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Cesàro Sequence and Exponential Partial Bell Polynomials

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Abstract

In this work we introduce and investigate the inverse Cesàro sequence. We compute the explicit formula with hint of exponential partial Bell polynomials. We apply the result to Bernoulli numbers, zeta function at even positive integers and Euler numbers to evaluate sums of products of these numbers as K. Dilcher [3].

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1 Introduction

The sequence c_n of complex numbers is a Cesàro sequence (see [7]) if it satisfies the recursion formula

$$c_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k c_k \tag{1}$$

It is easy to verify that $(-1)^n B_n$ and $\frac{(-1)^n}{n+1} G_{n+1}$ are Cesàro sequences, where B_n and G_n are Bernoulli numbers and Genocchi numbers defined respectively in means of exponential generating functions $\frac{t}{e^t-1}$ and $\frac{2t}{e^t+1}$. The identity (1) can be rewritten in the form

$$\left(1-(-1)^n\right) c_n = \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k c_k. \tag{2}$$

If n even we obtain

$$\sum_{k=0}^{n-1}\binom{n}{k}(-1)^k c_k = 0 \tag{3}$$

and if n odd we will have

$$c_n = \frac{1}{2}\sum_{k=0}^{n-1}\binom{n}{k}(-1)^k c_k. \tag{4}$$

Let $C(t) = \sum_{n\geq 0} c_n \frac{t^n}{n!}$ the corresponding exponential generating function. According to identity (1); c_n results from Cauchy product (see [4]) of e^t and $C(-t)$. Then $C(t)$ satisfies the identity

$$C(t) = e^t C(-t). \tag{5}$$

If $c_0 \neq 0$; $C^{-1}(t)$ is a generating function too. Regarding the identity (5) we will have

$$C^{-1}(t) = e^{-t}C^{-1}(-t) \text{ or } C^{-1}(-t) = e^t C^{-1}(t). \tag{6}$$

If $c_0 = 0$, $(C^\star)^{-1}$ is a generating function, where $C^\star(t) = t^{-1}C(t)$. Let us consider $c_n^{(1)}$ the sequence generated by the exponential generating function $C^{-1}(t) = \sum_{n\geq 0} c_n^{(1)} \frac{t^n}{n!}$, then $c_n^{(1)}$ satisfies the recursion formula

$$c_n^{(1)} = (-1)^n \sum_{k=0}^{n}\binom{n}{k} c_k^{(1)}. \tag{7}$$

Definition 1.1 *All the sequences satisfying the identity (7) with the first term not zero are so called inverse Cesàro sequences.*

Since we have $C^{-1}(t)C(t) = 1$ we obtain

$$\sum_{k=0}^{n}\binom{n}{k} c_k^{(1)} c_{n-k} = \left\lfloor \frac{1}{n+1} \right\rfloor, \tag{8}$$

where $\lfloor a \rfloor$ denotes the integer part of a real number a. Following the identity (8); the connection between Cesàro sequence and its inverse is

$$c_n^{(1)} = -\frac{1}{c_0}\sum_{k=0}^{n-1}\binom{n}{k} c_k^{(1)} c_{n-k}. \tag{9}$$

From the identity (6) follows $C^{-1}(t)C(-t) = e^{-t}$ and

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k c_k^{(1)} c_{n-k} = 1. \tag{10}$$

Furthermore

$$c_n^{(1)} = \frac{(-1)^n}{c_0} - \frac{(-1)^n}{c_0}\sum_{k=0}^{n-1}\binom{n}{k}(-1)^k c_k^{(1)} c_{n-k} \tag{11}$$

The second member of the expressions (9) and (11) depends on first terms $c_k^{(1)}$. Our interest in this work is to give explicit formula of $c_n^{(1)}$ in means of c_k without apparition of terms $c_k^{(1)}$. In this demarche appear exponential partial Bell polynomials $B_{n,k}$. These polynomials are defined in means of the generating function

$$\frac{1}{k!}\left(\sum_{m\geq 1} x_m \frac{t^m}{m!}\right)^k = \sum_{n\geq k} B_{n,k}\left(x_1,\cdots,x_{n-k+1}\right)\frac{t^n}{n!}. \tag{12}$$

The explicit formula of $B_{n,k}$ is

$$B_{n,k}\left(x_1,\cdots,x_{n-k+1}\right) = \frac{n!}{k!}\sum_{\pi_n(k)}\binom{k}{k_1,\cdots,k_n}\prod_{r=1}^{n-k+1}\left(\frac{x_r}{r!}\right)^{k_r}, \tag{13}$$

where $\pi_n(k)$ is the set of all $(k_1,\cdots,k_{n-k+1})$ such that $k_1+\cdots+k_{n-k+1}=k$ and $k_1+2k_2+\cdots+(n-k+1)k_{n-k+1}=n$. For more details about these polynomials we refer to the book [1] of L. Comtet.

