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Strong Proximinality in the Function Space $\mathbf{L}^{\varphi}(\mu, X)$

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Abstract

In this paper, we study strong proximinality in the function space $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$, where φ is a modulus function and ***X*** is a Banach space. We prove some new results in this direction. The main theorem is that: if G is a separable subspace of ***X***, then ***G*** is strongly proximinal in ***X*** if and only if $\mathbf{L}^{\varphi}(\mu, G)$ is strongly proximinal in $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$. An important result that follows directly is: for ***G*** a separable subspace of ***X*,** then ***G*** is strongly proximinal in ***X*** if and only if $\mathbf{L}^{p}(\mu, \mathbf{G})$ is strongly proximinal in $\mathbf{L}^{p}(\mu, \mathbf{X})$, for 0 < p <1.

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1. Introduction and some Preliminaries

Let (T, Σ, μ) be a finite measure space and **X** a Banach space. Let $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$ the space of all equivalence classes of strongly measurable **X**-valued functions where φ is a modulus function such that $\int_{\mathbf{T}} \varphi \| f(t) \| \, d\mu < \infty$.

It is important to mention here that this does not define a norm on $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$. However, in order to make $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$ a complete metric space, the following metric is defined on $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$;

For f and g in $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$ then $\boldsymbol{d}(f, g) = \| f - g \|_{\varphi} = \int_T \varphi \left\| f(t) - g(t) \right\| dt.$

An example of a modulus function is $\varphi(x) = x^p$, $0 < p < 1$. So, a special case from above, one can define the function space $\mathbf{L}^{p}(\mu, \mathbf{X})$, $0 < p < 1$, of equivalence classes of strongly measurable functions $f : \mathrm{T} \longrightarrow \mathbf{X}$, where $\int_{\mathbf{T}} \| f(t) \|^p \, d\mu < \infty$. For $f \in \mathbf{L}^{p}(\mu, \mathbf{X})$, $0 < p < 1$, let $\| f \|_p = \int_{\mathbf{T}} \| f(t) \|^p \, d\mu$ then $\mathbf{L}^{p}(\mu, \mathbf{X})$ is also a complete metric space via the metric

$$\boldsymbol{d}(f, \mathrm{g}) = \| f - g \|_p = \int_T \| f(t) - g(t) \|^p \, d\mu .$$

Let $\boldsymbol{G}$ be a non-empty subset of a Banach space $(\mathbf{X}, \|.\|)$ and let $x \in \mathbf{X}$. The set of best approximation points to x from $\boldsymbol{G}$ is defined by:

$$\mathcal{P}_{\mathrm{G}}(x) = \{y \in \boldsymbol{G} : \|x - y\| = \boldsymbol{d}(x, \boldsymbol{G})\}, \text{ where } \boldsymbol{d}(x, \boldsymbol{G}) = \boldsymbol{inf}\, \{\|x - \mathrm{z}\| : \mathrm{z} \in \boldsymbol{G}\}.$$

The set $\boldsymbol{G}$ is called *proximinal* (resp. *Chebyshev*) if $\mathcal{P}_{\mathrm{G}}(x)$ contains at least (resp. exactly) one point for every $x \in \mathbf{X}$. The mapping $\mathcal{P}_{\mathrm{G}} : \mathbf{X} \rightarrow 2^{\boldsymbol{G}}$, which associates with each $x \in \mathbf{X}$ the set $\mathcal{P}_{\mathrm{G}}(x)$, is called the *metric projection* of $\mathbf{X}$ onto $\boldsymbol{G}$.

For the basic theory on proximinality in normed spaces, see [10]. Proximinality in spaces of Bochner p-integrable function spaces $\mathbf{L}^{p}(\mu, \mathbf{X})$, where $1 \leq p < \infty$, had been studied by many authors, see [4], [5], [7], [9] and others later on. On the other hand, the authors in [6] studied proximinality in $\mathbf{L}^{p}(\mu, \mathbf{X})$, where $0 < p < 1$.

