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Hyper-Asymptotic Curves of a Weyl Hypersurface

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Abstract

In this paper, firstly, we obtained the differential equation of the hyper-asymptotic curves in W_n with respect to a rectilinear congruence. With the help of it, we defined the hyper-asymptotic curvature vector field and the hyper-asymptotic curve in W_n. Secondly, we described an asymptotic line of order p in W_n in W_{n+1} and a geodesic of order p in W_{n+1}. We gave necessary and sufficient condition to be an asymptotic line of a curve in W_n. And then we expressed the relation between geodesics in W_n and in W_{n+1}. Thirdly, we stated the relations among a hyper-asymptotic curve, an asymptotic line of second order and a geodesic of second order in W_n. Finally, we expressed the condition to be a geodesic of second order of a hyper-asymptotic curve.

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1 Introduction

A manifold with a conformal metric g_{ij} and a symmetric connection ∇_k satisfying the compatibility condition

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 \tag{1}$$

is called a Weyl space, which will be denoted by $W_n(g_{ij}T_k)$. The vector field T_k is named the complementary vector field. Under a renormalization of the metric tensor g_{ij} in the form

$$\tilde{g}_{ij} = \lambda^2 g_{ij} \tag{2}$$

the complementary vector field T_k is transformed by the law

$$\widetilde{T}_k = T_k + \partial_k \ln \lambda \tag{3}$$

where λ is a scalar function [1].

If, under the transformation (2), the quantity A is changed according to the rule

$$\widetilde{A} = \lambda^p A \tag{4}$$

then A is called a satellite of g_{ij} with weight $\{p\}$.

The prolonged covariant derivative of A is defined by [2]

$$\dot{\nabla}_k A = \nabla_k A - pT_k A. \tag{5}$$

Let $W_n(g_{ij}, T_k)$ be n-dimensional Weyl space and $W_{n+1}(g_{ab}, T_c)$ be $(n+1)$-dimensional Weyl space $(i, j, k = 1, 2, \ldots, n;\ a, b, c = 1, 2, \ldots, n+1)$. Let x^a and u^i be the coordinates of $W_{n+1}(g_{ab}, T_c)$ and $W_n(g_{ij}, T_k)$, respectively. The metrics of $W_n(g_{ij}, T_k)$ and $W_{n+1}(g_{ab}, T_c)$ are connected by the relations

$$g_{ij} = g_{ab} x_i^a x_j^b \tag{6}$$

where x_i^a is the covariant derivative of x^a with respect to u^i.

The prolonged covariant derivative of A with respect to u^k and x^c are $\dot{\nabla}_k A$ and $\dot{\nabla}_c A$, respectively. These are related by the conditions

$$\dot{\nabla}_k A = x_k^c \dot{\nabla}_c A. \tag{7}$$

Let the normal vector field n^a of $W_n(g_{ij}, T_k)$ be normalized by the condition $g_{ab} n^a n^b = 1$. The moving frame $\{x_i^a, n_a\}$ and its reciprocal $\{x_a^i, n^a\}$ are connected by the relations [1]

$$n^a n_a = 1,\ n_a x_i^a = 0,\ n^a x_a^i = 0,\ x_i^a x_a^j = \delta_i^j. \tag{8}$$

Since the weight of x_i^a is $\{0\}$, the prolonged covariant derivative of x_i^a relative to u^k is given by [1]

$$\dot{\nabla}_k x_i^a = \nabla_k x_i^a = w_{ik} n^a \tag{9}$$

where w_{ik} are the coefficients of the second fundamental form of $W_n(g_{ij}, T_k)$.