2 Explicit formula of inverse Cesàro sequence

The explicit formula of the inverse Cesàro sequence $c_n^{(1)}$ is given by following theorem

Theorem 2.1 *If $c_0 \neq 0$ we obtain $c_0^{(1)} = \frac{1}{c_0}$ and for $n \geq 1$;*

$$c_n^{(1)} = \sum_{k=1}^{n}(-1)^k k! c_0^{-1-k} B_{n,k}\left(c_1,\cdots,c_{n-k+1}\right). \tag{14}$$

If $c_0 = 0$ we have

$$c_n^{(1)} = \sum_{k=1}^{n}(-1)^k k!\left(\frac{c_1}{2}\right)^{-1-k} B_{n,k}\left(\frac{c_2}{2},\cdots,\frac{c_{n-k+2}}{n-k+2}\right). \tag{15}$$

In means of Theorem 2.1 and identities (9) and (11); some recursive formulae on exponential partial Bell polynomials are given in the following immediate corollary

Corollary 2.2 *We have*

$$\sum_{k=1}^{n}(-1)^k k! B_{n,k}\left(c_1,\cdots,c_{n-k+1}\right) = -\frac{1}{c_0}\sum_{k=1}^{n}\sum_{i=1}^{k}\binom{n}{k}(-1)^i i! B_{k,i}\left(c_1,\cdots,c_{k-i+1}\right)c_{n-k} \tag{16}$$

and

$$\sum_{k=1}^{n}(-1)^k k! B_{n,k}\left(c_1,\cdots,c_{n-k+1}\right) = \frac{(-1)^n}{c_0} - \frac{(-1)^n}{c_0}\sum_{k=0}^{n}\sum_{i=1}^{k}\binom{n}{k}(-1)^{n-k}(-1)^i i! B_{k,i}\left(c_1,\cdots,c_{k-i+1}\right)c_{n-k} \tag{17}$$

2.1 Proof of Theorem 2.1

Let $f(t) = \sum_{n\geq 0} a_n t^n$ a generating function without regarding if it is exponential or ordinary generating function, with the first coefficient $a_0 \neq 0$. Then for any complex number α; $f^{\alpha}(t)$ is a generating function too. Let $f^{\alpha}(t) = \sum_{n\geq 0} f^{\Delta}(n,k)t^n$, In our recent works [5, 6] we have obtained the following explicit formula of $f^{\Delta}(n,k)$:

$$f^{\Delta}(n,\alpha) = \sum_{k=1}^{n}\sum_{\substack{k_1+\cdots+k_n=k\\ k_1+2k_2+\cdots+nk_n=n}}\binom{\alpha}{k}\binom{k}{k_1,\cdots,k_n}a_0^{\alpha-k}a_1^{k_1}\cdots a_n^{k_n},\ n\geq 1,$$

where

$$\binom{\alpha}{k} = \frac{(\alpha)_k}{k!}$$

and

$$(\alpha)_k = \alpha\left(\alpha-1\right)\cdots\left(\alpha-k+1\right)$$

is a falling number. It is obvious to remark that for $j > n-k+1$, only $k_j = 0$. Then for $n \geq 1$ we have

$$f^{\Delta}(n,\alpha) = \frac{1}{n!}\sum_{k=1}^{n}(\alpha)_k a_0^{\alpha-k}B_{n,k}(1!a_1,\cdots,(n-k+1)!a_{n-k+1}). \tag{18}$$

If $\alpha = -1$ the last formula reduced to

$$f^{\Delta}(n,-1) = \frac{1}{n!}\sum_{k=1}^{n}(-1)_k a_0^{\alpha-k}B_{n,k}(1!a_1,\cdots,(n-k+1)!a_{n-k+1}). \tag{19}$$

