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On Fuzzy $\omega\alpha$ - Closed Set in Fuzzy Topological Spaces

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Abstract

This paper introduced new classes of fuzzy sets in fuzzy topological spaces, namely, the class of fuzzy $\omega\alpha$-closed set and fuzzy $\omega\alpha$-open set by fuzzification of the $\omega\alpha$-closed set and fuzzy $\omega\alpha$-open set in Topology. Fuzzy $\omega\alpha$-open sets were used to define fuzzy pre g^*-closed set. Lastly, some properties of the said fuzzy sets and its relationship to some existing class of fuzzy closed sets were proved.

Keywords: fuzzification of Topology, fuzzy set, fuzzy $\omega\alpha$-closed set, fuzzy $\omega\alpha$-open set, fuzzy pre g^*-closed set, fuzzy pre g^*-open set, fuzzy $\omega\alpha$-kernel

1 Introduction

As an attempt to handle mathematical phenomena which are innately vague or fuzzy in nature, L. A. Zadeh [12] brought out the concept of fuzzy sets in his classical paper in 1965. After such discovery, the dawn of fuzzy set theory has become a concern and a new tool for mathematicians working in several distinctive areas of Mathematics. It provides framework which is broader than that of classical set theory. Fuzzy topology is one such area, combining with topological structure, the ordered structure. This structure have been known and applied to computer programming, engineering and Physics as well as in forecasting. Nowadays, fuzzy topology have been used to some network analysis/inference problems, generalization of catalyzed modern theory of dynamical system. It was also used in flood prediction through incorporating fuzzy spatial objects in GIS application [4].

There were many types of fuzzy open and closed sets developed in Fuzzy topological spaces. Azad [1] introduced the concept of fuzzy semi-open set in Fuzzy Topology. Signal and Rajvansi [11] as well as Bin Shahna [3] developed the notion of fuzzy α-open set. Subsequent to the concept of pre-open set in topological spaces established by Masshour et. al [8], there had been studies by Bin Shahna [3] on fuzzy topology as fuzzy pre-open set. Meanwhile, Mishra and Thakur [9] introduced the fuzzy ω-closed set.

Recently, Benchalli, et al. [2] studied the notion of $\omega\alpha$-closed set and $\omega\alpha$-open set in topological spaces. Likewise, Jafari, et al. [7] defined a subset A of a topological space X is pre g^*-closed set and pre g^*-open set.

In this paper, the $\omega\alpha$-open set [2] and pre g^*-closed set and pre g^*-open set [7] are extended in fuzzy topological space by defining new class of fuzzy generalized sets such as fuzzy $\omega\alpha$-closed set.Fuzzy pre g^*-closed set was then defined and examine the properties.

2 Preliminary Notes / Materials and Methods

Let X be a nonempty set and I be the unit interval $[0, 1]$. A *fuzzy set* on X is given by the set

$$A = \{(x, A(x)) : \forall x \in X\}$$

where the mapping A from X on to I is the membership function of x in A and $\forall x \in X$, $A(x)$ denotes the *degree* or *grade* of membership of x. The null fuzzy set $\mathbf{0}$ is the mapping from X in to I which assumes only the value 0 and the whole fuzzy set $\mathbf{1}$ is a mapping from X on to I which takes the value 1 only. The union(resp. intersection) of family $\{A_\beta : \beta \in B\}$ of fuzzy sets on X is defined by the mapping sup A_β (resp. inf A_β). A fuzzy set A of X is a *subset* of a fuzzy set B if $A(x) \leq B(x)$ for each $x \in X$. The *complement* of fuzzy set A is given by $A^c = 1 - A(x)$. The operators *fuzzy closure* and *fuzzy interior* are denoted and defined by $Cl\ A = \bigcap\{F : F\ is\ fuzzy\ closed, A \subseteq F\}$ and $Int\ A = \bigcup\{O : O\ is\ fuzzy\ open, O \subseteq A\}$, respectively [12].