Strong Proximinality was first considered by Godefroy and Indumathi, see [2]. For some other works in this area see [1], [3] and the references there in. A proximinal subset $\boldsymbol{G}$ of $\mathbf{X}$ is called strongly proximinal at a point $x \in \mathbf{X} \setminus \boldsymbol{G}$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathcal{P}_{\mathrm{G}}(x, \delta) \subseteq \mathcal{P}_{\mathrm{G}}(x) + \varepsilon \mathrm{B}_{\mathbf{X}}$, where $\mathrm{B}_{\mathbf{X}}$ is the unit ball of $\mathbf{X}$ and $\mathcal{P}_{\mathrm{G}}(x, \delta) = \{\mathrm{z} \in \boldsymbol{G} : \| x - \mathrm{z} \| < \mathbf{d}(x, \boldsymbol{G}) + \delta\}$. The set $\mathcal{P}_{\mathrm{G}}(x, \delta)$ is sometimes called the set of near best approximation points to x. $\boldsymbol{G}$ is called *strongly proximinal* in $\mathbf{X}$ if it is strongly proximinal at each x, for all $x \in \mathbf{X} \setminus \boldsymbol{G}$.

However, an equivalent definition, that is easier to deal with, will be given in the next section: see **Definition 2.2**.

Strong proximinality is one of many other proximinality notions of sets in Banach spaces which lead to important properties of approximation theory in these spaces. Therefore, it is critical to study the existence and some properties of such sets in function spaces also. In a recent work, see [11], the author studied strong proximinality in the spaces of Bochner p-integrable functions $\mathbf{L}^{p}(\mu, \mathbf{X})$, $1 \leq p < \infty$. Part of the work was to show that: "If $\boldsymbol{G}$ is separable and strongly proximinal subspace in **X**, then $\mathbf{L}^{p}(\mu, \mathbf{G})$ is strongly proximinal in $\mathbf{L}^{p}(\mu, \mathbf{X})$, $1 \leq p < \infty$."

It is worth noting that proximinality in the $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$ space and as a special case for $\mathbf{L}^{p}(\mu, \mathbf{X})$, $0 < p < 1$, has been studied in [6]. On the other hand, strong proximinality for these spaces is not encountered anywhere.

This paper is devoted to study strong proximinality in the function space $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$. Our goal is to prove that the above result in [11], is still valid for $\mathbf{L}^{p}(\mu, \mathbf{X})$, $0 < p < 1$. In the following section, we introduce some results concerning strong proximinality theory in the $\mathbf{L}^{\varphi}$- space that are needed for the proofs of the main results. In the last section, we give our main results in which we prove that $\mathbf{L}^{\varphi}(\mu, \mathbf{G})$ is strongly proximinal in $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$, when $\boldsymbol{G}$ is separable in **X** if and only if $\boldsymbol{G}$ strongly proximinal in **X**. As corollary, we obtain that this result is also valid for $\mathbf{L}^{p}(\mu, \mathbf{X})$, $0 < p < 1$.

2. Some General Results

The first result that is of most importance in the theory of best approximation in function spaces is the Distance Theorem. We prove a distance formula in $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$. A similar formula have been obtained for $\mathbf{L}^{p}$- spaces, $1 \leq p < \infty$, [7].

Theorem 2.1. (Distance Theorem)

Let φ *be a modulus function and* $f \in \mathbf{L}^{\varphi}(\mu, \mathbf{X})$. *Then the real valued function* $\boldsymbol{d}(f(\cdot), \boldsymbol{G}) \in \mathbf{L}^{\varphi}(\mu)$ *and* $\boldsymbol{d}(f, \mathbf{L}^{\varphi}(\mu, \mathbf{G})) = \| \boldsymbol{d}(f(\cdot), \boldsymbol{G}) \|_{\varphi}$.

Proof. For $f \in \mathbf{L}^{\varphi}(\mu, \mathbf{X})$ then f is strongly measurable, hence there exists a sequence of simple functions say $\{S_n\}$ in $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$ such that

$$\lim_{n\to\infty} \| S_n(t) - f(t) \| = 0, \ a.e\ t \text{ in T}.$$

The Continuity of the distance function implies that

$$\lim_{n\to\infty} | \boldsymbol{d}(S_n(t), \boldsymbol{G}) - \boldsymbol{d}(f(t), \boldsymbol{G}) | = 0.$$