Let $\underset{r}{v}^a$ and $\underset{r}{v}^i$ be the contravariant components of the vector field $\underset{r}{v}$ relative to $W_{n+1}(g_{ab}, T_c)$ and $W_n(g_{ij}, T_k)$, respectively. Denoting the components of $\overset{r}{v}$ relative to $W_{n+1}(g_{ab}, T_c)$ and $W_n(g_{ij}, T_k)$ by $\overset{r}{v}_a$ and $\overset{r}{v}_i$, we have [3]

$$\underset{r}{v}^a = x_i^a \underset{r}{v}^i, \quad \overset{r}{v}_a = x_a^i \overset{r}{v}_i. \tag{10}$$

2 Preliminaries

A hyper-asymptotic curve in an Euclidean space of three dimensions was defined by Mishra [4, 5], who also studied the hyper-asymptotic curves in a hypersurface of a Riemannian space [6]. Mishra defined hyper-asymptotic curve as: A hyper-asymptotic curve is a curve on a surface which has the property that its rectifying plane at all points contain a line of a specified rectilinear congruence through that point. Rastogi [7] obtained the differential equation of hyper-asymptotic curves in a hypersurface of a Riemannian space by using different method from Mishra's method. For this, Rastogi used definition of an asymptotic curve of order p of Hayden [8] and definition of a geodesic of order p of Srivastava [9].

3 Hyper-Asymptotic Curves

Let v be a vector field of a rectilinear congruence in W_{n+1}. Let v^a be the contravariant components of v relative to W_{n+1}. Then we have

$$v^a = t^i x_i^a + r n^a \quad (i = 1, 2, \ldots, n;\ a = 1, 2, \ldots, n+1) \tag{11}$$

where t^i are the contravariant components of the vector field t in the hypersurface W_n and r is a parameter.

Let v^a be normalized by the condition

$$g_{ab} v^a v^b = 1. \tag{12}$$

Hence

$$\cos\theta = g_{ab} v^a n^b = r \tag{13}$$

where θ is the angle between v^a and n^b.

Using (12), we get

$$g_{ab} v^a v^b = 1 = g_{ij} t^i t^j + r^2 \tag{14}$$

$$1 - r^2 = \sin^2\theta = g_{ij} t^i t^j. \tag{15}$$

Let C be a curve with equations $x^i = x^i(s)$ in W_n (s is the arc length of C).

We know that if C is a curve in W_n, C is also a curve in W_{n+1}.

Let $\underset{1}{v}$ be the tangent vector field of C. Let us denote the contravariant components of $\underset{1}{v}$ relative to W_n and W_{n+1} by $\underset{1}{v}^i$ and $\underset{1}{v}^a$, respectively. There are $(n-1)$ normal vector fields of C relative to W_n in the form $\underset{2}{v}, \underset{3}{v}, \ldots, \underset{n}{v} : \underset{r}{v}$ $(r = 2, 3, \ldots, n)$. There are $(n-1)$ curvatures of C in the form $\underset{r}{K}$ $(r = 1, 2, \ldots, n-1)$ relative to W_n. Besides C has got n normal vector fields $\underset{r}{v}$ $(r = 2, 3, \ldots, n+1)$ and n curvatures $\bar{K}_r$ $(r = 1, 2, \ldots, n)$ relative to W_{n+1}.

Definition 3.1 *Geodesic surface is introduced as an analogue to the rectifying plane. It is defined by the tangent vector field* $\underset{1}{v}$ *of* C *and by the second normal vector field* $\underset{3}{v}$ *of* C *in* W_{n+1}.

If the vector field v lies in geodesic surface, it can be expressed as a linear combination of $\underset{1}{v}$ and $\underset{3}{v}$:

$$v^a = \alpha \underset{1}{v}^a + \beta \underset{3}{v}^a \ (a = 1, 2, \ldots, n+1) \tag{16}$$

where α and β are to be determined, $\underset{1}{v}^a$ and $\underset{3}{v}^a$ are the contravariant components of $\underset{1}{v}$ and $\underset{3}{v}$ relative to W_{n+1}.

Using (12) and (16), we have

$$t^i x_i^a + r n^a = \alpha \underset{1}{v}^a + \beta \underset{3}{v}^a \ (i = 1, 2, \ldots, n). \tag{17}$$

We know that

$$\underset{2}{v}^a = \underset{2}{v}^i x_i^a \tag{18}$$

where $\underset{2}{v}^i$ and $\underset{2}{v}^a$ are the contravariant components of the first normal vector field $\underset{2}{v}$ relative to W_n and W_{n+1}, respectively.

