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Generalization of Rodrigues' Formula in Weyl Space

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Abstract

In this paper, a generalization of Rodrigues's Formula in Weyl space is expressed.

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1 Introduction

A manifold with a conformal metric g_{ij} and a symmetric connection ∇_k satisfying the compatibility condition

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 \tag{1}$$

is called a Weyl space that will be denoted by $W_n(g_{ij}, T_k)$. The vector field T_k is named the complementary vector field.

The prolonged derivative and prolonged covariant derivative of A are, respectively defined by ([1, 5])

$$\dot{\partial}_k A = \partial_k A - pT_k A \tag{2}$$

and

$$\dot{\nabla}_k A = \nabla_k A - pT_k A \tag{3}$$

where A is a satellite of g_{ij} with weight $\{p\}$.

2 Generalization of Rodrigues' Formula

Let us take a point on the line v. Let us denote its coordinates by $x^a + tv^a$ where $v^a = t^i x_i^a + rn^a$ and $g_{ab}v^a v^b = 1$. Let $x^a + tv^a$ be describe a curve which is tangent to the line v. Then the prolonged covariant derivative of $x^a + tv^a$ in the direction of $\underset{1}{v}^k$, which is tangent vector field of a curve C at a point on W_2, satisfies the following condition:

$$\underset{1}{v}^a + \underset{1}{v}^k(\dot{\nabla}_k t)v^a + t\underset{1}{v}^k \dot{\nabla}_k v^a = mv^a \quad (a = 1,2,3; k = 1,2) \tag{4}$$

where m is to be determined.

Multiplying (4) by $g_{ab}v^b$, we get

$$g_{ab}\underset{1}{v}^a v^b + \underset{1}{v}^k \dot{\nabla}_k t + tg_{ab}(\underset{1}{v}^k \dot{\nabla}_k v^a)v^b = m \quad (b = 1,2,3) \tag{5}$$

or

$$g_{ab}\underset{1}{v}^a v^b + \underset{1}{v}^k \dot{\nabla}_k t = m \tag{6}$$

where $g_{ab}v^a v^b = 1$ and $(\underset{1}{v}^k \dot{\nabla}_k v^a)v^b = 0$.

Using (6) in (4), we have

$$\underset{1}{v}^a + \underset{1}{v}^k(\dot{\nabla}_k t)v^a + t\underset{1}{v}^k \dot{\nabla}_k v^a = (g_{cb}\underset{1}{v}^c v^b + \underset{1}{v}^k \dot{\nabla}_k t)v^a \quad (c = 1,2,3) \tag{7}$$

or

$$\underset{1}{v}^k(x_k^a + t\dot{\nabla}_k v^a - g_{cb}x_k^c v^b v^a) = 0 \tag{8}$$

where $\underset{1}{v}^c = \underset{1}{v}^k x_k^c$ [7].

We know that:

1)

$$g_{ab}v^a x_i^b = g_{ab}(t^j x_j^a + rn^a)x_i^b = g_{ij}t^j = t_i \quad (i,j = 1,2). \tag{9}$$

2)

$$\begin{aligned} g_{ab}(\dot{\nabla}_k v^a)(\dot{\nabla}_l v^b) &= g_{ab}(D_k^i x_i^a + D_k n^a)(D_l^j x_j^b + D_l n^b) \\ &= g_{ij}D_k^i D_l^j + D_k D_l \\ &= D_k^i D_l^j (g_{ij} + h_i h_l) \\ &= G_{kl} \qquad (l = 1,2) \end{aligned} \tag{10}$$

where $D_k = -h_i D_k^i$ [7].

3)

$$\begin{aligned} g_{ab}(\dot{\nabla}_k v^a)x_j^b &= g_{ab}(D_k^i x_i^a + D_k n^a)x_j^b \\ &= g_{ij}D_k^i \\ &= D_{jk} \quad [4]. \end{aligned} \tag{11}$$

