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Semi Regular Local Rings

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Abstract

A ring R is said to be right (left) semi regular local ring or (srl− ring) if for every a ∈ R, either a or (1 − a) is right (left) semi regular element. In this paper, we introduce some characterization and basic properties of this rings. Also, we studied the relation between semi regular local rings and Von Neumann regular local rings, Von Neumann regular rings, clean rings, exchange rings and suitable rings.

Keywords: Local rings, Von Neumann regular rings, Clean rings, Exchange rings and suitable rings.

1 Introduction

Throughout this paper, R denotes associative rings with identity; all modules are unitary. For a subset X of R, the right (left) annihilator of X is denoted by $r\ (X)$ and $(l\ (X))$. If $X = \{a\}$ we usually abbreviate it to $r\ (a)$, $(l(a))$. A right R− module M is called p−injective, if for any principal right ideal aR of R and any R−homomorphism of aR into M can be extended to one of an R into M. The ring R is called right p−injective if an R_R Is p−injective [8]. A ring R is said to be Von Neumann regular ring (or just regular) if and only if for each a in R, there exists b in R such that $a = aba$ [7]. Following [10] A ring R is said to be right (left) semi− regular ring, if and only if for each a in R, there exists b in R such that $a = ab\ (a = ba)$ and $r\ (a) = r\ (b)\ (l\ (a) = l\ (b))$ and b is called to associate element. A ring R is called local ring, if it has exactly one maximal ideal [4],[6]. A ring R is called Von Neumann regular local ring (VNL−ring) if for all $a \in R$, either at or $(1 - a)$ is a Von Neumann regular element [5], [2], [1]. A ring R is called clean ring, if each element a in R can be written as, a = e + u, where e is an idempotent element and u is unit element [3]. A ring R is said to be exchange ring, if for each element a in R, there exists idempotent element e in R such that, $e \in aR$ and $(1-e) \in (1-a)R$ [9]. Nicholson in [9], called a ring R a suitable ring, if for each element a in R, there exists idempotent element e in R such that $e - a \in R(a - a^2)$.

2 The Semi Regular Local Rings

In this section we give the definition of semi-regular local rings with some of its characterization and basic properties.

Definition 2.1: A ring R is said to be right (left) semi regular local ring or (srl−ring) if for every $a \in R$, either a or (1− a) is right (left) semi regular element. Now, we give the most important result of this section.

Proposition 2.2: If R is right semi regular local ring, then the associated elements are idempotent.

Proof: Let $a \in R$.Then either a or $(1- a)$ is semi regular element in R. If a is semi regular element, then there exists b in R such that $a = ab$ *and* $r(a) = r(b)$, so $a(1-b) = 0$, then $(1-\mathrm{b}) \in \mathrm{r(a)} = \mathrm{r(b)}$ and hence $b(1-b) = 0$. Thus $b = b^2$. Now, if $(1-a)$ is semi regular then there exists c in R such that $(1-\mathrm{a}) = (1-\mathrm{a})\mathrm{c}$ and $\mathrm{r}(1-\mathrm{a}) = \mathrm{r(c)}$.So $(1-a)(1-c) = 0$, then $(1-c) \in r(1-a) = r(c)$ and, hence $c(1-c) = 0$.Therefore $c = c^2$ ∎

Proposition 2.3: If R is a reduced right semi regular local ring, then the associated elements are unique.