With the help of Frenet's Formulae, the prolonged covariant derivative of $\underset{2}{v}^a$ in the direction of $\underset{1}{v}$ is

$$\underset{1}{v}^b \dot{\nabla}_b \underset{2}{v}^a = \underset{1}{v}^k x_k^b \dot{\nabla}_b \underset{2}{v}^a = \underset{1}{v}^k \dot{\nabla}_k (\underset{2}{v}^i x_i^a) = \underset{1}{v}^k (\dot{\nabla}_k \underset{2}{v}^i) x_i^a + w_{ik} \underset{2}{v}^i \underset{1}{v}^k n^a \tag{19}$$

$$-\bar{K}_1 \underset{1}{v}^a + \bar{K}_2 \underset{3}{v}^a = (-K_1 \underset{1}{v}^i + K_2 \underset{3}{v}^i) x_i^a + w_{ik} \underset{2}{v}^i \underset{1}{v}^k n^a \tag{20}$$

where w_{ik} are the coefficients of the second fundamental form. Let us denote $w_{ik} \underset{2}{v}^i \underset{1}{v}^k$ by $\underset{21}{\tau}$. Then we have

$$-\bar{K}_1 \underset{1}{v}^a + \bar{K}_2 \underset{3}{v}^a = (-K_1 \underset{1}{v}^i + K_2 \underset{3}{v}^i) x_i^a + \underset{21}{\tau} n^a. \tag{21}$$

Using definition of hyper-asymptotic curve of Mishra [6] for Weyl space, we have $\bar{K}_1 = 0$ and $K_1 = 0$. Therefore, from (21)

$$\bar{K}_2 \underset{3}{v}^a = K_2 \underset{3}{v}^i x_i^a + \underset{21}{\tau} n^a \tag{22}$$

is obtained.

Using (22), $\underset{3}{v}^a$ is obtained as follows:

$$\underset{3}{v}^a = \frac{1}{\bar{K}_2} (K_2 \underset{3}{v}^i x_i^a + \underset{21}{\tau} n^a). \tag{23}$$

From (17) and (23), we have

$$t^i x_i^a + r n^a = \alpha \underset{1}{v}^a + \beta \frac{1}{\bar{K}_2}(K_2 \underset{3}{v}^i x_i^a + \underset{21}{\tau} n^a). \tag{24}$$

Multiplying (24) by $g_{ab} x_j^b$ and summing on a and b, we get

$$t^i g_{ij} = \alpha \underset{1}{v}^i g_{ij} + \beta \frac{1}{\bar{K}_2} \underset{3}{v}^i g_{ij} \ (j = 1, 2, \ldots, n) \tag{25}$$

where $\underset{1}{v}^a = \underset{1}{v}^i x_i^a$, $g_{ab} x_i^a x_j^b = g_{ij}$ and $g_{ab} n^a x_j^b = 0$ $(b = 1, 2, \ldots, n+1)$.

Multiplying (25) by $\underset{1}{v}^j$ and summing on i and j, we obtain

$$g_{ij} t^i \underset{1}{v}^j = \alpha \tag{26}$$

where $g_{ij} \underset{1}{v}^i \underset{1}{v}^j = 1$ and $g_{ij} \underset{3}{v}^i \underset{1}{v}^j = 0$.

Multiplying (24) by $g_{ab} n^b$ and summing on a and b, we find

$$r = \beta \frac{1}{\bar{K}_2} \underset{21}{\tau} \tag{27}$$

where $g_{ab} n^a n^b = 1$.