Set $h_n(t) = \boldsymbol{d}(S_n(t), \boldsymbol{G})$. Then each h_n is a simple function and so $\boldsymbol{d}(f(\cdot), \boldsymbol{G})$ is measurable. Now, since $\boldsymbol{d}(f(t), \boldsymbol{G}) \leq \| f(t) - z \|$, for all $z \in \boldsymbol{G}$, thus for any g in in $\mathbf{L}^{\varphi}(\mu, \mathbf{G})$ we have,

$$\boldsymbol{d}(f(t), \boldsymbol{G}) \leq \| f(t) - g(t) \|, \ a.e.\ t \text{ in T and for all } g \in \mathbf{L}^{\varphi}(\mu, \mathbf{G}).$$

Then since φ is increasing, we get

$$\varphi\ \boldsymbol{d}(f(t), \boldsymbol{G}) \leq \varphi\ \| f(t) - g(t) \|.$$

Hence,
$$\int_{T} \varphi\ \boldsymbol{d}(f(t), \boldsymbol{G})\, dt \leq \int_{T} \varphi\ \| f(t) - g(t) \|\, dt.$$

Therefore,
$$\| \boldsymbol{d}(f(\cdot), \boldsymbol{G}) \|_{\varphi} \leq \| f - g \|_{\varphi}, \text{ for all } g \in \mathbf{L}^{\varphi}(\mu, \mathbf{G}).$$

Consequently $\boldsymbol{d}(f(\cdot), \boldsymbol{G}) \in \mathbf{L}^{\varphi}(\mu)$, then by taking the infimum over all g in $\mathbf{L}^{\varphi}(\mu, \mathbf{G})$, we get :

$$\| \boldsymbol{d}(f(\cdot), \boldsymbol{G}) \|_{\varphi} \leq \boldsymbol{d}(f, \mathbf{L}^{\varphi}(\mu, \mathbf{G})) \tag{1}$$

For the other direction, again $f \in \mathbf{L}^{\varphi}(\mu, \mathbf{X})$ is strongly measurable implies that for given $\varepsilon > 0$, there exists a simple function $f' \in \mathbf{L}^{\varphi}(\mu, \mathbf{X})$ and $\| f - f' \|_{\varphi} < \varepsilon/3$.

We can write f' in the form $f'(t) = \sum_{i=1}^{n} x_i \chi_{A_i}(t)$, where the A_i's are disjoint, measurable subsets of T satisfying $\cup A_i = T$ and $x_i \in \mathbf{X}$.

Now, for each $i = 1, 2, ..., n$, take $y_i \in \boldsymbol{G}$ such that $\| x_i - y_i \| < \boldsymbol{d}(x_i, \boldsymbol{G}) + \varepsilon/_{3\mu(T)}$.

Set $g(t) = \sum_{i=1}^{n} y_i \chi_{A_i}(t)$ then $g \in \mathbf{L}^{\varphi}(\mu, \mathbf{G})$ and

$$\boldsymbol{d}(f,\ \mathbf{L}^{\varphi}(\mu,\mathbf{G})) \ = \ inf\,\{\|\ f-h\|_{\varphi},\forall\, h\in \mathbf{L}^{\varphi}(\mu,\mathbf{G})\,\} \ \le \ \|\ f-g\ \|_{\varphi}.$$

But, $\|\ f-g\ \|_{\varphi} = \int_T \varphi\,\|f(t)-g(t)\|\ dt \ = \int_T \varphi\,\|\ f(t)-f'(t)+f'(t)-g(t)\|\ dt.$

Then by triangular inequality and φ is increasing,

$$\|\ f-g\ \|_{\varphi} \ \le \int_T \varphi\,(\|\ f(t)-f'(t)\ \| + \|f'(t)-g(t)\|)dt$$