Proof: Let $a \in R$,since R is right srl−ring .Then either a or $(1-a)$ is semi regular element in R .If a is semi regular ,then there exists an element $b \in R$ such that $a = ab$ and $r(a) = r(b)$. Now , assume that b is not unique associated element for a , then there exists an element $\bar{b}$ in R such that $a = a\bar{b}$ and $r(a) = r(\bar{b})$. Hence $ab = a\bar{b}$, then $a(b-\bar{b}) = 0$. So $(b-\bar{b}) \in r(a) = r(b) = r(\bar{b})$ .Then $b\left(b-\bar{b}\right) = 0$ and $\bar{b}(b-\bar{b}) = 0$ .Thus $b^2 = b\bar{b}$ and $\bar{b}b = \bar{b}^2$, then $b = b\bar{b}$ and $b\bar{b} = \bar{b}$. Since R is reduced, then $r(b) = l(b) = l(\bar{b})$ .Thus $(b-\bar{b}) \in l(b) = l(\bar{b})$, and hence $(b-\bar{b})\,b = 0$, then $b^2 = b\bar{b}$, and hence $b = b\bar{b}$.Therefore $b = \bar{b}b = b\bar{b} = \bar{b}$.Now, if $(1-a)$ is semi regular element, then there exists an element $c \in R$ such that $(1-a) = (1-a)c$ and $r(1-a) = r(c)$. Assume that c is not unique, then there exists $\bar{c}$ in R such that $(1-a) = (1-a)\,\bar{c}$ and $r(1-a) = r(\bar{c})$. Hence $(1-a)c = (1-a)\,\bar{c}$, then $(1-a)\,(c-\bar{c}) = 0$.Thus $(c-\bar{c}) \in r(1-a) = r(c) = r(\bar{c})$. Now $c(c-\bar{c}) = 0$, then $c^2 = c\bar{c}$ and $\bar{c}(c-\bar{c}) = 0$, thus $\bar{c}c = \bar{c}^2$ and hence $c = c\bar{c}$ and $\bar{c}c = \bar{c}$. Since R is reduced, then $r(c) = l(c) = l(\bar{c})$. Then $(c-\bar{c})c = 0$, that is $c^2 = c\bar{c}$ and $(c-\bar{c})\,\bar{c} = 0$, hence $c\bar{c} = \bar{c}^2$, so $c = c\bar{c}$ and $c\bar{c} = \bar{c}$. Therefore $c = c\bar{c} = c\bar{c} = \bar{c}$. ∎

Proposition 2.4: Any element a in a right semi regular local ring is regular if $Ra = Rb$ for any associated b of R.

Proof: Assume that $Ra = Rb$.Then $ra = b$ where $r \in R$ and $ra \in Ra$ and $b \in Rb$. Since b is idempotent element, then $Rb \oplus R(1-b) = R$ then $b + (1-b) = 1$ where $b \in Rb$ and $(1- b) \in R(1-b)$. Now $b + (1-b) = 1$, then $ra + (1-b) = 1$, and $ara + a(1-b) = a$, hence $ara = a$, since $(1-b) \in r(a)$ therefor a is regular element. Now, if $R(1-a) = Rb$, then assume that $s(1-a) =$ b, where $s(1-a) \in R(1-a)$ and $b \in Rb$. Now $Rb \oplus R(1- b) = R$, hence $b + (1-b) = 1$, then $s(1-a) + (1-b) = 1$, thus $(1-a)s(1-a) = (1-a)$. Therefore $(1-a)$ is regular element in R. ■

Proposition 2.5: The epimorphism image of right semi regular local ring is right semi regular local ring.