Using (26) and (27) in (25), we get

$$t^i g_{ij} = g_{kl} t^k \underset{1}{v}^l \underset{1}{v}^i g_{ij} + r \frac{K_2}{\underset{21}{\tau}} \underset{3}{v}^i g_{ij} \ (k, l = 1, 2, \ldots, n). \tag{28}$$

Multiplying (28) by g^{jm} and summing on j, we get

$$t^m = g_{kl} t^k \underset{1}{v}^l \underset{1}{v}^m + r \frac{K_2}{\underset{21}{\tau}} \underset{3}{v}^m \ (m = 1, 2, \ldots, n) \tag{29}$$

or

$$\frac{t^m}{r} = g_{kl} \frac{t^k}{r} \underset{1}{v}^l \underset{1}{v}^m + \frac{K_2}{\underset{21}{\tau}} \underset{3}{v}^m. \tag{30}$$

Taking h^m instead of $\frac{t^m}{r}$, we obtain

$$h^m = g_{kl} h^k \underset{1}{v}^l \underset{1}{v}^m + \frac{K_2}{\underset{21}{\tau}} \underset{3}{v}^m \tag{31}$$

or

$$K_2 \underset{3}{v}^m - \underset{21}{\tau} (h^m - g_{kl} h^k \underset{1}{v}^l \underset{1}{v}^m) = 0. \tag{32}$$

We shall call the differential equation of hyper-asymptotic curves for (32). For a congruence specified by the parameters h^m, solutions of the n equations (32) determine the hyper-asymptotic curves in W_n relative to that congruence.

Let us denote the left hand side of equation (32) by Γ^m which we shall call the contravariant components of the hyper-asymptotic curvature vector field:

$$\Gamma^m = K_2 \underset{3}{v}^m - \underset{21}{\tau} y^m = 0 \tag{33}$$

where $y^m = h^m - g_{kl} h^k \underset{1}{v}^l \underset{1}{v}^m$.

Definition 3.2 *A hyper-asymptotic curve of W_n with respect to a congruence can be defined as a curve whose hyper-asymptotic curvature vector field is a null vector field.*

4 An Asymptotic Line of Order p and A Geodesic of Order p

Definition 4.1 *We shall call an asymptotic line of order p ($p < n$) for a curve in W_n in W_{n+1} if $\bar{K}_r = K_r$ ($r = 1, 2, \ldots, p$).*

Theorem 4.2 *If C is an asymptotic line of order p in W_n in W_{n+1}, then $w_{ik} \underset{r}{v}^i \underset{1}{v}^k = 0$ ($r = 1, 2, \ldots, p$). The converse of this is also true.*

Proof:

Let C be an asymptotic line of order p in W_n in W_{n+1}. Therefore $\bar{K}_r = K_r$ where $\bar{K}_r$ and K_r are r-th curvatures of C relative to W_{n+1} and W_n, respectively.

From Frenet's Formulae,

$$\underset{1}{v}^b \dot{\nabla}_b \underset{r}{v}^a = -\bar{K}_{r-1} \underset{r-1}{v}^a + \bar{K}_r \underset{r+1}{v}^a \; (\bar{K}_0 = \bar{K}_{n+1} = 0) \text{ (in } W_{n+1}) \tag{34}$$

$$\underset{1}{v}^k \dot{\nabla}_k \underset{r}{v}^i = -K_{r-1} \underset{r-1}{v}^i + K_r \underset{r+1}{v}^i \; (K_0 = K_n = 0) \text{ (in } W_n) \tag{35}$$

are satisfied. Here, $\underset{1}{v}$ is the tangent vector field of C.

We know that $\underset{1}{v}^a = \underset{1}{v}^i x_i^a$.