And by a property of the modulus function (subadditive property), we get

$$\begin{aligned}
\|\ f-\mathrm{g}\ \|_{\varphi} \ &\le \int_T \varphi\,\|\ f(t)-f'(t)\ \|dt + \int_T \varphi\,\|f'(t)-g(t)\|dt \\
&= \|\ f-f'\ \|_{\varphi} + \int_T \varphi\,\|\ f'(t)-g(t)\ \|\ dt \\
&< \ \varepsilon/3 \ + \ \Big\|\sum_{\mathrm{i}=1}^{\mathrm{n}}\chi_{\mathrm{Ai}}(\mathrm{t})(x_{\mathrm{i}}-\mathrm{y}_{\mathrm{i}})\Big\|_{\varphi} \\
&< \varepsilon/3 \ + \sum_{\mathrm{i}=1}^{\mathrm{n}} \varphi\,\|\ x_{\mathrm{i}}-\mathrm{y}_{\mathrm{i}}\ \|\ \mu(\mathrm{A}_i) \\
&< \varepsilon/3 \ + \sum_{\mathrm{i}=1}^{\mathrm{n}} \varphi\,\ (\boldsymbol{d}(x_{\mathrm{i}}\,,\boldsymbol{G}\,)+\varepsilon_{/3\mu(\mathrm{T})})\ \ \mu(\mathrm{A}_i) \\
&\le \varepsilon/3 \ + \sum_{\mathrm{i}=1}^{\mathrm{n}} [\,\varphi\,\boldsymbol{d}(x_{\mathrm{i}}\,,\boldsymbol{G}) + \ \varphi\,(\varepsilon_{/3\mu(\mathrm{T})})]\ \mu(\mathrm{A}_i) \\
&< \varepsilon/3 \ + \sum_{\mathrm{i}=1}^{\mathrm{n}} [\,\varphi\,\boldsymbol{d}(x_{\mathrm{i}}\,,\boldsymbol{G}) + \varepsilon_{/3\mu(\mathrm{T})}]\ \mu(\mathrm{A}_i) \\
&\le \varepsilon/3 \ + \int_T \varphi\,\boldsymbol{d}(f'(t)\,,\boldsymbol{G})\ dt \ + \ \varepsilon/3 \\
&\le 2\varepsilon/3 \ + \int_T \varphi\,\boldsymbol{d}(f(t),\boldsymbol{G})\ dt + \int_T \varphi\,\|\ f(t)-f'(t)\|\ dt \\
&\le 2\varepsilon/3 \ + \int_T \varphi\,\boldsymbol{d}(f(t),\boldsymbol{G})\ dt + \|\ f-f'\ \|_{\varphi} \\
&\le \ \varepsilon \ + \|\ \boldsymbol{d}(f(\cdot),\boldsymbol{G}\,)\ \|_{\varphi}
\end{aligned}$$

So, as $\varepsilon \to 0$, and from (1) above we have,

$$\boldsymbol{d}(f,\ \mathbf{L}^{\varphi}(\mu,\mathbf{G})) = \|\ \boldsymbol{d}(f(\cdot),\boldsymbol{G})\ \|_{\varphi}.\ \blacksquare$$

In what follows, we give an alternative definition for strong proximinality of subsets in a Banach space which we will use in the proofs of our results.

Definition 2.2. (Strongly Proximinal)

A proximinal subset of $\boldsymbol{G}$ *of a Banach space* $\mathbf{X}$ *is said to be strongly proximinal if for each* $x \in \mathbf{X}$ *and for any minimizing sequence* $\{y_n\} \subseteq \boldsymbol{G}$ *for* x*, there is a subsequence* $\{y_{nk}\}$ of $\{y_n\}$ *and a sequence* $\{z_k\} \subseteq \mathcal{P}_G(x)$ *such that* $\|y_{nk} - z_k\| \to 0$. (*This is equivalent to*: $\boldsymbol{d}(y_{nk}, \mathcal{P}_G(x)) \to 0$), *where a minimizing sequence* $\{y_n\} \subseteq \boldsymbol{G}$ *for x is that satisfying*: $\lim_{n\to\infty} \| x - y_n \| = \boldsymbol{d}(x, \boldsymbol{G})$.

Lemma 2.3.

Let $\{g_n\}$ *in* $\mathbf{L}^{\varphi}(\mu, \mathbf{G})$ *be a minimizing sequence for* $f \in \mathbf{L}^{\varphi}(\mu, \mathbf{X})$ *then there exists a minimizing sequence in* $\boldsymbol{G}$ *for* $f(t) \in \mathbf{X}$*, a.e t in T.*

Proof. Let $\{g_n\}$ in $\mathbf{L}^{\varphi}(\mu, \mathbf{G})$ be a minimizing sequence for f in $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$ *i.e*

$$\|f - g_n\|_{\varphi} \to \boldsymbol{d}(f, \mathbf{L}^{\varphi}(\mu, \mathbf{G})).$$