Proof: Let $f : R \to \bar{R}$ be epimorphism function such that R is right (srl− ring) and let $\bar{1}$, y, $\bar{e}$ in $\bar{R}$. Then there exists 1, x, e in R such that $f(x) = y$ and $f(e) = \bar{e}$. Now since R is right (srl− ring), then either x or $1- x$ is semi regular. If $x = xe$ and $r(x) = r(e)$. Hence $y = f(x) = f(xe) = f(x)f(e) = y\,\bar{e}$ then $y = y\,\bar{e}$. Now, to prove that $r(y) = r(\bar{e})$. Let $a \in r(y)$ then $ya = 0$ hence $f(x)a = 0$, then $x\,f^{-1}(a) = 0$. Hence $f^{-1}(a) \in r(x) = r(e)$, thus $ef^{-1}(a) = 0$, then $f(e)a = 0$, thus $\bar{e}a = 0$, then $a \in r(\bar{e})$. Therefore $r(y) \subseteq r(\bar{e})$ ………..(1). Let $b \in r(\bar{e})$, then $\bar{e}b = 0$. Hence $f(e)b = 0$, thus $ef^{-1}(b) = 0$, then $f^{-1}(b) \in r(e) = r(x). So\; xf^{-1}(b) = 0$, hence $f(x)b = 0$. Thus $yb = 0$, then $b \in r(y)$. Therefore $r(\bar{e}) \subseteq r(y)$…… (2). Now, from (1) and (2), we have $r(\bar{e}) = r(y)$. Now, if $1-x$ is semi regular then $(1-x) = (1-x)e$ and $r(1-x) = r(e)$, then $(\bar{1}- y) = \bar{1} + (-y) = f(1) + f(-x) = f(1-x) = f\big((1-x)e\big) = f(1-x)f(e) = (\bar{1}-y)\bar{e}$. Hence $(\bar{1}- y) = (\bar{1}-y)\,\bar{e}$. Now to prove that $r(\bar{1}- y) = r(\bar{e})$. Let $c \in r(\bar{1}- y)$. Then $(\bar{1}- y)\,c = 0$, hence $f(1-x)c = 0$ and $(1-x)\,f^{-1}(c) = 0$, thus $f^{-1}(c) \in r(1-x) = r(e)$, then $ef^{-1}(c) = 0$and hence $f(e)\,c = 0$. Thus $\bar{e}c = 0$ then $c \in r(\bar{e})$, hence $r(\bar{1}- y) \subseteq r(\bar{e}) \dots\dots\dots.(3)$. Let $d \in r(\bar{e})$. Then $\bar{e}d = 0,$ hence $f(e)\,d = 0$. Thus $ef^{-1}(d) = 0$ then $f^{-1}(d) \in r(e) = r(1-x)$ and hence $(1-x)\,f^{-1}1(d) = 0$. Thus $f(1-x)\,d = 0$, then $(\bar{1}- y)\,d = 0$ that is $d \in r(\bar{1}- y)$. Therefore $r(\bar{e}) \subseteq r(\bar{1}- y)$ ………… (4). Now from (3) and (4) we get $r(\bar{e}) = r(\bar{1}- y)$. Hence either y or $\bar{1}-y$ is semi regular element in $\bar{R}$ .Therefore $\bar{R}$ is semi regular local ring. ■

Proposition 2.6: Let R be a ring, then R is right semi regular local ring if and only if either $r(a)$ or $r(1-a)$ is direct summand for all a in R.

Proof: Assume that $a \in R$ and $r(a)$ is direct summand .Then there exists $I \subset R$ such that $R = r(a) \oplus I$, then there are elements $b \in I$ and $d \in r(a)$, such that $d + b = 1$, and hence $ad + ab = a$, then
$a = ab$. Now, to prove that $r(a) = r(b)$. Let $x \in r(a)$ then $ax = 0$, hence