Taking prolonged covariant derivative of $\underset{1}{v}^a$ in the direction of $\underset{1}{v}$ and using (34) and (35), we get

$$\begin{aligned} \underset{1}{v}^b \dot{\nabla}_b \underset{1}{v}^a &= (\underset{1}{v}^k \dot{\nabla}_k \underset{1}{v}^i) x_i^a + w_{ik} \underset{1}{v}^i \underset{1}{v}^k n^a \\ -\bar{K}_0 \underset{0}{v}^a + \bar{K}_1 \underset{2}{v}^a &= (-K_0 \underset{0}{v}^i + K_1 \underset{2}{v}^i) x_i^a + w_{ik} \underset{1}{v}^i \underset{1}{v}^k n^a \\ \bar{K}_1 \underset{2}{v}^a &= K_1 \underset{2}{v}^i x_i^a + w_{ik} \underset{1}{v}^i \underset{1}{v}^k n^a \\ (\bar{K}_1 - K_1) \underset{2}{v}^a &= w_{ik} \underset{1}{v}^i \underset{1}{v}^k n^a. \end{aligned} \tag{36}$$

Since C is an asymptotic line of order p, $\bar{K}_1 = K_1$ or $\bar{K}_1 - K_1 = 0$ i.e. $w_{ik}\underset{1}{v}^i\underset{1}{v}^k = 0$ where $\underset{2}{v}^i x_i^a = \underset{2}{v}^a$ and $x_k^b\dot{\nabla}_b = \dot{\nabla}_k$.

Taking prolonged covariant derivative of $\underset{2}{v}^a = \underset{2}{v}^i x_i^a$ in the direction of $\underset{1}{v}$ and using Frenet Formulae, we get

$$\begin{aligned}\underset{1}{v}^b\dot{\nabla}_b\underset{2}{v}^a &= (\underset{1}{v}^k\dot{\nabla}_k\underset{2}{v}^i)x_i^a + w_{ik}\underset{2}{v}^i\underset{1}{v}^k n^a\\ -\bar{K}_1\underset{1}{v}^a + \bar{K}_2\underset{3}{v}^a &= (-K_1\underset{1}{v}^i + K_2\underset{3}{v}^i)x_i^a + w_{ik}\underset{2}{v}^i\underset{1}{v}^k n^a\\ (-\bar{K}_1 + K_1)\underset{1}{v}^a + (\bar{K}_2 - K_2)\underset{3}{v}^a &= w_{ik}\underset{2}{v}^i\underset{1}{v}^k n^a. \end{aligned}\tag{37}$$

Since $-\bar{K}_1 + K_1 = 0$ and $\bar{K}_2 - K_2 = 0$, we have $w_{ik}\underset{2}{v}^i\underset{1}{v}^k = 0$ where $\underset{3}{v}^i x_i^a = \underset{3}{v}^a$.

Similarly, for $\underset{p-1}{v}^a = \underset{p-1}{v}^i x_i^a$

$$\begin{aligned}\underset{1}{v}^b\dot{\nabla}_b\underset{p-1}{v}^a &= (\underset{1}{v}^k\dot{\nabla}_k\underset{p-1}{v}^i)x_i^a + w_{ik}\underset{p-1}{v}^i\underset{1}{v}^k n^a\\ -\bar{K}_{p-2}\underset{p-2}{v}^a + \bar{K}_{p-1}\underset{p}{v}^a &= (-K_{p-2}\underset{p-2}{v}^i + K_{p-1}\underset{p}{v}^i)x_i^a + w_{ik}\underset{p-1}{v}^i\underset{1}{v}^k n^a\\ (-\bar{K}_{p-2} + K_{p-2})\underset{p-2}{v}^a + (\bar{K}_{p-1} - K_{p-1})\underset{p}{v}^a &= w_{ik}\underset{p-1}{v}^i\underset{1}{v}^k n^a. \end{aligned}\tag{38}$$

Since $-\bar{K}_{p-2} + K_{p-2} = 0$ and $\bar{K}_{p-1} - K_{p-1} = 0$, we get $w_{ik}\underset{p-1}{v}^i\underset{1}{v}^k = 0$ where $\underset{p}{v}^i x_i^a = \underset{p}{v}^a$.

