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Numerical Approximations of 2-Point Adams Predictor-Corrector Block Method for Neutral Delay Using Newton's Divided Difference Formulation

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Abstract

In this paper, the new 2-point Adams predictor-corrector block method will be developed by using Newton's divided difference formulation. The two direct approximate solutions $y(x_{n+1})$ and $y(x_{n+2})$ will be calculated at each integration concurrently. The approximation of neutral delay $y'(x-\tau)$ is also carried out by using the central divided difference technique with the sufficient interpolation points. The formulation of the variable step size strategy is varying to constant, half and double to optimize the performance of the algorithm. Numerical results are presented to validate the accuracy and efficiency of the block method.

Keywords: Variable step size, Newton's divided differences Interpolation, Numerical approximations, Neutral delay differential equations

1. Introduction

In this paper, the approximate solution of the first order neutral delay differential equation (NDDE) will be considered and given as follows

$$\begin{aligned} &y'(x) = f(x, y(x), y(x-\tau), y'(x-\tau)), \qquad && a \le x \le b \\ &y(x) = \phi(x), && x \le a \end{aligned} \tag{1}$$

where $\tau > 0$ is a time delay and $\phi(x)$ is an initial function. The time delays can be a constant, time-dependent, or state-dependent. A substantial improvement in computational implementation and numerical analysis has been done during the last decades for various NDDEs. Numerical methods have been proposed for the approximation of NDDE because analytical solution can be computed only in a special cases of Eq. (1). Banks and Kappel [4] developed spline-based approximation methods for functional differential equations (FDEs) of linear systems. Kappel and Kunisch [8] presented a general approximation technique for linear autonomous neutral functional differential equation (NFDDE). Qin et al. [11] developed and extended the study on continuous Galerkin finite element methods (CGFEMs) for the numerical solution of NDDE.

Jackiewicz [6] examined the Adams method order two (AM2), order three (AM3), and order four (AM4) for the special case of NFDEs. Jackiewicz and Lo [7] implemented a variable step size and variable order algorithm based on the formulation of Adams method variable step for the numerical solution of NFDEs. Aziz and Majid [3] developed a technique to solve discontinuity in functional differential equations using 1-point implicit method. The methods discussed above can provide one numerical solution at each integration step corresponding to a value of the given interval.

The block methods based on predictor-corrector technique are very effective numerical methods in terms of function evaluations than the methods discussed above. Block methods calculate more than one numerical solutions at single integration step in [2] and [12]. Majid et al. [9] used direct two-point block method to solve the general second-order of ordinary differential equations (ODEs). In their study they used Lagrange interpolation polynomial for the approximation of $y(x_{n+i})$, $i = 1, 2$. Ishak and Ramli [5] studied the first order NDDEs system based on variable step size fixed order three-point one block method. Their study strategy to the limit of integration for the second and third points is x_n to x_{n+2} and x_n to x_{n+3}. Aziz et al. [1] proposed the multistep block method for solving the delay differential equations (DDEs) of small and vanishing lag. In their study, the two numerical solutions y_{n+1} and y_{n+2} obtained using the limit of integration on the interval x_n to x_{n+1} and x_{n+1} to x_{n+2}.

In this paper, a new C programming code is developed based on the variable step size strategy of Adams predictor-corrector methods. Our strategy for the limit of integration of the first and second points is x_n to x_{n+1} and x_{n+1} to x_{n+2} is same as in [1] but the formulation is using Newton divided difference interpolation technique.

2. Formulation of the Method

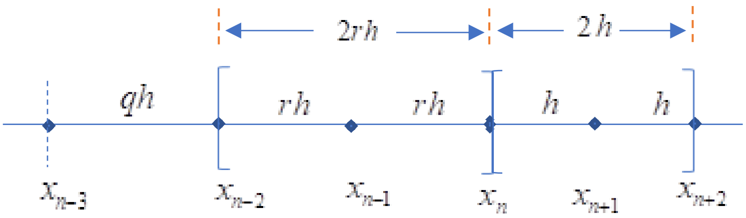

Figure 1: 2-point Adams predictor-corrector block method

In 2-point Adams predictor-corrector block method (2ABM5), the interval $[a,b]$ is divided into subintervals as shown in the Figure 1. Two numerical values $y(x_{n+i})$, $i=1,2$ can be obtained by using 2ABM5 simultaneously. For the sake of simplicity, consider the general first-order ordinary differential equation (ODE).

$$y' = f(x, y(x)), \qquad [a,b], \tag{2}$$

with initial conditions $y(x_0)=y_0$. Formulas of the proposed method were derived using Newton' divided difference interpolation formula. The interpolation points involved for the predictor are (x_{n-2+k}, f_{n-2+k}), $k=0,...,4$. The two values $y(x_{n+i})$ $i=1,2$ can be obtained by integrating Eq. (2) on the interval $\left[x_{n+j}, x_{n+j+1}\right]$, where $j=0,1$ respectively, gives

$$y(x_{n+j+1}) = y(x_{n+j}) + \int_{x_{n+j}}^{x_{n+j+1}} f(x, y(x))dx, \quad j=0,1. \tag{3}$$