Multiplying (8) by $g_{ad}(\dot{\nabla}_l v^d)$ $(d = 1, 2, 3)$ and using (9), (10) and (11), we obtain

$$\underset{1}{v}^k (D_{kl} + tG_{kl}) = 0. \tag{12}$$

Eliminating the parameter t in (12), we have

$$\begin{aligned} &(D_{11}G_{12} - D_{12}G_{11})\underset{1}{v}^1 \underset{1}{v}^1 + \\ &+ (D_{11}G_{22} - D_{22}G_{11})\underset{1}{v}^1 \underset{1}{v}^2 + \\ &+ (D_{21}G_{22} - D_{22}G_{21})\underset{1}{v}^2 \underset{1}{v}^2 = 0 \end{aligned} \tag{13}$$

or implicitly

$$\varepsilon^{jl} D_{ij} G_{kl} \underset{1}{v}^i \underset{1}{v}^k = 0 \tag{14}$$

or

$$e^{jl} D_{ij} G_{kl} \underset{1}{v}^i \underset{1}{v}^k = 0 \tag{15}$$

where $\varepsilon^{jl} = e^{jl}/\sqrt{g}$, $\quad e^{12} = 1$, $\quad e^{21} = -1$, $\quad$ and $\quad e^{11} = e^{22} = 0$.

The curves satisfying the equation (15) on W_2 constitute the net of curves. This net is named as the intersector net.

Using (10) and (11), we get from (15)

$$e^{jl} g_{hi} D_j^h D_k^m D_l^s (g_{ms} + h_m h_s) \underset{1}{v}^i \underset{1}{v}^k = 0 \quad (m, s = 1, 2) \tag{16}$$

or

$$(De^{hs} g_{hi} g_{ms} D_k^m - De^{hs} g_{hi} D_k g_{sq} h^q) \underset{1}{v}^i \underset{1}{v}^k = 0 \quad (q = 1, 2) \tag{17}$$

where $e^{jl} D_s^h D_l^s = De^{hs}$, $\quad D = |D_j^h|$ $\quad$ and $\quad -D_k = D_k^m h_m$ or

$$(Dge_{im} D_k^m - Dge_{iq} D_k h^q) \underset{1}{v}^i \underset{1}{v}^k = 0 \tag{18}$$

where $e^{hs} g_{hi} g_{ms} = ge_{im}$ [7], $\quad g = |g_{ij}|$.

Taking m instead of q in the second term of (18), we get

$$e_{im}(D_k^m - D_k h^m) \underset{1}{v}^i \underset{1}{v}^k = 0 \tag{19}$$

where D and g are nonvanishing.

(19) is equivalent to (15).

On the other hand, the prolonged covariant derivative of Y: $v^a = t^i x_i^a + rn^a$ in the direction of $\underset{1}{v}^k$ is

$$\underset{1}{v}^k \dot{\nabla}_k v^a = \underset{1}{v}^k (\dot{\nabla}_k t^i - r\omega_{kl} g^{il}) x_i^a + \underset{1}{v}^k (t^i \omega_{ik} + \dot{\nabla}_k r) n^a \tag{20}$$

or

$$\underset{1}{v}^k\dot{\nabla}_k v^a = \underset{1}{v}^k(\dot{\nabla}_k t^i - r\omega_{kl}g^{il} - t^j\omega_{jk}\frac{t^i}{r} - \frac{1}{r}t^i\dot{\nabla}_k r)x_i^a + \underset{1}{v}^k(\frac{t^i}{r}\omega_{ik} + \frac{1}{r}\dot{\nabla}_k r)v^a \quad (21)$$

where $n^a = \frac{1}{r}(v^a - t^i x_i^a)$.

Let us denote the tangential component of (21) by $-DY$:

$$DY = \underset{1}{v}^k(-\dot{\nabla}_k t^i + r\omega_{kl}g^{il} + t^j\omega_{jk}\frac{t^i}{r} + \frac{1}{r}t^i\dot{\nabla}_k r)x_i^a. \quad (22)$$

DY is tangent to C if and only if the following equation is satisfied for some scalar q:

$$\underset{1}{v}^k(\dot{\nabla}_k t^i - r\omega_{kl}g^{il} - t^j\omega_{jk}\frac{t^i}{r} - \frac{1}{r}t^i\dot{\nabla}_k r)x_i^a = qx_k^a\underset{1}{v}^k. \quad (23)$$

Multiplying (23) by $g_{ab}x_m^b$, we have

$$\underset{1}{v}^k(\dot{\nabla}_k t_m - r\omega_{km} - t^j\omega_{jk}\frac{t_m}{r} - \frac{1}{r}t_m\dot{\nabla}_k r) = qg_{km}\underset{1}{v}^k \quad (24)$$

where $g_{ab}x_i^a x_m^b = g_{im}, \quad g_{im}g^{il} = \delta_m^l \quad$ and $\quad g_{im}t^i = t_m$.