Finally for $\underset{p}{v}^a = \underset{p}{v}^i x_i^a$

$$\begin{aligned}\underset{1}{v}^b\dot{\nabla}_b\underset{p}{v}^a &= (\underset{1}{v}^k\dot{\nabla}_k\underset{p}{v}^i)x_i^a + w_{ik}\underset{p}{v}^i\underset{1}{v}^k n^a\\ -\bar{K}_{p-1}\underset{p-1}{v}^a + \bar{K}_p\underset{p+1}{v}^a &= (-K_{p-1}\underset{p-1}{v}^i + K_p\underset{p+1}{v}^i)x_i^a + w_{ik}\underset{p}{v}^i\underset{1}{v}^k n^a\\ (-\bar{K}_{p-1} + K_{p-1})\underset{p-1}{v}^a + (\bar{K}_p - K_p)\underset{p+1}{v}^a &= w_{ik}\underset{p}{v}^i\underset{1}{v}^k n^a \end{aligned}\tag{39}$$

Since $-\bar{K}_{p-1} + K_{p-1} = 0$ and $\bar{K}_p - K_p = 0$, we have $w_{ik}\underset{p}{v}^i\underset{1}{v}^k = 0$ where $\underset{p+1}{v}^i x_i^a = \underset{p+1}{v}^a$.

Hence, while C is an asymptotic line of order p in W_n in W_{n+1}, $w_{ik}\underset{r}{v}^i\underset{1}{v}^k = 0$ $(r = 1, 2, \ldots, p)$ is obtained.

Conversely, suppose that $w_{ik}\underset{1}{v}^i\underset{1}{v}^k = 0$. From here, we get $\bar{K}_1 - K_1 = 0$ or $\bar{K}_1 = K_1$ i.e. C is an asymptotic line of order 1 in W_n in W_{n+1}. Let $w_{ik}\underset{2}{v}^i\underset{1}{v}^k = 0$. $-\bar{K}_1 + K_1 = 0$ and $\bar{K}_2 - K_2 = 0$ or $\bar{K}_1 = K_1$ and $\bar{K}_2 = K_2$ i.e. C is an asymptotic line of order 2 in W_n in W_{n+1}. If $w_{ik}\underset{p-1}{v}^i\underset{1}{v}^k = 0$, $-\bar{K}_{p-2} + K_{p-2} = 0$ and $\bar{K}_{p-1} - K_{p-1} = 0$ or $\bar{K}_{p-2} = K_{p-2}$ and $\bar{K}_{p-1} = K_{p-1}$ i.e. C is an asymptotic line of order $(p-1)$ in W_n in W_{n+1}. If $w_{ik}\underset{p}{v}^i\underset{1}{v}^k = 0$,

$-\bar{K}_{p-1} + K_{p-1} = 0$ and $\bar{K}_p - K_p = 0$ or $\bar{K}_{p-1} = K_{p-1}$ and $\bar{K}_p = K_p$ i.e. C is an asymptotic line of order p in W_n in W_{n+1}. The proof is completed.

Definition 4.3 *A curve in W_{n+1} is called a geodesic of order p of W_{n+1} if at every point of the curve, its first p curvatures relative to W_{n+1} are all zero i.e. $\bar{K}_r = 0$ $(r = 1, 2, \ldots, p)$.*

From Frenet's Formulae, we write

$$\underset{1}{v}^b \dot{\nabla}_b \underset{r}{v}^a = -\bar{K}_{r-1} \underset{r-1}{v}{}^a + \bar{K}_r \underset{r+1}{v}{}^a (r = 1, 2, \ldots, n+1), \quad \bar{K}_0 = \bar{K}_{n+1} = 0.$$

Using definition of a geodesic of order p of W_{n+1} and from above equation, we get

$$\underset{1}{v}^b \dot{\nabla}_b \underset{r}{v}^a = 0 \ (r = 1, 2, \ldots, p).$$

From here:

Corollary 4.4 *Let C be a geodesic of order p of W_{n+1} and $\underset{1}{v}$ be the tangent vector field of C. Then tangent vector field $\underset{1}{v}$ and $(p-1)$ normal vector fields $v_2, v_3, \ldots, v_p$ are undergo parallel displacement along the curve C.*

