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A Note on Symmetric Orthogonal Circulant Matrices

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Abstract

We prove that there is no symmetric orthogonal circulant matrix with complex entries and that the number of distinct symmetric orthogonal circulant matrices of order n is $2^{\lfloor n/2 \rfloor + 1}$.

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1 Introduction and preliminaries

A symmetric orthogonal circulant matrix is the simplest orthogonal matrix such that the sum of elements in each column equals 1. Orthogonal matrices with this property can be applied to data anonymization in the field of statistics [3]. Craigen and Kharaghani [1] proved that there is no circulant Hermitian complex Hadamard matrix of order larger than 4. In this paper we study the relationship between the eigenvalues of an orthogonal circulant matrix, a real orthogonal circulant matrix, and a symmetric orthogonal circulant matrix. By using these relationships, we prove that there is no symmetric orthogonal circulant matrix with complex entries and that the number of distinct symmetric orthogonal circulant matrices of order n is $2^{\lfloor n/2 \rfloor + 1}$.

We denote by $(a_{ij})_{0 \leq i.j \leq n-1}$ the $n \times n$ matrix whose $(i+1, j+1)$-th entry is a_{ij}. We also denote by $\mathrm{circ}(a_0, \ldots, a_{n-1})$ the circulant matrix whose first row is $(a_0 \ldots a_{n-1})$, by $\mathrm{diag}(d_0, \ldots, d_{n-1})$ the diagonal matrix whose i-th diagonal element is d_i, and by F_n the $n \times n$ Fourier matrix $({\omega_n}^{ij})_{0 \leq i,j \leq n-1}$, where $\omega_n =$

$e^{2\pi\sqrt{-1}/n}$. It is well known [2] that an $n \times n$ matrix C_n is circulant if and only if there exist $d_0, \ldots, d_{n-1} \in \mathbb{C}$ such that

$$\begin{aligned}
C_n &= F_n \mathrm{diag}(d_0, \ldots, d_{n-1}) {F_n}^{-1} = \frac{1}{n} F_n (d_i {\omega_n}^{-ij})_{0 \le i,j \le n-1} \\
&= \frac{1}{n} \left(\sum_{k=0}^{n-1} {\omega_n}^{ik} d_k {\omega_n}^{-kj} \right)_{0 \le i,j \le n-1} \\
&= \frac{1}{n} \mathrm{circ} \left(\sum_{k=0}^{n-1} d_k {\omega_n}^{-0k}, \sum_{k=0}^{n-1} d_k {\omega_n}^{-1k}, \ldots, \sum_{k=0}^{n-1} d_k {\omega_n}^{-(n-1)k} \right).
\end{aligned}$$

From this property, we shall derive our results.

2 Main results

We write

$$\frac{1}{n} \mathrm{circ} \left(\sum_{k=0}^{n-1} d_k {\omega_n}^{-0k}, \ldots, \sum_{k=0}^{n-1} d_k {\omega_n}^{-(n-1)k} \right) = \begin{pmatrix} v_0 \\ \vdots \\ v_{n-1} \end{pmatrix}.$$

Then it holds that

$$\begin{aligned}
v_0 \cdot v_0 &= \frac{1}{n^2} \sum_{i=0}^{n-1} \left(\sum_{k=0}^{n-1} d_k {\omega_n}^{-ik} \right) \left(\sum_{k=0}^{n-1} d_k {\omega_n}^{-ik} \right) \\
&= \frac{1}{n^2} \sum_{i=0}^{n-1} \left(\sum_{0 \le s,t \le n-1} d_s d_t {\omega_n}^{-i(s+t)} \right) \\
&= \frac{1}{n} \left({d_0}^2 + \sum_{t=1}^{n-1} d_t d_{n-t} \right),
\end{aligned} \tag{1}$$

$$\begin{aligned}
v_0 \cdot v_i &= \frac{1}{n^2} \sum_{j=0}^{n-1} \left(\sum_{k=0}^{n-1} d_k {\omega_n}^{-jk} \right) \left(\sum_{k=0}^{n-1} d_k {\omega_n}^{-(j-i)k} \right) \\
&= \frac{1}{n^2} \sum_{j=0}^{n-1} \left(\sum_{0 \le s,t \le n-1} d_s d_t {\omega_n}^{-j(s+t)+it} \right) \\
&= \frac{1}{n} \left({d_0}^2 + \sum_{t=1}^{n-1} d_t d_{n-t} {\omega_n}^{it} \right)
\end{aligned} \tag{2}$$

for $1 \le i \le n-1$. From these equatios, the following lemma is derived.