$abx = 0$ and $bx \in r(a)$. But $bx \in I$ and $r(a) \cap I = 0$, then $bx = 0$ and $x \in r(b)$. Therefore $r(a) \subseteq r(b)$...(1). Now, let $y \in r(b)$ then $by = 0$, hence $aby = 0$, thus $ay = 0$, hence y ∈ r(a). Therefore $r(b) \subseteq r(a)$ …. (2). Now from (1) and (2) we have $r(a) = r(b)$, then a is semi regular element. Now ,assume that $(1 - a) \in R$ and $r(1 - a)$ is direct summand .Then there exists $J \subset R$ such that $R = r(1 - a) \oplus J$ .Now let $c \in J$ and $f \in r(1 - a)$ such that $1 = f + c$, then $(1 - a) = (1 - a)f + (1 - a)$c, and hence $(1 - a) = (1 - a)c$. Now to prove that $r(1 - a) = r(c)$. Now let $w \in r(1 - a)$, then $(1 - a)w = 0$, hence $(1 - a)cw = 0$. Thus $cw \in r(1 - a)$. But $cw \in J$ and $J \cap r(1 - a) = 0$ then $cw = 0$, hence $w \in r(c)$. Therefore $r(1 - a) \subseteq r(c)$……… (3). Now, let $z \in r(c)$, then $cz = 0$ and hence $(1 - a)cz = 0$. Thus $(1 - a)z = 0$, then $z \in r(1 - a)$, and therefore $r(c) \subseteq r(1 - a)$ …… (4). Now from (3) and (4), we have $r(c) = r(1 - a)$ that is 1-a is semi regular element. Therefore, R is right semi regular local ring. Conversely, assume that R is right semi regular local ring. Then either a or $(1 - a)$ is semi regular element. If a is semi regular element, then there exists b in R such that a = ab and r(a) = r(b). Since $a(1 - b) = 0$, then $(1 - b) \in r(a), then$ $1 = b + (1 - b)$, hence $R = bR + (1 - b)R$, thus $R = bR + r(a)$. To prove $bR \cap r(a) = 0$. Now let $x \in bR \cap r(a)$, then $x \in bR$ and $x \in r(a)$, hence $x = br$ for some $r \in R$ and $ax = 0$. Since, $x \in r(a) = r(b)$, then $bx = 0$ and hence $b.br = 0$, hence $br = 0$, thus $x = 0$. Therefore $bR \cap r(a) = 0$ .Hence $r(a)$ is direct summand .Now ,if $1 - a$ is semi regular if $(1 - a)$ is semi regular then there exists $0 \neq c \in R$ such that $(1 - a) = (1 - a)c$ and $r(1 - a) = r(c)$ since $(1 - a)(1 - c) = 0$ and $(1 - c) \in r(1 - a)$, then $1 = c + (1 - c)$ hence $R = cR + (1 - c)R$ therefore $R = cR + r(1 - a)$. To prove $r(1 - a) \cap cR = 0$. Now let $y \in r(1 - a) \cap cR$ then $y \in r(1 - a)$ and $y \in cR$, hence $(1 - a)y = 0$ and $y = cr$ for some $r \in R$. Since $y \in r(1 - a) = r(c)$, then $cy = 0$ and $c.cr = 0$, hence $cr = 0$, and thus $y = cry = 0$, then $r(1 - a) \cap cR = (0)$ .Therefore $r(1 - a)$ is direct summand. ■

Proposition 2.7: Let R be an abelian and right semi regular local ring. Then any element x in R can be written as $x = s + a$ or $1 - x = s + a$,where s is semi regular element and a is nilpotent element.

Proof: Assume that $x \in R$ is semi regular element, then there exists $e \in R$ such that $x = xe$ and $r(x) = r(e)$. Now, $x = xe + (x - xe)$ .We will prove that xe is semi regular element and $(x - xe)$ is nilpotent element. Now $xe = xe.e$, to prove $r(xe) = r(e)$. Now let $b \in r(xe)$, then $xeb = 0$ and $eb \in r(x) = r(e)$, hence $e.eb = 0$ and $eb = 0$, then $b \in r(e)$, and hence $r(xe) \subseteq r(e)$…….(1). Now let $c \in r(e)$, then $ec = 0$. Hence $xec = 0$, thus $c \in r(xe)$, therefore $r(e) \subseteq r(xe)$……… (2). Now from (1) and (2) we have $r(xe) = r(e)$. Then xe is semi regular element. Now, $(x - xe)^n = (x(1 - e))^n = x^n (1 - e)n^n = 0$. Then $(x - xe)$ is nilpotent element. Now if $1 - x$ is semi regular element, then there exists $c \in R$ such that $(1 - x) = (1 - x)c$ and $(1 - x) = r(c)$. Then , $1 - x = [(1 - x)c + (1 - x) - (1 - x)c]$ Now we will prove that $(1 - x)c$ is semi regular element and $(1-x)$– $(1-\ x)c$ is nilpotent element. Then $(1-\ x)c=(1-x)c.c$ and to prove $r((1-x)c)=r(c)$.Now let $a\in r((1-x)c)$, then $(1-x)ca=0$, hence $ca\in r(1-x)=r(c)$ thus $cca=0$. Therefore $ca=0$, then $a\in r(c)$ and hence $r((1-x)c)\subseteq r(c)$...(3). Now let $d\in r(c)$, then $cd=0$, and hence $(1-x)cd=0$ thus $d\in r((1-x)c)$. Therefore $r(c)\subseteq r((1-x)c)$…(4). Now from (3) and (4) we have $r(c)=\ r((1-x)c)$, hence $(1-x)c$ is semi regular element. Now, $[(1-x)-(1-\ x)c]^n=[(1-x)(1-c)]^n=[(1-x)^n(1-c)^n]=0$.Ttherefore $[(1-x)-(1-\ x)c]$ is nilpotent element. ■

3 The relation between right semi regular local ring and another ring

In this section we give the relation between right semi regular local ring and VNL−ring, local ring, Von Neumann ring, clean ring, exchange ring and suitable ring.