Theorem 4.5 *If a curve (which lies in W_n) is a geodesic of order p $(p < n)$ of W_{n+1} then it is both a geodesic of order p and an asymptotic line of order p in W_n. The converse of this is also true.*

Proof:

Let C (which lies in W_n) be a geodesic of order p of W_{n+1}. Then $\bar{K}_r = 0$ $(r = 1, 2, \ldots, p)$. We know that $\underset{1}{v}^a = \underset{1}{v}^i x_i^a$. From Frenet's Formulae, $\bar{K}_1 \underset{2}{v}^a = K_1 \underset{1}{v}^i x_i^a + w_{ik} \underset{1}{v}^i \underset{1}{v}^k n^a$. Since $\bar{K}_1 = 0$, we get $K_1 = 0$ and $w_{ik} \underset{1}{v}^i \underset{1}{v}^k = 0$. Since $K_1 = 0$, C is a geodesic of order 1 in W_n. Furthermore, since $\bar{K}_1 = K_1$ (besides $w_{ik} \underset{1}{v}^i \underset{1}{v}^k = 0$), C is an asymptotic line of order 1 in W_n.

From $\underset{2}{v}^a = \underset{2}{v}^i x_i^a$ and $0 = K_2 \underset{3}{v}^i x_i^a + w_{ik} \underset{2}{v}^i \underset{1}{v}^k n^a$, we get $K_2 = 0$ and $w_{ik} \underset{2}{v}^i \underset{1}{v}^k = 0$. Since $K_2 = 0$, C is a geodesic of order 2 of W_n. Since $\bar{K}_2 = K_2$ and $w_{ik} \underset{2}{v}^i \underset{1}{v}^k = 0$, C is an asymptotic of order 2 of W_n.

Repeating the above procedure, from $\bar{K}_r = 0$ $(r = 1, 2, \ldots, p)$, we obtain $K_r = 0$ and $w_{ik} \underset{r}{v}^i \underset{1}{v}^k = 0$ $(r = 1, 2, \ldots, p)$. Since $K_r = 0$, C is a geodesic of order p of W_n. Since $\bar{K}_r = K_r$ and $w_{ik} \underset{r}{v}^i \underset{1}{v}^k = 0$, C is an asymptotic line of order p of W_n.

Conversely, if C is a geodesic of order p and an asymptotic line of order p in W_n, then $K_r = 0$ and $w_{ik} \underset{r}{v}^i \underset{1}{v}^k = 0$ $(r = 1, 2, \ldots, p)$. From here, $\bar{K}_r = 0$ $(r = 1, 2, \ldots, p)$ is obtained i.e. C is a geodesic of order p in W_{n+1}. The proof is completed.

With the help of (33), Theorem 4.2 and Definition 4.3:

Theorem 4.6 *If C is a hyper-asymptotic curve and an asymptotic line of order 2 in W_n, then C is a geodesic of order 2 in W_n or v^m is a null vector field.*

Proof:
If C is a hyper-asymptotic curve, from (33) $\Gamma^m = K_2 \underset{3}{v}^m - \underset{21}{\tau} y^m = 0$ is valid. Since C is also an asymptotic line of order 2 in W_n, using Theorem 4.2, $\underset{21}{\tau} = w_{ik} \underset{2}{v}^i \underset{1}{v}^k = 0$ is obtained. Since both $\Gamma^m = 0$ and $\underset{21}{\tau} = 0$, we have $K_2 = 0$ i.e. C is a geodesic of order 2 of W_n or $\underset{3}{v}^m$ is a null vector field.

Theorem 4.7 *If C is a hyper-asymptotic curve and a geodesic of order 2 in W_n, then C is an asymptotic line of order 2 in W_n or y^m is a null vector field.*

Proof:
Let C be a hyper-asymptotic curve and a geodesic of order 2 in W_n. Then, with the help of (33), $\Gamma^m = 0$ and $K_2 = 0$ are valid. Thus $\underset{21}{\tau} = w_{ik} \underset{2}{v}^i \underset{1}{v}^k = 0$ i.e. C is an asymptotic line of order 2 in W_n or y^m is a null vector field.