This result is a consequence of Faà di Bruno formula (see [2])

$$\left(g\circ h\right)^{(n)}(t) = \sum_{k=1}^{n}\left(g^{(k)}\circ h(t)\right)B_{n,k}\left(h^{(1)}(t),\cdots,h^{(n-k+1)}(t)\right),$$

used for computing successive derivatives of the composition $g \circ f$ of two derivative functions until order n. In our case,we reproduce explicitly the proof by taking $g(t) = t^{-1}$ and $f(t) = C(t)$ and we obtain

$$\frac{d^n C^{-1}(t)}{dt^n} = \sum_{k=1}^{n} (-1)_k C^{-1-k}(t) B_{n,k} \left(C^{(1)}(t), \cdots, C^{(n-k+1)}(t) \right).$$

If $c_0 \neq 0$, $C^{-1}(t)$ is a generating function. Then

$$C^{-1}(t) = \sum_{n \geq 0} \frac{d^n C^{-1}(t)}{dt^n}|_{t=0} \frac{t^n}{n!}$$

and

$$C^{-1}(t) = c_0^{-1} + \sum_{n \geq 1} \sum_{k=1}^{n} (-1)_k c_0^{-1-k} B_{n,k} \left(c_1, \cdots, c_{n-k+1} \right) \frac{t^n}{n!}$$

In the special case $c_0 = 0$, the identity (5) becomes

$$C^{\star}(t) = e^t C^{\star}(-t)$$

where $C^{\star}(t) = \sum_{n \geq 0} \frac{c_{n+1}}{n+1} \frac{t^n}{n!}$ with the first term is not zero. We apply again the same technique to obtain the second identity (15).

3 Application to Bernoulli and Euler numbers

The Cesàro sequence $(-1)^n B_n$ is identic to Bernoulli numbers B_n except for $n = 1$. Without lost generality, in this section by computing the inverse Cesàro sequence $B_n^{(1)}$ we revisit the sums of products of Bernoulli numbers (see [3]) and obtaining similar explicit formula including the closed form of sums of products of zeta functions. The zeta function is defined over the set of complex numbers formally by the series

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}. \tag{20}$$

The values of ζ at even positive integers is given via the well-known Euler formula

$$\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}. \tag{21}$$

K. Dilcher investigated the sum

$$S_N(n) =: \sum \binom{2n}{2j_1, 2j_2, \cdots, 2j_N} B_{2j_1} B_{2j_2} \cdots B_{2j_N} \tag{22}$$

and provide (see [3, Theorem 1]) that

$$S_N(n) = \frac{(2n)!}{(2n-N)!} \sum_{k=0}^{\lfloor \frac{N-1}{2} \rfloor} b_k^{(N)} \frac{B_{2N-2k}}{2N-2k}. \tag{23}$$

$b_k^{(N)}$ is a sequence of rational numbers defined recursively by $b_0^{(1)} = 1$ and

$$b_k^{(N+1)} = -\frac{1}{N} b_k^{(N)} + \frac{1}{4} b_{k-1}^{(N-1)}. \tag{24}$$

In our case, we investigate the new sum

$$S(n) = \sum_{k=1}^{n} (-1)^k \sum \binom{k}{k_2, \cdots, k_{2a}} \prod_{r=1}^{a} \left(\frac{B_{2r}}{(2r)!} \right)^{k_{2r}} \tag{25}$$

in order to give its explicit formula. $\sum$ is the sum over all $(k_2, k_4, \cdots, k_{2a})$ where $2a$ is the greatest even number inferior to $n-k+1$; such that $k_2 + \cdots + k_{2a} = k$ and $2k_2 + \cdots + 2ak_{2a} = n$. Then n must be even as we will see later.

The Euler numbers are defined in means of the generating function

$$\frac{2}{e^t + 1} = \sum_{n \geq 0} E_n \frac{t^n}{n!}. \tag{26}$$

The first few values are $E_0 = 1, E_1 = -\frac{1}{2}, E_3 = \frac{1}{4}, \cdots$ and $E_{2n} = 0$ for $n \geq 1$. The expression of $S(n)$ depends in these numbers and we have

Theorem 3.1

$$S(n) = -\frac{2E_{n+1}}{(n+1)!}. \tag{27}$$

Regarding the characteristics of numbers E_n, the expression (27) becomes

$$S(2n) = -\frac{2E_{2n+1}}{(2n+1)!} \text{ and } S(2n+1) = 0.$$

Proof. Let

$$C(t) = \frac{te^t + t}{2\left(e^t - 1\right)} = \sum_{n \geq 0} B_{2n} \frac{t^{2n}}{(2n)!}$$

and

$$C^{-1}(t) = \sum_{n \geq 0} B_n^{(1)} \frac{t^n}{n!}.$$

Then

$$C^{-1}(t) = \frac{2(e^t - 1)}{te^t + t} = \frac{2}{t} \left(1 - \frac{2}{e^t + 1} \right) = \frac{2}{t} \left(1 - \sum_{n \geq 0} E_n \frac{t^n}{n!} \right) = -2 \sum_{n \geq 0} \frac{E_{n+1}}{n+1} \frac{t^n}{n!}.$$