Proposition 3.1: Let R be a local ring. Then R is right semi regular local ring.

Proof: Since R is local ring then by proposition (10.1.3) in [5] either a or $1-a$ is invertible element. If a is invertible, then there exists b in R such that $ab=1$ and hence $aba=a$. Let $e=ba$. Then $ae=a$. To prove that $r(a)=r(e)$ let $x\in r(a)$, hence $ax=0$, and $bax=0$ thus $ex=0$ hence $x\in r(e)$ therefore $r(a)\subseteq r(e)$……(1). Now, let $y\in r(e)$ then $ey=0$ hence $bay=0$ thus $abay=0$, then $ay=0$, hence $y\in r(a)$, therefore $r(e)\subseteq r(a)$…(2). From (1) and (2) we have $r(a)=r(e)$. Therefore a is right semi regular in R .Now ,if $1-a$ is invertible element then there exists c in R such that $(1-a)c=1$, then $(1-a)c(1-a)=(1-a)$ .Let $d=c(1-a)$. Then $(1-a)d=(1-a)$, now to prove that $r(1-a)=r(d)$. Let $v\in r(1-a)$. Then $(1-a)v=0$, hence $c(1-a)v=0$ and $dv=0$, then $v\in r(d)$ and therefore $r(1-a)\subseteq r(d)$….(3). Now, let $w\in r(d)$ then $dw=0$ hence $c(1-a)w=0$ thus $(1-a)c(1-a)\,w=0$ then $(1-a)w=0$. Then $w\in r(1-a)$ therefore $r(d)\subseteq r(1-a)$…. (4). From (3) and (4) we have $r(d)\ =\ r(1-a)$. Then $1-a$ is right semi regular element, and therefore R is right semi regular local ring. ■

Corollary 3.2: Every Von Neumann regular ring is right semi regular local ring.

Proposition 3.3: Let R be VNL− ring. Then R is right semi regular local ring.

Proof: By the same proof of proposition (3.1)

It's clearly that, the converse of proposition (3.3) is not true.

Now we give a necessary condition to get the converse of above proposition.

Proposition 3.4: If R is right semi regular local ring and R_R is P−injective, then R is VNL−ring.

Proof: Let $a \in R$, since R is right semi regular local ring. Then either a or $1-a$ is a semi regular element. If a is semi regular element, then there exists b in R such that $a = ab$ and $r(a) = r(b)$. To prove a is regular we need to prove that $Ra = l(r(a))$ for any a in R. Let $x \in l(r(a))$. Now define $f: aR \longrightarrow R$ by $f(ar) = xr$ for all $r \in R$. We shall prove first that f is well define. If $ar_1 = ar_2$, then $a(r1 - r2) = 0$ and, hence $(r_1 - r_2) \in r(a)$ but $r(a) = r(l(r(a)) \subseteq r(x)$. Thus $(r_1 - r_2) \in r(x)$, so $xr_1 = xr_2$ .Therefore f is well define. Now, since R is P−injective then there exists c in R such that $f(ar) = car$ and $x = f(a) = ca \in Ra$, this prove that $l(r(a)) \subseteq Ra$. The converse inclusion is easy, so $Ra = l(r(a)) = l(r(b)) = Rb$. Then $Ra = Rb$, and by proposition (2.4) we have a is regular element. Now, If $1 - a$ is semi regular element then there exists d in R such that $(1 - a) = (1 - a)d$ and $r(1-a) = r(d)$. To prove $(1-a)$ is regular we need to prove that $R(1-a) = l(r(1-a))$ for any $(1-a)$ in R. Let $y \in l(r(1-a))$. Now, define $f: (1-a)R \longrightarrow R$ by $f(1-a)r) = yr$ for all $r \in R$. We shall prove first f is well defined. If $(1-a)r_1 = (1-a)r_2$, then $(1-a)(r_1 - r_2) = 0$ and, hence $(r1 - r2) \in r(1-a)$. But $r(1-a) = r(l(r(1-a)) \subseteq r(y)$. Thus $(r_1 - r_2) \in r(y)$ and hence $yr_1 = yr_2$. Therefore f is well define. Now, since R is P−injective then there exists k in R such that $f\big((1-a)r\big) = k(1-a)r$, then $y = f(1-a) = k(1-a) \in R(1-a)$. This prove that $l(r(1-a)) \subseteq R(1-a)$,the converse inclusion is easy, so $R(1-a) = l(r(1-a)) = l(r(d)) = Rd$. Then $R(1-a) = Rd$, then by proposition (2.4) we have $1-a$ is regular element. Therefore, R is VNL−ring ■