Then by Distance theorem, we have

$$\|f - g_n\|_{\varphi} \longrightarrow \|\boldsymbol{d}(f(\cdot), G)\|_{\varphi}.$$

This implies,

$$\| f - g_n \|_{\varphi} - \|\boldsymbol{d}(f(\cdot), G)\|_{\varphi} \to 0$$

By a property: convergence in $\mathbf{L}^{\varphi}$ implies pointwise convergence *a.e.* t in T for subsequences. Hence, $\exists$ a subsequence $\{g_{nk}(t)\}$ such that:

$$\|f(t) - g_{nk}(t)\| - \boldsymbol{d}(f(t), \boldsymbol{G}) \to 0, \quad a.e.\ t \text{ in T}.$$

Therefore, $\{g_{nk}(t)\}$ in $\boldsymbol{G}$ is a minimizing sequence for $f(t)$ in $\mathbf{X}$, *a.e.* t in T. ■

3. Strong Proximinality of $\mathbf{L}^{\varphi}(\mu, \mathbf{G})$ in $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$

In this section, we give our main results in which we prove the strong proximinality of $\mathbf{L}^{\varphi}(\mu, \mathbf{G})$ in $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$, whenever $\boldsymbol{G}$ is separable and strongly proximinal in $\mathbf{X}$.

Theorem 3.1. *Let* ***G*** *be separable subspace of* **X.** *If* ***G*** *is strongly proximinal* in **X** *then* $\mathbf{L}^{\varphi}(\mu, \mathbf{G})$ *is strongly proximinal in* $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$.

Proof. Let ***G*** be strongly proximinal in **X** then by definition ***G*** ***i***s proximinal in **X** and since ***G*** is separable in **X** then $\mathbf{L}^{\varphi}(\mu, \mathbf{G})$ is proximinal in $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$**.** By **Theorem 2.5.** in [6] and **Theorem 3.4.** in [9].

Now, assume $\mathbf{L}^{\varphi}(\mu, \mathbf{G})$ is not strongly proximinal in $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$. Hence, $\exists$ f in $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$, $\varepsilon > 0$ and a sequence $\{g_n\} \subseteq \mathbf{L}^{\varphi}(\mu, \mathbf{G})$ such that $\{g_n\}$ a minimizing sequence for f in $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$ but $\mathbf{d}(g_n, \mathcal{P}_{\mathbf{L}^{\varphi}(\mu,\mathbf{G})}(f)) \geq \varepsilon$. So, by the **Lemma 2.3.,** $\exists$ a subsequence $\{g_{nk}(t)\}$ in ***G*** which is a minimizing sequence for $f(t)$ in **X**, *a.e* t in T. i.*e*.

$$\| f(t) - g_{nk}(t) \| \quad \to \quad \boldsymbol{d}(f(t), \boldsymbol{G}), \quad a.e \ t \text{ in T}.$$

Since **G** is strongly proximinal in X, this implies $\boldsymbol{d}(g_{nk}(t), \mathcal{P}_{\mathbf{G}}(f(t))) \to 0$, *a.e* t in T. But the distance function $\boldsymbol{d}(g_{nk}(.), \mathcal{P}_{\mathbf{G}}(f(.)))$: T $\to$ **R** is continuous, hence measurable and since $\varphi\, \boldsymbol{d}(g_{nk}(t), \mathcal{P}_{\mathbf{G}}(f(t))) \leq \| f(t) - g_{nk}(t) \| \leq 2\ \varphi \| f(t) \|$, *a.e* t in T, then

$$\boldsymbol{d}(g_{nk}(.), \mathcal{P}_{\mathbf{G}}(f(.))) \in \mathbf{L}^{\varphi}(\mu) \text{ and } \int_{T} \varphi\, \boldsymbol{d}(g_{nk}(t), \mathcal{P}_{\mathbf{G}}(f(t)))\, dt \to 0.$$

This means that, $\| \boldsymbol{d}(g_{nk}(.), \mathcal{P}_{\mathbf{G}}(f(.))) \|_{\varphi} \to 0$. Again by Distance theorem, we get $\boldsymbol{d}(g_n, \mathcal{P}_{\mathbf{L}^{\varphi}(\mu,\mathbf{G})}(f)) \to 0$ which contradicts the assumption above. ■

Theorem 3.2. *Let* $\mathbf{L}^{\varphi}(\mu, \mathbf{G})$ *is strongly proximinal in* $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$. *Then* **G** *is strongly proximinal in* **X**.