From this identity lead $B_n^{(1)} = -\frac{2E_{n+1}}{n+1}$. Since E_n vanish for even positive numbers except zero, then $B_{2n+1}^{(1)} = 0$ and $B_{2n}^{(1)} = -\frac{2E_{2n+1}}{2n+1}$. But in means of identity (14) Theorem 2.1 we will have

$$B_n^{(1)} = \frac{2E_{n+1}}{n+1} = \sum_{k=1}^{n} (-1)^k k! B_{n,k}\left(B_2, \cdots, B_{2a}\right).$$

But

$$B_{n,k}\left(B_2, \cdots, B_{2a}\right) = \frac{n!}{k!} \sum \binom{k}{k_2, \cdots, k_{2a}} \prod_{r=1}^{a} \left(\frac{B_{2r}}{(2r)!}\right)^{k_{2r}}.$$

Then identity (27) Theorem 3.1 follows.

According to Euler formula we have

$$\frac{B_{2r}}{(2r)!} = (-1)^{r-1} \frac{2\zeta(2r)}{(2\pi)^{2r}}.$$

Furthermore the following corollary is immediate.

Corollary 3.2

$$\sum_{k=1}^{n} \sum \binom{k}{k_2, \cdots, k_{2a}} 2^k \prod_{r=1}^{a} \zeta^{k_{2r}}(2r) = \frac{2(2\pi)^n E_{n+1}}{(-1)^{\frac{n}{2}} (n+1)!}. \tag{28}$$

In the literature the Genocchi numbers G_n are defined by

$$\frac{2t}{e^t + 1} = \sum_{n \geq 0} G_n \frac{t^n}{n!}.$$

These numbers are introduced and studied by Angelo Genocchi (1817-1889). We have $G_0 = 0$, $G_1 = 1$ and $G_{2n+1} = 0$, $n > 0$. G_{2n} are odd integer and related to tangent hyperbolic by

$$t \tanh \frac{t}{2} = -\sum_{n \geq 1} G_{2n} \frac{t^{2n}}{(2n)!}.$$

Since we have (the proof is left as an exercise)

$$t \tanh \frac{t}{2} = \sum_{n \geq 1} 2\left(2^{2n} - 1\right) B_{2n} \frac{t^{2n}}{(2n)!}$$

we obtain

$$G_{2n} = -2\left(2^{2n} - 1\right) B_{2n}. \tag{29}$$

$G_0 = 0$ implies that $C(t) = \frac{2t}{e^t+1}$ is not invertible. to escape the problem we consider the function $C^{\star}(t) = \frac{2}{e^t+1}$ which generates numbers $\frac{G_{n+1}}{n+1}$; we deduce that $\frac{G_{n+1}}{n+1} = E_n$ and $(-1)^n E_n$ is a Cesàro sequence. We can easily verify that $\frac{2}{e^{-t}+1} = \frac{2e^t}{e^t+1}$. The series expansion of $\frac{e^t+1}{2}$ is

$$\frac{e^t + 1}{2} = 1 + \sum_{n\geq 1} \frac{1}{2}\frac{t^n}{n!}.$$

From the last identity follows the inverse Cesàro sequence $E_n^{(1)}$ which is given by $E_0^{(1)} = 1$ and $E_n^{(1)} = \frac{1}{2}$ for $n \geq 1$. We have already proved the following theorem.

Theorem 3.3 *For $n \geq 1$ we have*

$$\sum_{k=1}^{n} (-1)^k \sum \binom{k}{k_1, \cdots, k_{2a+1}} \prod_{r=1}^{a} \left(\frac{E_{2r+1}}{(2r+1)!}\right)^{k_{2r+1}} = \frac{1}{2n!}. \tag{30}$$

the symbol $\sum$ means that the sum is over all $(k_3, k_5, \cdots, k_{2a+1})$ where $2a + 1$ is the greatest odd number inferior to $n - k + 1$; such that $k_3 + \cdots + k_{2a+1} = k$ and $3k_2 + \cdots + (2a + 1)k_{2a+1} = n$.

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