken irrational number I_p. Respectively $|I_p| < |I|$, where $|I|$ is the cardinality of the set of the irrational numbers I, also $|I| < |I_s|$, where $|I_s|$ is the generalized numerical and digital cardinality of the set of the generalized set I_s of the irrational numbers, as well as $|I_s| < c_s$, where c_s is the generalized numerical and digital cardinality of the quantitatively generalized set R_s of the real numbers. If it is proved that $\aleph_q < \aleph_{0s}$, then the arrangement of the chronal countable infinite countal cardinalities is connected by means of one's own respectively generalised countable infinite numerical and digital cardinalities with the arrangement of the spatial generalized uncountable infinite numerical and digital cardinalities close to c_s in this way:

$$p_i < \aleph_{0k} < \aleph_0 < \aleph_{qu} < \aleph_q < \aleph_{0s} < \aleph_{qus} < \aleph_{qs} < |I_p| < |I| < |I_s| < c_s. \qquad (5)$$

In a positional countal system the set I of the irrational numbers consists of all the sorts of infinitely digitally determined numbers with unpredictable distribution of these digits. With this is mind they fill up to continuity the set of the real numbers R, which together with them includes the set of the countal natural numbers N and the set of the fractional rational numbers. Therefore, the very set I turns out to be interrupted as by the set of the countal natural numbers N, so by the illimitably more dense set of the fractional rational numbers. In addition, the set I of the irrational numbers consists of illimitable quantity of kinds of subsets I_n, where *n* is a countal natural number for marking of the respective kind of their determining what are the various takings of a root, takings logs and the other kinds of functions that produce them. About the kinds of subsets I_n and I_{ns} follows to be determined how far it is possible their respective cardinalities to be unambiguously arranged according their size, or to be presented as a sheaf of rays from $|I_p|$ to c_s. Because of that in the interval between $|I_p|$ and c_s of the series of inequalities (5), the cardinalities $|I_n|$ and $|I_{ns}|$ can be suitably structured.

5 Short survey

At the end of 19th century Georg Cantor found the border $\aleph_0$ about cardinality between the enumerable finite countal sets and the uncountable infinite numbers sets. By means of the idea of quantization of the energy, at the same time Max Planck found the border between the microworld with sizes less than an atom and the mesoworld with sizes from an atom to a galaxy. The discoveries of physics in the middle of the 20th century outlined the borders between the microworld, mesoworld and megaworld with sizes bigger than a galaxy so clearly that made me compose the infinite sequence with self-increasing countal natural numbers in a maximum rapid way:

$$1^1,\ 2^{2^2},\ 3^{3^{3^3}},\ \ldots,\ n^{n^{\cdot^{\cdot^{\cdot^{n}}}}},\ \ldots \qquad (6)$$

and from it terms to form the infinite succession of infinite sequences with self-increasing in a maximum rapid way terms, but now on the base of every next so formed countal natural number:

$$2^{2^{2}} = 2^4 = 16,\ 16^{16^{\cdot^{\cdot^{16}}}},\ \ldots \qquad (7)$$

$$3^{3^{3^{3}}} = 3^{3^{27}} = 3^{7625537484987},\ \ldots \qquad (8)$$

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at which it is seen that the every next term of such a sequence is becoming more and more incomparably bigger in respect to the base from which it is formed. The idea for the last kind of sequences then arose for illustration of the finite however more and more incomparable qualitative transitions from the material nature to the living nature and from the living nature to the consciousizing nature. Now I can see that the properties of the infinite sequence (6) correspond to the properties of the living nature to locally increase its organization and the properties of the infinite sequences (7), (8), (9), … correspond to the properties of the consciousizing nature to be supporting on itself in its march towards achieving the world. Therefore the tendency toward accelerated unleashing of the expansion in more and more new spheres of manifestations as in the material, so in the living, and mostly in the consciousizing nature is a firm indication that as the living so and the consciousizing nature are not inconsiderable accidents in the evolution of the Universe. At that the last two natures are images and likenesses of the material nature as accelerators of its evolution, and as finite existences.