Corollary 4.8 *Let us denote the magnitude of vector field Γ by K_h : $K_h^2 = g_{ij}\Gamma^i\Gamma^j$*

Let us calculate it:

$$\begin{aligned} K_h^2 &= g_{ij}\Gamma^i\Gamma^j \\ K_h^2 &= g_{ij}(K_2 \underset{3}{v}^i - \underset{21}{\tau} y^i)(K_2 \underset{3}{v}^j - \underset{21}{\tau} y^j) \\ K_h^2 &= K_2^2 - 2K_2 \underset{21}{\tau} g_{ij} \underset{3}{v}^i y^j + \underset{21}{\tau}^2 g_{ij} y^i j^j \end{aligned} \tag{40}$$

is obtained where $g_{ij} \underset{3}{v}^i \underset{3}{v}^j = 1$.

Using $y^i = h^i - g_{kl} h^k \underset{1}{v}^l \underset{1}{v}^i$, we get

$$\begin{aligned} g_{ij} y^i y^j &= g_{ij}(h^i - g_{kl} h^k \underset{1}{v}^l \underset{1}{v}^i)(h^j - g_{kl} h^k \underset{1}{v}^l \underset{1}{v}^j) \\ g_{ij} y^i y^j &= g_{ij} h^i h^j - (g_{ij} h^i \underset{1}{v}^j)^2 \end{aligned} \tag{41}$$

where $g_{ij} \underset{1}{v}^i \underset{1}{v}^j = 0$.

Using $h^i = \frac{t^i}{r}$, (13) and (15), we have

$$g_{ij} h^i h^j = \frac{1}{r^2} g_{ij} t^i t^j = \frac{\sin^2\theta}{\cos^2\theta} = \mathrm{tg}^2\theta. \tag{42}$$

Let α be the angle between h^i and $\underset{1}{v}^j$. Then

$$\cos\alpha\,\mathrm{tg}\theta = g_{ij}h^i\underset{1}{v}^j. \tag{43}$$

Using (41), (42) and (43), we have

$$g_{ij}y^iy^j = \mathrm{tg}^2\theta - \cos^2\alpha\,\mathrm{tg}^2\theta = \mathrm{tg}^2\theta\sin^2\alpha. \tag{44}$$

Since $\sphericalangle(h^i, \underset{3}{v}^j) = \frac{\pi}{2} - \alpha$, we get

$$\cos(\frac{\pi}{2} - \alpha)\,\mathrm{tg}\theta = \sin\alpha\,\mathrm{tg}\theta = g_{ij}h^i\underset{3}{v}^j. \tag{45}$$

Using (45), we obtain

$$\begin{aligned} g_{ij}\underset{3}{v}^iy^j &= g_{ij}\underset{3}{v}^i(h^j - g_{kl}h^k\underset{1}{v}^l\underset{1}{v}^j) \\ g_{ij}\underset{3}{v}^iy^j &= g_{ij}\underset{3}{v}^ih^j = \sin\alpha\,\mathrm{tg}\theta \end{aligned} \tag{46}$$

where $g_{ij}\underset{3}{v}^i\underset{1}{v}^j = 0$.

Using (44) and (46) in (40), we get

$$\begin{aligned} K_h^2 &= K_2^2 - 2K_2\underset{21}{\tau}\sin\alpha\,\mathrm{tg}\theta + \underset{21}{\tau}^2(\mathrm{tg}^2\theta\sin^2\alpha) \\ K_h^2 &= (K_2 - \underset{21}{\tau}\,\mathrm{tg}\theta\sin\alpha)^2 \\ K_h &= K_2 - \underset{21}{\tau}\,\mathrm{tg}\theta\sin\alpha. \end{aligned} \tag{47}$$

Corollary 4.9 *If $\theta = 0$, the hyper-asymptotic curve is a geodesic of order 2 in W_n.*

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