Consider a polynomial $P_4(x) = f(x, y(x))$ in Eq. (3) then, the integral becomes

$$y(x_{n+j+1}) = y(x_{n+j}) + \int_{x_n+j}^{x_{n+j+1}} P_4(x)dx, \quad j=0,1. \tag{4}$$

The Newton's divided differences interpolation polynomial for the corrector formula is

$$P_4(x) = \sum_{k=0}^{4} f(x_{n+k-2,k}) \prod_{i=0}^{k-1} (x - x_{n+i-2}), \tag{5}$$

with

$$\begin{aligned} P_4(x) = {} & f(x_{n-2}) + f(x_{n-2}, x_{n-1})(x-x_{n-2}) + f(x_{n-2}, x_{n-1}, x_n)(x-x_{n-2})(x-x_{n-1}) \\ & + f(x_{n-2}, x_{n-1}, x_n, x_{n+1})(x-x_{n-2})(x-x_{n-1})(x-x_n) \\ & + f(x_{n-2}, x_{n-1}, x_n, x_{n+1}, x_{n+2})(x-x_{n-2})(x-x_{n-1})(x-x_n)(x-x_{n+1}), \end{aligned} \tag{6}$$

where $f(x_{n-2}, x_{n-1}, x_n, x_{n+1}, x_{n+2}) = \dfrac{f(x_{n-1}, x_n, x_{n+1}, x_{n+2}) - f(x_{n-2}, x_{n-1}, x_n, x_{n+1})}{x_{n+2} - x_{n-2}}$.

By computing the divided differences of Eq. (6) and substituting $s=\frac{x-x_{n+2}}{h}$ in Eq. (4), then the general corrector formula at the step ratios $r=\frac{1}{2}$, $r=2$ and $r=1$ is,

$$y(x_{n+j+1})=y(x_{n+j})+h\sum_{j=0}^{4}\beta_j f_{n-2+j}, \quad j=0,1. \tag{7}$$

At $r=\frac{1}{2}$,

$$\begin{aligned} y(x_{n+1})&=y(x_n)+\frac{h}{1800}(145f_{n-2}-704f_{n-1}+1635f_n+755f_{n+1}-31f_{n+2})\\ y(x_{n+2})&=y(x_{n+1})+\frac{h}{1800}(-305f_{n-2}+1216f_{n-1}-1515f_n+1805f_{n+1}+599f_{n+2}) \end{aligned} \tag{8}$$

At $r=2$,

$$\begin{aligned} y(x_{n+1})&=y(x_n)+\frac{h}{14400}(37f_{n-2}-335f_{n-1}+7455f_n+7808f_{n+1}-565f_{n+2})\\ y(x_{n+2})&=y(x_{n+1})+\frac{h}{14400}(-53f_{n-2}+415f_{n-1}-2895f_n+11648f_{n+1}+5285f_{n+2}) \end{aligned} \tag{9}$$

At $r=1$,

$$\begin{aligned} y(x_{n+1})&=y(x_n)+\frac{h}{720}(11f_{n-2}-74f_{n-1}+456f_n+346f_{n+1}-19f_{n+2})\\ y(x_{n+2})&=y(x_{n+1})+\frac{h}{720}(-19f_{n-2}+106f_{n-1}-264f_n+646f_{n+1}+251f_{n+2}) \end{aligned} \tag{10}$$

Similarly, the one order less predictor formulas were derived using the interpolation points (x_{n-3+k}, f_{n-3+k}), $k=0,\dots,3$.

3. Implementation of the Method

This study focuses on the numerical approximation of NDDEs using 2ABM5. The order of the corrector formula is considered as the order of the proposed method. To calculate two numerical solutions of $y(x_{n+i})$, $i=1,2$, four prior values obtained using Euler method, namely (x_{n-3+k}, f_{n-3+k}) with $k=0,\dots,3$, will be used. The numerical values $y(x_{n+i})$, $i=1,2$, will be calculated using the predictor and corrector formula. Furthermore, the iteration of the corrector can be repeated several times to produce the desired level of accuracy for $y(x_{n+i})$, $i=1,2$.

The approximation of retarded delay $y(x-\tau)$ is computed directly using the Newton divided difference interpolation of *y* values. Meanwhile, the neutral delay $y'(x-\tau)$ is computed using Newton divided difference interpolation of *f* values. Both terms are very important as the requirements for computing $f(x,y(x),y(x-\tau),y'(x-\tau))$ of Eq. (1).

In the implementation of Newton divided differences interpolation, if $x-\tau \le a$ then the initial value function $y(x) \le \phi(x)$ will be used. If the delay falls on the first block, the points for interpolation are x_0, x_1, x_2, x_3, x_4 and x_5. If $x-\tau$ falls in the intermediate block then the points are $x_{k-2}, x_{k-1}, x_k, x_{k+1}, x_{k+2}$ and x_{k+3}. If $x-\tau$ falls at the end of the block then the points are $x_{s-5}, x_{s-4}, x_{s-3}, x_{s-2}, x_{s-1}$ and x_s will be selected. In this study the $PIE(CIE)^s$ mode proposed by Aziz et al. [1] will be applied, where s is the run times, **P** stand for predictor, **I** stand for interpolation, **E** stand for evaluation and **C** stand for corrector.