A family τ of fuzzy sets A of X is called a **fuzzy topology** on X if $\mathbf{0}$, $\mathbf{1}$ belong to τ and τ is closed with respect to arbitrary union and finite intersection. The members of τ are called *fuzzy open sets* in τ(or simply fuzzy open sets) and and their complement are *fuzzy closed sets* [5].

Definition 2.1 *A fuzzy set A of a fuzzy topological space (X, τ) is called:*

i. **fuzzy semi-open** *[1] if* $A \subseteq Cl[Int\ A]$.

ii. **fuzzy ω-closed set** *[9] if* $Cl\ A \subseteq O$, *whenever* $A \subseteq O$ *and* O *is fuzzy semi-open set.*

iii. **fuzzy ω-open** *[9] if its complement,* A^c, *is fuzzy* ω*-closed.*

iv. **fuzzy α-open set** *[3] if* $A \subseteq Int\{Cl[Int\ A]\}$.

v. **fuzzy α-closed** *[3] if* A^c *is fuzzy* α*-open.*

vi. **fuzzy pre-open** *[3] if* $A \subseteq Int[Cl\ A]$.

vii. **fuzzy pre-closed** *[3] if the complement of* A *is fuzzy pre-open.*

viii. **fuzzy gp-closed** *[6] if* $fpCl\ A \subseteq O$ *whenever* $A \subseteq O$ *and* O *is fuzzy open set.*

Theorem 2.2 *([5]) Let* $\{A_\beta : \beta \in B\}$ *be a family of fuzzy sets on a fuzzy topological space* X. *Then the following hold:*

i. $\bigcup[Int\ A_\beta] \subseteq Int[\bigcup A_\beta]$;

ii. $\bigcup[Cl\ A_\beta] \subseteq Cl[\bigcup A_\beta]$.

Lemma 2.3 *Let* A *be a fuzzy set on a toplogical space* X.

i. If A *is fuzzy closed, then* A *is fuzzy* ω*-closed [9].*

ii. If A *is fuzzy open, then* A *is fuzzy* α*-open [3].*

iii. If A *is fuzzy* α*-closed, then* A *is fuzzy pre-closed [3].*

The following properties were obtained as consequences of the previous definitions, theorem and lemma.

Definition 2.4 *([10]) The intersection of all fuzzy* α*-closed sets containing a fuzzy set* A *of a fuzzy topological space* X *is called a* **fuzzy α-closure** *of* A, *denoted by* $f\alpha Cl\ A$. *That is*

$$f\alpha Cl\ A = \bigcap \{F : F\ is\ fuzzy\ \alpha - closed, A \subseteq F\}.$$

Definition 2.5 *([3]) The intersection of all fuzzy pre-closed sets containing the fuzzy set* A *of a fuzzy topological space is called the* **fuzzy pre-closure** *of* A, *denoted by* $fpCl\ A$. *That is,*

$$fpCl\ A = \bigcap \{F : F\ is\ fuzzy\ pre - closed\ and A \subseteq F\}.$$

Definition 2.6 *([3]) The union of all fuzzy pre-open sets that are contained in A is called* **fuzzy pre-interior** *of A, denoted by $fpInt\ A$. That is,*

$$fpInt\ A = \bigcup \{O : O\ is\ fuzzy\ pre-open\ and\ O \subseteq A\}.$$

Lemma 2.7 *Let A be a fuzzy set on a topological space X. Then*

i. *The intersection of fuzzy α-closed sets is a fuzzy α-closed set. In particular, $f\alpha Cl\ A$ is fuzzy α-closed [11].*

ii. *$A = f\alpha Cl\ A$ if and only if A is fuzzy α-closed [11].*

iii. *The intersection of fuzzy pre-closed sets is fuzzy pre-closed. In particular, $fpCl\ A$ is fuzzy pre-closed.*

iv. *$A = fpCl\ A$ if and only if A is fuzzy pre-closed.*

v. *Any union of fuzzy α-closed sets is a fuzzy α-closed [11].*

Proof:

iii. Let $\{F_i : i \in B\}$ be any family of fuzzy pre-closed sets.Then $\{F_i^c : i \in B\}$ is a family of fuzzy pre-open sets. That is, $F_i^c \subseteq Int[Cl\ F_i^c]$. In effect

$$\begin{aligned} \cup F_i^c &\subseteq \cup(Int[Cl\ F_i^c]) \\ &\subseteq Int[\cup(Cl\ F_i^c)] \\ &\subseteq Int[Cl\ (\cup F_i^c)]. \end{aligned}$$