Multiplying (24) by g^{ms}, we get

$$g^{ms}\underset{1}{v}^k(\dot{\nabla}_k t_m - r\omega_{km} - t^j\omega_{jk}\frac{t_m}{r} - \frac{1}{r}t_m\dot{\nabla}_k r) = q\underset{1}{v}^s \quad (25)$$

where $g_{km}g^{ms} = \delta_k^s$.

Multiplying (25) by $\varepsilon_{sh}\underset{1}{v}^h$, we obtain

$$\varepsilon_{sh}g^{ms}\underset{1}{v}^k(\dot{\nabla}_k t_m - r\omega_{km} - t^j\omega_{jk}\frac{t_m}{r} - \frac{1}{r}t_m\dot{\nabla}_k r)\underset{1}{v}^h = 0 \quad (26)$$

where $\varepsilon_{sh}\underset{1}{v}^s\underset{1}{v}^h = 0$ (the directions coincide), or

$$\varepsilon_{sh}\underset{1}{v}^k(\dot{\nabla}_k t^s - r\omega_{km}g^{ms} - t^j\omega_{jk}\frac{t^s}{r} - \frac{1}{r}t^s\dot{\nabla}_k r)\underset{1}{v}^h = 0 \quad (27)$$

or

$$\varepsilon_{sh}\underset{1}{v}^k(D_k^s - h^s D_k)\underset{1}{v}^h = 0 \quad (28)$$

where $D_k^s = \dot{\nabla}_k t^s - r\omega_{km}g^{ms}, \quad D_k = t^j\omega_{jk} + \dot{\nabla}_k r \quad$ [2] and $h^s = t^s/r$.

(26) is equivalent to (28). From here, we have seen that C is a curve of the intersector net.

Let κ denote the magnitude (signed magnitude) of DY in the direction of $\underset{1}{v}$ which is the tangent vector field of the curve C of the intersector net. Then, we have

$$\kappa\underset{1}{v}^i = \underset{1}{v}^k(-\dot{\nabla}_k t^i + r\omega_{kl}g^{il} + t^j\omega_{jk}\frac{t^i}{r} + \frac{1}{r}t^i\dot{\nabla}_k r). \quad (29)$$

Using (29) in (22), we have

$$DY + \kappa \underset{1}{v}^i x_i^a = DY + \kappa \underset{1}{v}^a = 0 \tag{30}$$

(30) is a generalization of Rodrigues's formula.

Remark: Generalization of Rodrigues' formula in E^3 was obtained by Pan [6].

If $Z : v^a = n^a$, then $\underset{1}{v}^k \dot{\nabla}_k v^a = \underset{1}{v}^k \dot{\nabla}_k n^a$. Using (20) and (23), we have

$DZ = \underset{1}{v}^k \dot{\nabla}_k n^a = -\underset{1}{v}^k \omega_{kl} g^{il} x_i^a$ and $-\underset{1}{v}^k \omega_{kl} g^{il} x_i^a = q\underset{1}{v}^i x_i^a$. From here, we get $\underset{1}{v}^k (w_{kl} - \underset{11}{\kappa} g_{kl}) = 0$ where $-\underset{11}{\kappa} = q$, $\underset{11}{\kappa} = \omega_{ij} \underset{1}{v}^i \underset{1}{v}^j$ and $\underset{11}{\kappa}$ is the normal curvature of W_2. This is the equation of lines of curvature [3] i.e. The net of lines of curvature is the intersector net of the normal congruence. Let κ denote the magnitude of DZ along a curve of the intersector net of Z. Since $\underset{1}{v}^k \omega_{kl} g^{il} = \kappa \underset{1}{v}^i$, we have $DZ = \kappa \underset{1}{v}^a$ or $DZ + \kappa \underset{1}{v}^a = 0$ or $\underset{1}{v}^k \dot{\nabla}_k n^a + \kappa \underset{1}{v}^a = 0$. This is a special case of (30).

Theorem 2.1 *If the tangential component of the prolonged covariant derivative of a congruence along a curve C is tangent to the curve C then C is a curve of the intersector net of the congruence.*

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