Lemma 2.1. *C_n is an orthogonal circulant matrix if and only if there exist $d_0, \ldots, d_{n-1} \in \mathbb{C}$ satisfying the following conditions:*
(i) ${d_0}^2 = 1$ *and* $d_i d_{n-i} = 1$ *for* $1 \leq i \leq n-1$.
(ii)

$$C_n = \frac{1}{n}\mathrm{circ}\left(\sum_{k=0}^{n-1} d_k {\omega_n}^{-0k}, \sum_{k=0}^{n-1} d_k {\omega_n}^{-1k}, \ldots, \sum_{k=0}^{n-1} d_k {\omega_n}^{-(n-1)k}\right).$$

Proof. By (1) and (2), a circulant matrix C_n is orthogonal if and only if $d_0, \ldots, d_{n-1} \in \mathbb{C}$ such that

$$ {d_0}^2 + \sum_{t=1}^{n-1} d_t d_{n-t} = n, \tag{3}$$

$$ {d_0}^2 + \sum_{t=1}^{n-1} d_t d_{n-t} {\omega_n}^{it} = 0 \tag{4}$$

for $1 \leq i \leq n-1$. (4) is equivalent to asserting that ${d_0}^2 + \sum_{t=1}^{n-1} d_t d_{n-t} x^t$ is divisible by $\sum_{t=0}^{n-1} x^t$. Hence (3) and (4) is equivalent to (i). □

Furthermore, we have the following lemmas

Lemma 2.2. *C_n is a real orthogonal circulant matrix if and only if there exist $d_0, \ldots, d_{n-1} \in \mathbb{C}$ satisfying the conditions of Lemma 2.1 and*
(iii) $d_i = \overline{d_{n-i}}$ *for* $1 \leq i \leq n-1$.

Proof. An orthogonal circulant matrix C_n is a real matrix if and only if

$$\sum_{k=0}^{n-1} d_k {\omega_n}^{-ik} = \sum_{k=0}^{n-1} \overline{d_k} {\omega_n}^{ik} \tag{5}$$

for $0 \leq i \leq n-1$. (5) is equivalent to asserting that $(d_0 - \overline{d_0}) + \sum_{i=1}^{n-1}(d_{n-i} - \overline{d_i})x^i$ is divisible by $x^n - 1$. Hence (5) is equivalent to (iii). Note that the condition $d_0 = \overline{d_0}$ can be omitted because of ${d_0}^2 = 1$. □

Lemma 2.3. *C_n is a symmetric orthogonal circulant matrix if and only if there exist $d_0, \ldots, d_{n-1} \in \mathbb{C}$ satisfying the following conditions*
(iv) $d_i = \pm 1$ *for* $0 \leq i \leq n-1$,
(v) $d_i = d_{n-i}$ *for* $1 \leq i \leq n-1$,
and the condition (ii) *in Lemma 2.1.*

Proof. An orthogonal circulant matrix C_n is symmetric if and only if

$$\sum_{k=0}^{n-1} d_k {\omega_n}^{-ik} = \sum_{k=0}^{n-1} d_k {\omega_n}^{-(n-i)k} \tag{6}$$

for $1 \le i \le n-1$. (6) is equivalent to asserting that $\sum_{i=1}^{n-1}(d_{n-i} - d_i)x^i$ is divisible by $\sum_{t=0}^{n-1} x^t$. Hence (6) is equivalent to (v), so that (i) can be replaced by (iv). □

These lemmas imply the following theorems.

Theorem 2.4. *A symmetric orthogonal circulant matrix is a real matrix.*

Proof. Suppose that C_n is a symmetric orthogonal circulant matrix. By (iv) and (v), we have $d_i = d_{n-i} = \overline{d_{n-i}}$ for $1 \le i \le n-1$. By Lemma 2.2, the theorem follows. □

Theorem 2.5. *The number of distinct symmetric orthogonal circulant matrices of order n is $2^{\lfloor n/2 \rfloor + 1}$.*

Proof. Let C_n and ${C_n}'$ be symmetric orthogonal circulant matrices and let

$${C_n}' = \frac{1}{n}\mathrm{circ}\left(\sum_{k=0}^{n-1} {d_k}'{\omega_n}^{-0k}, \sum_{k=0}^{n-1} {d_k}'{\omega_n}^{-1k}, \ldots, \sum_{k=0}^{n-1} {d_k}'{\omega_n}^{-(n-1)k}\right).$$

If $C_n = {C_n}'$, then $\mathrm{diag}(d_0, \ldots, d_{n-1}) = \mathrm{diag}({d_0}', \ldots, {d_{n-1}}')$. Hence, by Lemma 2.3, we see that there is a one-to-one correspondence between the set of all symmetric orthogonal circulant matrices and

$$\{(d_0, \ldots, d_{\lfloor n/2 \rfloor}) \mid d_0 = \pm 1, \ldots, d_{\lfloor n/2 \rfloor} = \pm 1\} = \{\pm 1\}^{\lfloor n/2 \rfloor + 1}.$$

□

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