Observe, that after accommodating infinitely many new gests at once all Hilbert's Grand Hotel rooms are again occupied, whereas after accommodating at once all new gests that arrive simultaneously by infinitely many coaches with infinitely many gests each, incredibly many rooms in this Hotel remain vacant. However, the vacant rooms with a small number will be unimaginably increased if for moving every of the available guests and for accommodating the new gests in the second case are used the rooms not with the numbers of the terms from the sequences with the increasing powers of the prime numbers at the infinite succession of these numbers, but the rooms with the numbers of the terms from the last kind of infinite sequences (7), (8), (9), …

The arduous road from the finite toward the infinite is strewn with insights and delusions. After the delimitation of the finite from the infinite according to the difference between their qualities made by Galilei, Bolzano's work shines with its deep insight into these qualities. He set himself the great task yet in §1, by a preliminary analysis of the notions in this field to create a base, on which to reveal the seeming paradoxicality of the qualities of the infinite in comparison with the qualities of the finite. In his dispute with Spinoza, with Kant and with Hegel, as an average man, Bolzano didn't succeed in overcoming some unilaterality of his time, restricting in §10 the infinite only to the quantitative determinateness of the things, as this is usually accepted by mathematicians. And yet in §11 he presented the idea that a magnitude which always may be taken bigger in some relation "… may besides remain itself always simply a finite magnitude, which for example is right for every numerical magnitude 1, 2, 3, …". The difference between the notions magnitude and number as made in §6 and §16 is sufficient for the level of that time at considered topic.

In §20 after the demonstrative example of unambiguous and irreversible combining between the elements of two evidently different according to cardinality infinite sets, in §21, §22 and §24 Bolzano showed, that whereas at the finite sets such a correspondence is enough to ensure equality between their cardinalities, at the uncountable infinite numbers sets the relation between their cardinalities is determined by a relation between the respective sizes of the same for them parameter as in the case are the covered by them lengths on the number line. However, at his trial in §23 to explain the found difference between the properties of the enumerable finite countal sets and the properties of the uncountable infinite numbers sets by difference in the distances between the elements of the uncountable infinite numbers sets with different cardinality, he was inconsistent in relation to determining the quantity of their elements. Because the difference in the distances between the elements of the uncountable infinite numbers sets with different cardinality, which occupy different in sizes extents, implicitly presumes sameness between quantities of their elements, which is in contradiction with the difference between their cardinalities. In the case we cannot accept that the density of the points in the interval from 0 to 5 is different from the density of the points in the interval from 0 to 12, when the latter interval contains the former. Classical is the inconsistency in §33, of such a precise analytic as Bolzano, where he compares the quantities of countable infinite countal sets. Indeed at the initial estimation for the quantity of terms from the sequence of the squares of the countal natural numbers in comparison with the quantity of terms from their first powers, he revealed some hesitation. Ultimately, however, he incorrectly accepted that these quantities are equal, which accepting did not render the increasingly setting wide apart between the terms from the sequence of the squares of the natural numbers in comparison with the close following of the terms from the sequence of the natural numbers, as well as the participation in the sequence of the natural numbers in addition to the terms of the sequence of the squares of this numbers as well as the situated between them terms of the sequence of the natural numbers, which aren't exact squares of other natural numbers. Because of that, at the summing he incorrectly reckoned the consecutive terms of each of the sequences as corresponding, at that he reached to a wrong conclusion for the correlation between these sums. This mistake shows only the importance of the countable infinite countal cardinality $\aleph_0$ as a unit of measurement for initial delimiting by cardinality of the countable infinite countal sets.

When I wrote the sequences a_1), a_2), …, a_k), …; a), b), c), …, and d), e), f), …, not to mention the sequences (7), (8), (9), …, I noticed how easy and precisely I measured the relations between the infinite quantities of the elements from the first kind of sequences and how more difficult was to be done this for the next kinds of sequences. Bolzano's intuition is right to compare the sum of the terms of such magnitudes presented in different ways for determining the relation between their quantities. And just at the revealing of the right way for determining these sums I used the border $\aleph_0$ later found by Cantor as a unit measure to be sure that the quantity of elements of the set of the squares of the countal natural numbers is less than the whole quantity of elements of the set N.

Remarkable is the inference in §38 that the continuum can be explained only by accepting the fact, that "… every entity has, and must have, many properties which aren't inherent to its components". At that "… if we try to clarify the notion, which we call *uninterupted extent*, or *continuum*, we will be compelled to acknowledge that the continuum exists there, and only there, where a collection of elementary topics exists (points in time or in space, or substances), positioned in such a way that every one of them at every arbitrarily little distance has at least one topic neighboring it.". In short, besides the infinity of a set of elements it is necessary that they be positioned in a special way one in relation to the other, so that the respective extent can be obtained.