This means that $\cup F_i^c$ is fuzzy pre-open. Therefore,

$$(\cup F_i^c)^c = \cap(F_i^c)^c = \cap F_i$$

is fuzzy pre-closed. Furthemore, $fpCl\ A$, being the intersection of fuzzy pre-closed sets containing A, is fuzzy pre-closed.

iv. Let A be an fuzzy pre-closed set. Since

$$fpCl\ A = \bigcap \{F : F\ is\ fuzzy\ pre-closed\ and\ A \subseteq F\},$$

we have $A \subseteq fpCl\ A$. Moreover, A is one of the F's which implies that $fpCl\ A \subset A$. Thus, $A = fpCl\ A$. Conversely, suppose

$$fpCl\ A = \bigcap \{F : F\ is\ fuzzy\ pre-closed\ and\ A \subseteq F\}.$$

By (iii), $fpCl\ A$ is fuzzy pre-closed. Hence, $A = fpCl\ A$ is fuzzy pre-closed.

The following results are necessary in proving the main results.

Lemma 2.8 *Let A and B be fuzzy sets on X and $A \subseteq B$. Then*

i. $fpCl\ A \subseteq fpCl\ B$ *[3];*

ii. $f\alpha Cl\ A \subseteq f\alpha Cl\ B$ *[11].*

Lemma 2.9 *Let A be a fuzzy set on X. Then the followng statements are true.*

i. $fpCl\ A \subseteq f\alpha Cl\ A$*;*

ii $fpCl[fpCl\ A] = fpCl\ A$*;*

iii. $(fpInt\ A)^c = fpCl\ (A^c)$.

Proof:

i. By definition,

$$fpCl\ A = \bigcap \{F : F\ is\ fuzzy\ pre-closed,\ A \subseteq F\}$$

and

$$f\alpha Cl\ A = \bigcap \{G : G\ is\ fuzzy\ \alpha-closed,\ A \subseteq G\}.$$

By Lemma 2.7 (i), $f\alpha Cl\ A$ is fuzzy α-closed, by Lemma 2.3 (iii), $f\alpha Cl\ A$ is fuzzy pre-closed. In effect, $f\alpha Cl\ A$ is one of the F's. Therefore, that $fpCl\ A \subseteq f\alpha Cl\ A$.

ii. By Lemma 2.7 (iv), if B is fuzzy pre-closed, then $fpCl\ B = B$. Also, from Lemma 2.7 (iii), $fpCl\ A$ is fuzzy pre-closed. Hence,

$$fpCl[fpCl\ A] = fpCl\ A.$$

iii. By definition,

$$fpInt\ A = \bigcup \{O : O\ is\ fuzzy\ pre-open,\ O \subseteq A\}.$$

Then

$$\begin{aligned}(fpInt\ A)^c &= (\bigcup \{O : O\ is\ fuzzy\ pre-open,\ O \subseteq A\})^c \\ &= \bigcap \{O : O\ is\ fuzzy\ pre-open,\ O \subseteq A\}^c \\ &= \bigcap \{O^c : O^c\ is\ fuzzypre-closed,\ A^c \subseteq O^c\} \\ &= fpCl\ (A^c).\end{aligned}$$

Example 2.10 *Let* $X = \{a, b, c\}$, $B = \{(a, 0.7), (b, 0.3), (c, 1)\}$, $D = \{(a, 0.7), (b, 0), (c, 0)\}$. *Consider the topology on* X *given by* $\tau = \{\bar{\mathbf{0}}, \bar{\mathbf{1}}, B, D\}$.

The fuzzy closed sets are $\mathbf{0}, \mathbf{1}, \mathbf{B}' = \{(\mathbf{a}, \mathbf{0.3}), (\mathbf{b}, \mathbf{0.7}), (\mathbf{c}, \mathbf{0})\}$, *and* $D' = \{(a, 0.3), (b, 1), (c, 1)\}$.