Proposition 3.5: Let R be a VNL−ring. Then for each proper ideal I of R, the quotient ring R/I is right semi regular local ring.

Proof: Let $a + I \in R/I$ where $a \in R$. Since R is VNL−ring, then either a or $(1-a)$ is regular element. If a is regular element, then there exists b in R such that $a = aba$. Let $e = ba$ then $ae = a$. That is $a + I = ae + I = (a+I)(e+I)$. Now, to prove $r(a+I) = r(e+I)$. Let $x + I \in r(a+I)$, then $(a+I)(x+I) = I$. Thus $ax + I = I$ and hence $(b+I)(ax+I) = I$ and then $bax + I = I$ hence $ex + I = I$, that is $(e+I)(x+I) = I$ thus $x + I \in r(e+I)$. Therefore $r(a+I) \subseteq r(e+I)\ldots$ (1). Now, let $y + I \in r(e+I)$, then $(e+I)(y+I) = I$ that is $ey + I = I$. Thus $(a+I)(ey+I) = I$, hence $aey + I = I$. Then $abay + I = I$, hence $ay + I = I$ and thus $(a+I)(y+I) = I$. Hence $y + I \in r(a+I)$. Therefore $r(e+I) \subseteq r(a+I)$... (2). From (1) and (2) we have $r(e+I) = r(a+I)$. Then $a + I$ is right semi regular element in R. Now, if $1-a$ is regular element then there exists c in R such that $(1-a)c\,(1-a) = (1-a)$. Let $d = c(1-a)$, then $(1-a)d = (1-a)$. To prove that $r((1-a)+I) = r(d+I)$, let $z + I \in r((1-a)+I)$, $z \in R$ then $((1-a)+I)(z+I) = I$ hence $(1-a)z + I = I$, thus $(c+I)((1-a)z+I) = I$, therefore $c(1-a)z + I = I$. Then $dz + I = I$, hence $(d+I)(z+I) = I$ thus $z + I \in r(d+I)$ then $r((1-a)+I) \subseteq r(d+I)\ldots$(3). Now, let $w + I \in r(d+I)$, then $(d+I)(w+I) = I$ hence $dw + I = I$. Thus $((1-a)+I)(dw+I) = I$, then $(1-a)dw + I = I$, hence $(1-a)c(1-a)w + I = I$, then $(1-a)w + I = I$ hence

$((1-a)+I)(w+I)=I$ and $w+I\in r((1-a)+I)$, thus $r(d+I(\subseteq r((1-a)+I)$….(4). From (3) and (4) we have $r(d+I)=r((1-a)+I)$. Then $(1-a)+I$ is right semi regular element in R. Therefore R/I is right semi regular local ring. ■

Finally, we closed this section by the following theorem.

Theorem 3.6: Let R be a commutative semi regular local ring and R is P−injective. Then:

1)R is clean ring.

2)R is exchange ring.

3) R is suitable ring.

Proof: 1) By proposition (3.4) we have that R is a VNL−ring, and by proposition (5.1) (2) in [8], either $a=ue$ or $a=1+ue$ where $u\in U(R)$ and $e\in Idem(R)$. Now, if $a=ue$ by Theorem (2.2) in [8]. Thus $a=(ue+e-1)+(1-e)$ with $(ue+e-1(\in U(R)$ since $(ue+e-1)(u^{-1}e+e-1)=1$ and $(1-e)\in Idem(R)$, so $a\in cln(R)$. Now, if $a=1+ue=(ue+1-e)+e$ with $ue+1-e\in U(R)$ since $(ue+1-e)(u^{-1}e+1-e)=1$ and $e\in Idem(R)$, then $a\in cln(R)$. Therefore R is clean ring.