The step size strategy for the numerical solution of NDDEs with 2ABM5 is adopted from [10]. The criteria for convergence and step size strategy are set based on local truncation error $LTE = \left| y_{n+1}^{(k+1)} - y_{n+1}^{(k)} \right|$. If $LTE \le TOL$ then the step is accepted and the new step size will be doubled as $h_{new} = 2h_{old}$. If the $LTE > TOL$ then the step size is rejected and the current step size is half up to $h_{new} = \frac{1}{2} h_{old}$. In this case, the computation restart until the end of the interval. The new maximum step size computed from the formula

$$h_{new} = 0.5\left(\frac{TOL}{2.0LTE}\right)^{\left(\frac{1.0}{4.0}\right)} h_{old}, \tag{11}$$

where 0.5 is a safety factor used to avoid too many failure steps.

4. Numerical Problems

To validate the proposed method accuracy and efficiency, the following numerical problems are presented. The notations are stated as below:

TOL:	The prescribed tolerance.	**MTD:**	Method applied.
TS:	The number of total steps.	**MAXERR:**	The maximum error.
FS:	The number of failure steps.	**AVERR:**	The average error.
FNC:	The number of function calls.	**TIME:**	Computation time (μs).

Problem 1.

$$y'(x) = 1 + y(x) - 2y^2(x/2) - y'(x-\pi), \qquad x \in [0,\pi],$$
$$y(x) = \cos(x), \qquad x \in [-\pi, 0].$$

The exact solution

$$y(x) = \cos(x), \qquad x \in [-\pi, \pi].$$

Problem 2.

$$y'(x) = \exp(1-2x^2)\, y(x^2)(y'(x-1/(1+x)))^{1+x}, \qquad x \in [0,1],$$

$$y(x) = \exp(x), \qquad x \in [-1,0].$$

The exact solution is

$$y(x) = \exp(x), \qquad x \in [-1,1].$$

Problem 3

$$y'(x) = y(x) + y(x-1) - \frac{1}{4} y'(x-1), \qquad x \in [0,1],$$

$$y(x) = -x, \qquad x \le 0.$$

Exact Solution:

$$y(x) = -\frac{1}{4} + x + \frac{1}{4} e^x, \qquad x \in [0,1].$$

Table 1: Numerical Results for Problem 1

TOL	MTD	TS	FS	FNC	MAXERR	AVERR	TIME
2^{-2}	2ABM5	7	0	27	3.50E(-03)	1.08E(-03)	1.0E(+03)
2^{-4}	2ABM5	12	0	44	2.60E(-05)	8.01E(-06)	2.0E(+03)
2^{-6}	2ABM5	25	0	86	2.40E(-07)	8.06E(-08)	4.0E(+03)
2^{-8}	2ABM5	53	0	169	2.27E(-09)	7.48E(-10)	8.0E(+03)
2^{-10}	2ABM5	109	1	335	7.11E(-10)	1.67E(-10)	2.3E(+04)

Table 2: Numerical Results for Problem 2

TOL	MTD	TS	FS	FNC	MAXERR	AVERR	TIME
2^{-2}	2ABM5	4	0	17	1.54E(-03)	1.05E(-03)	3.0E(+03)
2^{-4}	2ABM5	8	0	26	2.04E(-05)	1.16E(-05)	4.0E(+03)
2^{-6}	2ABM5	13	0	40	2.03E(-07)	1.13E(-07)	6.0E(+03)
2^{-8}	2ABM5	23	0	68	1.98E(-09)	1.12E(-09)	1.2E(+04)
2^{-10}	2ABM5	46	0	135	2.02E(-11)	1.17E(-11)	1.6E(+04)

Table 3: Numerical Results for Problem 3

TOL	MTD	TS	FS	FNC	MAXERR	AVERR	TIME
2^{-2}	2ABM5	4	0	17	4.63E(-04)	3.77E(-04)	2.0E(+03)
2^{-4}	2ABM5	8	0	26	4.81E(-06)	3.87E(-06)	4.0E(+03)
2^{-6}	2ABM5	14	0	42	5.13E(-08)	4.01E(-08)	6.0E(+03)
2^{-8}	2ABM5	25	0	73	4.52E(-10)	3.89E(-10)	1.8E(+04)
2^{-10}	2ABM5	50	0	147	4.03E(-12)	3.74E(-12)	2.0E(+04)

5. Discussion and Conclusion

In this study, a 2-point Adams block method has been developed for solving the neutral delay differential equations with constant delays. The numerical approximation of the two terms $y(x-\tau)$ and $y'(x-\tau)$ of Eq. (1) were obtained by the method of divided difference. Numerical results of Tables 1-3 show that the proposed method is suitable to solve NDDEs based on the MAXERR for the prescribed tolerance and the function evaluations. Iterations of the algorithm have been carried out using the variable step size technique.

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