The fuzzy semi-open sets of X *are* $\mathbf{0}, \mathbf{1}, \mathbf{B}, \mathbf{D}$ *and* $C = \{(a, C(a)), (b, C(b)), (c, C(c))\}$ *where* $C(a) \geq 0.7$, *for any* $C(b)$ *and* $C(c)$.

The fuzzy ω*-closed sets of* X *obtained are the following:*
$\mathbf{0}, \mathbf{1}, \mathbf{B}', \mathbf{D}'$, $K = \{(a, K(a)), (b, K(b)), (c, K(c))\}$ *where* $K(a) < 0.3$, $K(b) = K(c) = 1$ *and*
$L = \{(a, L(a)), (b, L(b)), (c, L(c))\}$ *where* $L(a) < 0.3, L(b) = 0.7, L(c) = 0$.

Meanwhile, the fuzzy ω*-open sets of are* $\bar{\mathbf{0}}, \bar{\mathbf{1}}, B, D$, $K' = \{(a, K'(a)), (b, K'(b)), (c, K'(c))\}$ *where* $K'(a) > 0.7, K'(b) = K'(c) = 0$ *and* $L' = \{(a, L'(a)), (b, L'(b)), (c, L'(c))\}$ *where* $L'(a) > 0.7, L'(b) = 0.3, L'(c) = 1$.

Furthemore, the fuzzy α*-open sets of* X *are as follows:*
$\mathbf{0}, \mathbf{1}, \mathbf{B}, \mathbf{D}$ *and* $C = \{(a, C(a)), (b, C(b)), (c, C(c))\}$ *where* $C(a) \geq 0.7$ *and for any* $C(b)$ *and* $C(c)$.

On the other hand, the fuzzy α*-closed sets of* X *are* $\mathbf{0}, \mathbf{1}, \mathbf{B}', \mathbf{D}'$ *and* $C' = \{(a, C'(a)), (b, C'(b)), (c, C'(c))\}$ *where* $C'(a) \leq 0.3$, *for any* $C'(b)$ *and* $C'(c)$.

The fuzzy pre-open sets in X *are* $\mathbf{0}, \mathbf{1}, \mathbf{B}, \mathbf{D}$ *and* $E = \{(a, E(a)), (b, E(b)), (c, E(c))\}$ *where* $E(a) > 0.3$, *for any* $E(b)$ *and* $E(c)$.

The fuzzy pre-closed sets in X *are* $\mathbf{0}, \mathbf{1}, \mathbf{B}', \mathbf{D}'$ *and* $E' = \{(a, E'(a)), (b, E'(b)), (c, E'(c))\}$ *where* $E'(a) < 0.7$, *for any* $E'(b)$ *and* $E'(c)$.

The following sets are the fuzzy gp*-closed sets in* X*:*
$\mathbf{0}, \mathbf{1}, \mathbf{B}', \mathbf{D}'$, $E' = \{(a, E'(a)), (b, E'(b)), (C, E'(c))\}$ *where* $E'(a) < 0.7$, *for any* $E'(b)$ *and* $E'(c)$,
$G = \{(a, G(a)), (b, G(b)), (c, G(c))\}$ *where* $G(a) > 0.7$, *for any* $G(b)$ *and* $G(c)$ *and*
$H = \{(a, H(a)), (b, H(b)), (c, H(c))\}$ *where* $H(a) = 0.7$, $H(b) > 0.3$ *and for any* $H(c)$.

3 Results and Discussion

In this section, new classes of fuzzy sets in fuzzy topology namely, fuzzy $\omega\alpha$-closed sets and fuzzy $\omega\alpha$-open sets as well as fuzzy pre g^*-closed sets and fuzzy pre g^*-open sets were introduced. These main results of the paper were obtained through fuzzification of topological sets such as $\omega\alpha$-closed sets and $\omega\alpha$-open sets [8] as well as pre g^*-closed sets and pre g^*-open sets [9] .