Grounded also is Bolzano's attempt yet at the beginning of the same paragraph to examine together and not as actual or as actuality time and space, as well as matter as their concrete determinateness. Because of the very notion of actuality contains in oneself as chronal (in a sense of different moments for events at one and the same place) and spatial (in a sense of different places for events at one and the same moment), so and subjectal (in a sense of factor which engenders these events) determinateness. Thus, he reaches to a determination of time not as a variable thing but as a thing in which all changes take place (§39), i.e. as a thing which is out of the objectively existing universal chronospace, which thing is a human idea of it. With this in mind he accepts the ideas in general as existing out of the objective universal chronospace but only in us men, irrespective of that we ourselves exist in this chronospace. In this sense we may consider as the evolution of the material nature so and the mediated by biological information evolution of the living nature as existing in the objective universal chronospace, but for the mediated as well by the thinking evolution of the consciousizing nature, our ideas have already quite different kind of existence and development. Because, in contrast to the immediate consecutive causing of the course of events in the material nature and the mediated by the inherited biological information primary facilities and activities of the organisms in the living nature to influence the course of their own evolution, the course of the events at the consciousizing nature is mediated yet also by the consciousal inheriting of the past, the transitory awareness of the present and the conscious building of the future with which the purposefully evolution of the Universe is accomplished. From this point of view we come to the inference that the qualitatively different kinds of evolution of the material nature, of the living nature and of the consciousizing nature determine as essentially different according to kind universal material, local zonsal (of the living nature – from ζῶον – a living being, such are the plants and the animals) and comprehensively consciousal chronospaces, so and essentially different according to kind possibilities functionaries in the so engendered chronospaces. Bulgarian Apostle of liberty Vasil Levski wrote "…time is in us and we are in time, it reverses us and we reverse it" [3].

In this way Bolzano closely reaches to Kant's idea for time and space as forms of our consciousness, however, he immediately opposes categorically to their only subjective existence. Because still in §13, at determining the notion for infinite, he divided the existing things into existing in the objective time and space and into ideas, which exist not in this time and space. Namely, in this sense he doesn't accept that there isn't time and space in general. In §40 the indescernibleness of the notion time from the notion place is pointed out, where he says "For me space like time *isn't a property of the substanses*, but only refering to them determination, namely: those determinations in the created [by God – note M.] substanses which give the base for this, that, ruling its propertes they evoke at some time certain in other determination changes, [them – note M.] I call *places,* at which these substances are found; the aggregation yet of all places I call *space*, all space.".

Undoubtedly, one more detailed analysis of this last work of Bolzano may reveal many of his achievements undiscovered yet. Here I will mark only one of them. "In two perfectly similar to each other extents, the sets of their points must be in such a relation namely, in what their sizes are" (§41). The points, for example of two concentric circumferences with different radii are in unambiguous and irreversible correspondence, which is established by constructing any radius from the center to them. However if we suppose that the quantities of their points are equal, at infinite decreasing the radius of the fewer of circumference it is coming to the ridiculous inference, that in their center there are as many infinite points, when it consists of only one point.

I defined the infinity in general as determinateness which is idealized as unlimited, to ground the idea that the basic qualities of a standardly defined set are most easily revealed when their elements are countal natural numbers. To the question “Teacher, you mention the infinity very often. Can you give us a more clear concept for it?” the classical Greek philosopher answered “Son, make an attempt to imagine the human foolishness and you will derive a concept for the infinite”. Due to the polysemy this determination is not only more comprising than mine, but it is deeper, since it pays attention at least to yet two essential sides of the problem. Because of the infinity is in us, men, since just in the human nature is inherent the quality to idealize illimitably. At that namely we possess notional knowledge, which undoubtedly is power, because it reveals us improbable possibilities for acting. However, in conjunction with this it conceals delusion just for our inclination to idealize the respective possibilities illimitably and so inevitably to surpass the boundaries of their validity. Isn’t Democritus maxim music:

For us, people:
there is the color
there is the sweet
there is the bitter –
actually:
there are only indivisible particles and the empty.

And that is why Nils Boor for that reason believed that the cause, on account of which the art can enrich us, consists of its capacity to remind us about harmonies which are unapproachable for the systematic analysis.

The durability of the changed in the total evolution of the material nature leads to diversification. The living nature represents a local evolution on the base of material nature where the durability and the change of its species are diversified and accelerated in the presented those specimens. The way for realizing the durability and the change at these two natures consists of relatively reiteration, which begins from the spin of the micro particles, passes through the rotation of the planets, the stars and the galaxies, at that it is incarnated in the material and in the living structures and function of the species and of the specimens. In the comprehensive evolution of the consciousizing nature the durability is based on the consciousal inheritance. At the evolution of the consciousizing nature the striving to achieve a harmony between our own nature and the Universe give a meaning to our existence. Because of only we can impart a meaning to our acts. This is what induced Ecclesiastes to write its maxim “Vanity of vanities, says the Preacher, vanity of vanities! All is vanity.” According the sense of which he should not have written it.

I wrote this treatment to show that logical paradoxes do not exist, but there is only not understood breaking of the requirements for comprised, consistent and reasonable enough system of the notions.

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Received: February 12, 2020; Published: February 28, 2020