2) By proposition (3.4) we have that R is a VNL−ring. And by proposition (5.1) (2) in [1], either $a=ue$ or $a=1+ue$ where $u\in U(R)$ and $e\in Idem(R)$.
If $a=ue$ and $f=e$ where $f^2=f$, then
$(a-f)u = au-fu=(ue)u-eu=u^2e^2-eu=(ue)^2-eu=a^2-a$(since $e^2=e$ and $ue=eu$)
That is $(a-f)u=a^2-a, then\ a-f=(a^2-a)u^{-1}, a-f=a^2u^{-1}-au^{-1}$.
Hence $f = a+au^{-1}-a^2u^{-1}$, thus $f = a(1+u^{-1}-au^{-1})$, therefore $f=at$, hence $f=aR$
where $t = (a+au^{-1}-a^2u^{-1})\in R$. Therefore $e\in aR$.
Now, $1-f=1-(a+au^{-1}-a^2u^{-1})$, then $1-f=1-a-au^{-1}+a^2u^{-1}$
$1-f=(1-a)(1-au^{-1})$, hence $1-f=(1-a)L$,
where $L=(1-au^{-1})\in R$.
Therefore $1-f\in(1-a)R$.
Now, if $a=1+ue$ and $f = 1-e$, then $(a-f)u=au-fu$
$(a-f)u=(1+ue)u-(1-e)u= u+u^2e-u+eu$
$(a-f)u=1+eu+ue+u^2e^2-1-ue$ (by addition $1,-1,ue,-ue$)
$= (1+ue)^2-(1+ue)=a^2-a$
$(a-f)u=a^2-a$, then $a-f=(a^2-a)u^{-1}$, and hence
$f=a+au^{-1}-a^2u^{-1}$

$f = a(1 + u^{-1} - au^{-1})$, therefore $f = ab$, where $b = (a + au^{-1} - a^2u^{-1}) \in R$. Therefore $f \in aR$.
Now, $1 - f = 1 - (a + au^{-1} - a^2u^{-1})$, then $1 - f = 1 - a - au^{-1} + a^2u^{-1}$
$1 - f = (1 - a)(1 - au^{-1})$, hence $1 - f = (1 - a)k$, where $k = (1 - au^{-1}) \in R$.
Therefore $1 - f \in (1 - a)R$. Hence R is exchange ring.

3) By proposition (3.4) we have that R is a VNL−ring, and by proposition (5.1) (2) in [8], either $a = ue$ or $a = 1 + u$e, where $u \in U(R)$ and $e \in Idem(R)$.
If $a = ue$ and $f = e$, where $f^2 = f$, then $u(a - f) = ua - uf$
$= u(ue) - ue = u^2e^2 - ue$, since $(e = e^2)$
Now, $u(a - f) = (ue)^2 - ue$,then $u(a - f) = a^2 - a$ and $a - f = u^{-1}(a^2 - a)$, hence
$f - a = u^{-1}(a - a^2)$, thus $f - a = R(a - a^2)$, where $f - a \in R(a - a^2)$.
Now, if $a = 1 + ue$ and $f = 1 - e$, where $f^2 = f$.
Then $u(a - f) = ua - uf = u(1 + ue) - u(1 - e)$
$u(a - f) = u + u^2e - u + ue$.
By addition $1, -1, ue, -ue$ and since $(e = e^2)$
$u(a - f) = 1 + ue + ue + u^2e^2 - 1 - ue$, then $u(a - f) = (1 + ue)^2 - (1 + ue)$. $u(a - f) = a^2 - a$, hence $a - f = u^{-1}(a^2 - a)$ and thus$f - a = u^{-1}(a - a^2)$, hence
$f - a = R(a - a^2)$, where $f - a \in R(a - a^2)$. Therefore, R is suitable ring. ■

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