Definition 3.1 *A fuzzy set A of a fuzzy topological space is said to be* **fuzzy $\omega\alpha$-closed** *if $f\alpha Cl\ A \subseteq O$, whenever $A \subseteq O$ and O is fuzzy ω-open set.*

Theorem 3.2 *Every fuzzy α-closed set is fuzzy $\omega\alpha$-closed.*

Proof: Let A be a fuzzy α-closed set and O be a fuzzy ω-open set with $A \subseteq O$. Since A is fuzzy α-closed, by Lemma 2.7 (ii), $f\alpha Cl\ A = A$. This means that $f\alpha Cl\ A = A \subseteq O$. Therefore, A is fuzzy $\omega\alpha$-closed.

Example 3.3 *Consider the topological space X and the fuzzy sets in Example 2.10. We obtain the fuzzy $\omega\alpha$-closed sets in X as follows:*
$\mathbf{0}, \mathbf{1}, \mathbf{B}', \mathbf{D}'$ *and* $C' = \{(a, C'(a)), (b, C'(b)), (c, C'(c))\}$ *where* $C'(a) \leq 0.3$, *for any* $C'(b)$ *and* $C'(c)$ *and* $M = \{(a, M(a)), (b, M(b)), (c, M(c))\}$ *where* $M(a) > 0.3$, $M(b) > 0.3$ *and for any* $M(c)$.

Definition 3.4 *Let A be a fuzzy set of X. Then A is said to be* **fuzzy $\omega\alpha$-open** *if A^c is a fuzzy $\omega\alpha$-closed set.*

Example 3.5 *Consider the fuzzy topological space in Example 2.10. From Example 3.3, we attain the following fuzzy $\omega\alpha$-open sets in X.*
$\mathbf{0}, \mathbf{1}, \mathbf{B}, \mathbf{D}$, $C = \{(a, C(a)), (b, C(b)), (c, C(c))\}$ *where* $C(a) \geq 0.7$, *for any* $C(b)$ *and* $C(c)$ *and*
$M' = \{(a, M'(a)), (b, M'(b)), (c, M'(c))\}$ *where* $M'(a) < 0.7$, $M'(b) < 0.7$, *for any* $M'(c)$.

Definition 3.6 *Let X be a fuzzy topological space. A fuzzy set A of X is called* **fuzzy pre g^*–closed set** *if $fpCl\ A \subseteq O$, whenever $A \subseteq O$ and O is a fuzzy $\omega\alpha$-open set in X.*

Theorem 3.7 *Every fuzzy pre-closed set is fuzzy pre* g**-closed set.*

Proof: Let A be a fuzzy pre-closed set and O be a fuzzy $\omega\alpha$-open set with $A \subseteq O$. Since A is fuzzy pre-closed, by Lemma 2.7 (iv), $fpCl\ A = A$. This implies that $fpCl\ A = A \subseteq O$. Therefore, A is fuzzy pre g^*-closed.

Example 3.8 *Consider Example 2.10 with* $\tau = \{\mathbf{0}, \mathbf{1}, B, D\}$ *where* $B = \{(a, 0.7), (b, 0.3), (c, 1)\}$ *and* $D = \{(a, 0.7), (b, 0), (c, 0)\}$. *From Example 3.5, the fuzzy* $\omega\alpha$*-open sets in* X *are* $\mathbf{0}, \mathbf{1}, B, D$, $C = \{(a, C(a)), (b, C(b)), (c, C(c))\}$ *where* $C(a) \geq 0.7$, *for any* $C(b)$ *and* $C(c)$ *and* $M' = \{(a, M'(a)), (b, M'(b)), (c, M'(c))\}$ *where* $M'(a), M'(b) < 0.7$, *for any* $M'(c)$. *Then the following are the fuzzy pre* g*-*closed sets in* X*:*

$$\mathbf{0}, \mathbf{1}, B', D' \text{ and } E' = \{(a, E'(a)), (b, E'(b)), (c, E'(c)\}$$

where $E'(a) < 0.7$, *for any* $E'(b)$ *and* $E'(c)$.