Proof. If $\mathbf{L}^{\varphi}(\mu, \mathbf{G})$ is strongly proximinal $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$ then by definition $\mathbf{L}^{\varphi}(\mu, \mathbf{G})$ is proximinal in $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$. Hence, G is proximinal in X. See [6].

Now, let $\varepsilon > 0$ and $\{ y_n \}$ be a minimizing sequence in **G** for $x \in \mathbf{X}$. i.*e*:

$$\| x - y_n \| \to \boldsymbol{d}(x, \mathrm{G}).$$

Define $f \in \mathbf{L}^{\varphi}(\mu, \mathbf{X})$ as $f(t) = \boldsymbol{\chi}_{\mathbf{T}}(t) \cdot x$ and $\{g_n\} \subseteq \mathbf{L}^{\varphi}(\mu, \mathbf{G})$ such that for each n, $g_n(t) = \boldsymbol{\chi}_{\mathbf{T}}(t)\, y_n$, where, $\boldsymbol{\chi}_{\mathbf{T}}(\cdot)$ is the characteristic function. Then

$$\| f - g_n \|_{\varphi} = \int_T \varphi \| f(t) - g_n(t) \| dt = \int_T \varphi \| x - y_n \| dt = \varphi \| x - y_n \| \mu(T) .$$

But since, $\{ y_n \}$ is a minimizing sequence in **G** for x and φ continuous then

$$\varphi \| x - y_n \| \to \varphi \; \boldsymbol{d}(x, G) .$$

Also,

$$\varphi \; \boldsymbol{d}(x, G) \,.\, \mu(T) = \int_T \varphi \; d\,(x, \mathbf{G})\, dt = \int_T \varphi \; d\,(f(\mathbf{t}), \mathbf{G})\, dt = \| d(f(\cdot), G) \|_{\varphi} .$$

Now, by the above argument and the Distance theorem, we have $\{g_n\} \subseteq \mathcal{P}_{\mathbf{L}^{\varphi}(\mu, \mathbf{G})}(f)$ satisfying

$$\| f - g_n \|_{\varphi} \to \boldsymbol{d}(f, \; \mathbf{L}^{\varphi}(\mu, \mathbf{G})).$$

This means that $\{g_n\}$ is a minimizing sequence in $\mathbf{L}^{\varphi}(\mu, \mathbf{G})$ for f , hence by the given that $\mathbf{L}^{\varphi}(\mu, \mathbf{G})$ is strongly proximinal in $\mathbf{L}^{\varphi}(\mu, \mathbf{X})$ then $\exists$ $\{g_{nk}\} = \{ \chi_{\mathbf{T}}(t) \,.\, y_{n\boldsymbol{k}} \}$ a subsequence of $\{g_n\}$, such that

$$\boldsymbol{d}(g_{nk} \, , \; \mathcal{P}_{\mathbf{L}^{\varphi}(\mu, \mathbf{G})}(f) \,) = \| \; d(g_{nk}(\cdot), \; \mathcal{P}_{\mathbf{G}}(f(.)) \,) \, \|_{\varphi} \;\; \to 0$$

But, $f(t) = \chi_{\mathbf{T}}(t) \,.\, x$ and $g_{nk}(t) = \chi_{\mathbf{T}}(t) \,.\, y_{n\boldsymbol{k}}$, where $\{ y_{n\boldsymbol{k}} \}$ is a subsequence of $\{ y_n \}$ in $\boldsymbol{G}$ satisfying

$$\| d(y_{nk}, \mathcal{P}_G(x)) \| \;\; \to 0.$$

Hence, $\boldsymbol{G}$ is strongly proximinal in **X**. ■

Theorem 3.3. *Let* **G** *be separable subspace of* **X.** *G is strongly proximinal in* **X** *if and only if* $\mathbf{L}^{p}(\mu, \mathbf{G})$ *is strongly proximinal in* $\mathbf{L}^{p}(\mu, \mathbf{X})$, *for* $0 < p < 1$.

Proof. Taking $\varphi(x) = x^p$ **,** $0 < p < 1$ and apply the above two theorems. ■

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