Theorem 3.9 *If* A *is fuzzy pre* g*-*closed, then* A *is fuzzy* gp*-closed.*

Proof: Suppose A is a fuzzy pre g^*-closed set and let O be a fuzzy open set where $A \subseteq O$. By Lemma 2.3 (ii), O is fuzzy α-open. This means that O^c is fuzzy α-closed. But by Theorem 3.2, O^c fuzzy $\omega\alpha$-closed. In effect, O is fuzzy $\omega\alpha$-open. Since A is fuzzy pre g^*-closed, $fpCl\ A \subseteq O$. Consequently, A is fuzzy gp-closed.

Theorem 3.10 *If* A *is fuzzy* α*-closed, then* A *is fuzzy pre* g*-*closed.*

Proof: Assume that A is fuzzy α-closed. By Lemma 2.3 (iii), A is fuzzy pre-closed. Hence, by Theorem 3.7, A is fuzzy pre g^*-closed.

Definition 3.11 *Let* X *be a fuzzy topological space and* A *a fuzzy set of* X. *The intersection of all fuzzy* $\omega\alpha$*-open sets* O *in* X *where* $A \subseteq O$ *is called* **fuzzy $\omega\alpha$-kernel** *of* A *and is denoted by* $f\omega\alpha - ker\ A$. *That is,*

$$f\omega\alpha - ker\ A = \cap \{O : O \text{ } is \text{ } fuzzy \text{ } \omega\alpha - open \text{ } and \text{ } A \subseteq O\}.$$

Theorem 3.12 *The intersection of fuzzy* $\omega\alpha$*-open sets of* X *is fuzzy* $\omega\alpha$*-open. In particular, for a fuzzy set* A, $f\omega\alpha - ker\ A$ *is fuzzy* $\omega\alpha$*-open.*

Proof: Let $\{O_i : i \in I\}$ be a family of fuzzy $\omega\alpha$-open sets of X. We show that $\cap \{O_i : i \in I\}$ is fuzzy $\omega\alpha$-open set. That is, $\cup \{O_i^c : i \in I\}$ is fuzzy $\omega\alpha$-closed set. Let U be fuzzy ω-open set with $\cup O_i^c \subseteq U$. Since O_i^c is fuzzy $\omega\alpha$-closed, for each $i \in I$, by Definition 3.1, $f\alpha Cl\ O_i^c \subseteq V$, whenever V is a fuzzy ω-open set and $O_i^c \subseteq V$. Then

$$\cup (f\alpha Cl\ O_i^c) \subseteq \cup V$$

and

$$\cup O_i^c \subseteq \cup V.$$

Since $O_i^c \subseteq f\alpha Cl\ O_i^c$, we have

$$\cup O_i^c \subseteq \cup(f\alpha Cl\ O_i^c).$$

By Lemma 2.8 (ii),

$$f\alpha Cl(\cup O_i^c) \subseteq f\alpha Cl[\cup f\alpha Cl\ O_i^c].$$

By Lemma 2.3(v) and Lemma 2.3(i),

$$\cup(f\alpha Cl\ O_i^c)$$

is fuzzy α-closed. Thus, by Lemma 2.7 (ii),$f\alpha Cl[\cup f\alpha Cl\ O_i^c] = \cup(f\alpha Cl\ O_i^c)$. It follows that

$$f\alpha Cl(\cup O_i^c) \subseteq \cup(f\alpha Cl\ O_i^c) \subseteq \cup V.$$

Since $\cup O_i^c \subseteq \cup V$ and

$$\cup O_i^c \subseteq U,$$

U is one of the $\cup V$'s and hence,

$$f\alpha Cl(\cup O_i^c) \subseteq \cup(f\alpha Cl\ O_i^c) \subseteq U.$$

Also, since U is fuzzy ω-open, we have $\cup O_i^c$ is fuzzy $\omega\alpha$-closed. Consequently,

$$\cap O_i = (\cup O_i^c)^c$$

is fuzzy $\omega\alpha$-open.Thus,

$$f\omega\alpha - ker\ A = \cap \{U : U\ is\ fuzzy\ \omega - open\ and\ A \subseteq U\}$$

is fuzzy $\omega\alpha$-open.

Theorem 3.13 *A fuzzy set A is fuzzy $\omega\alpha$-open if and only if*

$$f\omega\alpha - ker\ A = A.$$

Proof: Suppose A is fuzzy $\omega\alpha$-open. By Definition 3.11,

$$A \subseteq f\omega\alpha - ker\ A.$$

Since A is fuzzy $\omega\alpha$-open, A is one of the fuzzy $\omega\alpha$-open sets O such that $A \subseteq A$. Therefore,

$$f\omega\alpha - ker\ A = A.$$

On the other hand, let

$$A = f\omega\alpha - ker\ A = \cap \{O : O\ is\ fuzzy\omega\alpha - open\ and\ A \subseteq O\}.$$

By Theorem 3.12, $f\omega\alpha - ker\ A$ is fuzzy $\omega\alpha$-open. Hence, $A = f\omega\alpha - ker\ A$ is fuzzy $\omega\alpha$-open.

Theorem 3.14 *If $A \subseteq B$, then $f\omega\alpha - ker\ A \subseteq f\omega\alpha - ker\ B$.*

Proof: Let A and B be fuzzy sets of X and $A \subseteq B$. By Definition 3.11,

$$f\omega\alpha - ker\ A = \cap \{O : O\ is\ fuzzy\ \omega\alpha - open,\ A \subseteq O\}$$

and

$$f\omega\alpha - ker\ B = \cap \{V : V\ is\ fuzzy\ \omega\alpha - open,\ B \subseteq V\}.$$

It follows that

$$A \subseteq f\omega\alpha - ker\ A$$

and

$$B \subseteq f\omega\alpha - ker\ B.$$

Since $A \subseteq B$,

$$A \subseteq f\omega\alpha - ker\ B.$$

By Theorem 3.12,

$$f\omega\alpha - ker\ B$$

is fuzzy $\omega\alpha$-open and is one of the O's. Hence,

$$f\omega\alpha - ker\ A \subseteq f\omega\alpha - ker\ B.$$

Example 3.15 *Consider the fuzzy topological space and fuzzy sets in Example 2.10. We compute the $f\omega\alpha - ker\ A$ as follows.*

$$f\omega\alpha - ker\ \mathbf{0} = \mathbf{0} \qquad f\omega\alpha - ker\ \mathbf{1} = \mathbf{1}$$
$$f\omega\alpha - ker\ B = B \qquad f\omega\alpha - ker\ D = D$$

For $C = \{(a, C(a)), (b, C(b)), (c, C(c))\}$ where $C(a) \geq 0.7$, for any $C(b)$ and $C(c)$, $f\omega\alpha - ker\ C = C$.
For $M' = \{(a, M'(a)), (b, M'(b)), (c, M'(c))\}$ where $M'(a), M'(b) < 0.7$ and for any $M'(c)$, $f\omega\alpha - ker\ M' = M'$.

The above results follows from Theorem 3.13 since $\mathbf{0}, \mathbf{1}, B, D, C$ and M' are fuzzy $\omega\alpha$-open sets of X.

Now, consider the fuzzy set

$$N = \{(a, N(a)), (b, N(b)), (c, N(c))\}$$

where $N(a) < 0.7$ and $N(b) \geq 0.7$ and for any $N(c)$. Then the fuzzy $\omega\alpha$-open sets containing N are $\mathbf{1}$ and C where $C(a) \geq 0.7$, $C(b) \geq N(b)$ and $C(c) \geq N(c)$. Thus, $f\omega\alpha - ker\ N = C \cap \mathbf{1} = C$.

Theorem 3.16 *If $fpCl\ A \subseteq f\omega\alpha$-$ker\ A$, then A is fuzzy pre g*-closed.*

Proof: Suppose $fpCl\ A \subseteq f\omega\alpha - ker\ A$. By Definition 3.11,

$$f\omega\alpha - ker\ A \subseteq O,$$

for each fuzzy $\omega\alpha$-open set O with $A \subseteq O$. Thus,

$$fpCl\ A \subseteq f\omega\alpha - ker\ A \subseteq O.$$

Accordingly, A is fuzzy pre g*-closed.

Theorem 3.17 *Every fuzzy $\omega\alpha$-closed set is fuzzy gp-closed.*

Proof: Let A be a fuzzy $\omega\alpha$-closed set and V be a fuzzy open set with $A \subseteq V$. Since A is fuzzy $\omega\alpha$-closed, $f\alpha Cl\ A \subseteq O$, for each fuzzy ω-open set O and $A \subseteq O$. By Lemma 2.9 (i),

$$fpCl\ A \subseteq f\alpha Cl\ A \subseteq O.$$

Furthermore, since V is fuzzy open, V^c is fuzzy closed. From Lemma 2.3(i), is fuzzy ω-closed. This means that V is fuzzy ω-open. Since $A \subseteq V$, we have V is one of the O's. Hence,

$$fpCl\ A \subseteq V.$$

Therefore, A is fuzzy *gp*-closed.

Definition 3.18 *Let A be a fuzzy set of X. Then A is said to be* **fuzzy pre *g**-open** *if A^c is a fuzzy pre* g**-closed set.*

Example 3.19 *Consider the fuzzy topological space in Example 2.10. Then from Example 3.8, the fuzzy pre* g**-open sets in X are as follows:*

$$\mathbf{0}, \mathbf{1}, B, D \text{ and } E = \{(a, E(a)), (b, E(b)), (c, E(c))\}$$

where $E(a) > 0.3$, for any $E(b)$ and $E(c)$.

Theorem 3.20 *A fuzzy set A is fuzzy pre* g**-open if and only if $F \subseteq fpInt\ A$ whenever $F \subseteq A$ and F is fuzzy $\omega\alpha$-closed set in X.*

Proof: Let O be a fuzzy $\omega\alpha$-open set and $A^c \subseteq O$. We want to show that $fpCl\ A^c \subseteq O$ so that A^c is fuzzy pre g*-closed and hence, A is fuzzy pre g*-open. Now, O^c is fuzzy $\omega\alpha$-closed and $O^c \subseteq A$. By the hypothesis, $O^c \subseteq fpInt\ A$. Taking the complement, we obtain

$$(fpInt\ A)^c \subseteq O.$$

By Lemma 3.14,

$$fpCl\ A^c = (fpInt\ A)^c \subseteq O.$$

Thus, A is fuzzy pre g^*-open.

Conversely, let A be a fuzzy pre g^*-open set and F be a fuzzy $\omega\alpha$-closed set with $F \subseteq A$. Then A^c is fuzzy pre g^*-closed and F^c is fuzzy $\omega\alpha$-open with $A^c \subseteq F^c$. Hence,

$$fpCl\ A^c \subseteq F^c.$$

Therefore,

$$F \subseteq [fpCl\ A^c] = fpInt\ A.$$

Figure 1 The following diagram explains the relationship between fuzzy $\omega\alpha$-closed set, fuzzy pre g^*-closed set and some existing fuzzy closed sets.

$$\begin{tikzcd}
\text{fuzzy }\omega\text{-closed set} & & \text{fuzzy closed set} \arrow[ll] \arrow[rr] \arrow[dl] \arrow[dr] & & \text{fuzzy semi-closed set} \\
& \text{fuzzy pre-closed set} \arrow[dl] \arrow[dr] & & \text{fuzzy }\alpha\text{-closed set} \arrow[ll] \arrow[dr] & \\
\text{fuzzy pre }g^*\text{-closed set} \arrow[rr] & & \text{fuzzy }gp\text{-closed set} & & \text{fuzzy }\omega\alpha\text{-closed set} \arrow[ll]
\end{tikzcd}$$

4 Conclusion

The fuzzification of Topology primarily the sets $\omega\alpha$-closed sets and $\omega\alpha$-open sets [8] yield a new class of fuzzy sets namely fuzzy $\omega\alpha$-closed set and fuzzy $\omega\alpha$-open set. Moreover, these newly discovered fuzzy sets were used to define fuzzy pre g^*-closed set. Properties of the aforementioned new fuzzy sets in fuzzy topology were established.

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