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Pseudo-Lines of Curvature of Weyl Space

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Abstract

In this paper, we have defined pseudo-line of curvature of Weyl space by means of an associate vector field. Besides, we have defined pseudo-principal directions and we have given condition of orthogonality of them.

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Keywords: Weyl space, pseudo-line of curvature, pseudo-principal direction

1 Introduction

A manifold with a conformal metric g_{ij} and a symmetric connection ∇_k satisfying the compatibility condition

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 \tag{1}$$

is called a Weyl space, which will be denoted by $W_n(g_{ij}, T_k)$. The vector field T_k is named the complementary vector field.

The prolonged derivative and prolonged covariant derivative of A are, respectively defined by [1, 3]

$$\dot{\partial}_k A = \partial_k A - pT_k A \tag{2}$$

and

$$\dot{\nabla}_k A = \nabla_k A - pT_k A \tag{3}$$

where A is a satellite of g_{ij} with weight $\{p\}$.

2 Preliminaries

Let C be a curve in W_2. Let $\underset{1}{v}^a$ and $\underset{1}{v}^i$ be the components of tangent vector field of C at a point P with respect to W_3 and W_2, respectively $(a = 1, 2, 3; i = 1, 2)$.

Let v^a be the contravariant components of a vector field along line of congruence passing through the point P and let it be normalized in the form $g_{ab}v^a v^b = 1 \quad (b = 1, 2, 3)$. v^a can be written as

$$v^a = t^i x_i^a + rn^a. \tag{4}$$

Let Z^a be linearly depend on $\underset{1}{v}^a$ and v^a and let it be orthogonal to $\underset{1}{v}^a$. Then Z^a can be written in the following form [6]:

$$Z^a = \alpha \underset{1}{v}^a + \beta v^a \tag{5}$$

where α and β are to be determined.

Multiplying (5) by $g_{ab}\underset{1}{v}^b$ and summing on a and b, we get

$$\begin{aligned} g_{ab}Z^a \underset{1}{v}^b &= \alpha + \beta g_{ab} v^a \underset{1}{v}^b \quad (b = 1, 2, 3) \\ 0 &= \alpha + \beta \cos\theta \end{aligned} \tag{6}$$

where θ is the angle between v^a and $\underset{1}{v}^b$.

Multiplying (5) by $g_{ab}v^b$ and summing on a and b, we have

$$g_{ab}Z^a v^b = \alpha \cos\theta + \beta \tag{7}$$

or

$$\begin{aligned} \cos(\theta - \frac{\pi}{2}) &= \quad \sin\theta = \alpha\cos\theta + \beta \\ \cos(\theta + \frac{\pi}{2}) &= -\sin\theta = \alpha\cos\theta + \beta. \end{aligned} \tag{8}$$

By means of (6) and (8), we get

$$\alpha = -\frac{\cos\theta}{\sin\theta}, \quad \beta = \frac{1}{\sin\theta} \quad \text{or} \quad \alpha = \frac{\cos\theta}{\sin\theta}, \quad \beta = -\frac{1}{\sin\theta}. \tag{9}$$

Using these values in (5), we obtain

$$Z^a = e\left(-\frac{\cos\theta}{\sin\theta}\underset{1}{v}^a + \frac{1}{\sin\theta}v^a\right) \tag{10}$$

or

$$Z^a = e\frac{1}{\sin\theta}[(-\cos\theta \underset{1}{v}^i + t^i)x_i^a + rn^a] \tag{11}$$

where $e = 1$ or $e = -1$ and $\underset{1}{v}^a = x_i^a \underset{1}{v}^i$.

Definition 2.1 *The vector field Z^a is called associate vector field.*

3 Pseudo Line of Curvature

Definition 3.1 *The curve is called pseudo-line of curvature if the associate vector fields intersect at its consecutive points.*

In that case, let P and Q be the consecutive points of a pseudo-line of curvature C and Z^a and $Z^a + \underset{1}{v}^k\dot{\nabla}_k Z^a$ be the associate vector fields at P and Q, respectively.

These associate vector fields intersect if and only if $\underset{1}{v}^a$, Z^a and $Z^c + \underset{1}{v}^k\dot{\nabla}_k Z^c$ are coplanar, i.e.

$$\varepsilon_{abc}\underset{1}{v}^a Z^b(Z^c + \underset{1}{v}^k\dot{\nabla}_k Z^c) = 0 \quad (k = 1,2; \quad c = 1,2,3) \tag{12}$$

or

$$\varepsilon_{abc}\underset{1}{v}^a Z^b(\underset{1}{v}^k\dot{\nabla}_k Z^c) = 0. \tag{13}$$

From (13),

$$\begin{aligned}
&\varepsilon_{abc}\underset{1}{v}^j x^a_j \frac{1}{\sin\theta}\{(-\cos\theta\underset{1}{v}^i + t^i)x^a_i + rn^a\} \\
&\{-\frac{(\dot{\nabla}_k\theta)\cos\theta}{\sin^2\theta}[(-\cos\theta\underset{1}{v}^h + t^h)x^c_h + rn^c] + \\
&+\frac{1}{\sin\theta}\{[(\dot{\nabla}_k\theta)\sin\theta\underset{1}{v}^h - \cos\theta(\dot{\nabla}_k\underset{1}{v}^h) + \dot{\nabla}_k t^h]x^c_h + \\
&+(-\cos\theta\underset{1}{v}^h + t^h)\omega_{hk}n^c + (\dot{\nabla}_k r)n^c - r\omega_{kl}g^{hl}x^c_h\}\}\underset{1}{v}^k = 0 \quad (j,h,l = 1,2)
\end{aligned} \tag{14}$$

where $\underset{1}{v}^a = \underset{1}{v}^j x^a_j$ [7], $\dot{\nabla}_k x^c_h = \omega_{hk}n^c$ [4] and $\dot{\nabla}_k n^c = b^h_k x^c_h = -\omega_{kl}g^{hl}x^c_h$, or

$$\begin{aligned}
&\{r\varepsilon_{abc}x^a_j x^b_i n^c \underset{1}{v}^j \frac{1}{\sin\theta}(-\cos\theta\underset{1}{v}^i + t^i)(-\frac{\cos\theta}{\sin^2\theta})(\dot{\nabla}_k\theta) + \\
&+\varepsilon_{abc}x^a_j x^b_i n^c \underset{1}{v}^j \frac{1}{\sin\theta}(-\cos\theta\underset{1}{v}^i + t^i)(-\cos\theta\underset{1}{v}^h + t^h)\omega_{hk} + \\
&+\varepsilon_{abc}x^a_j x^b_i n^c \underset{1}{v}^j \frac{1}{\sin^2\theta}(-\cos\theta\underset{1}{v}^i + t^i)(\dot{\nabla}_k r) - \\
&-r\varepsilon_{acb}x^a_j x^c_h n^b \underset{1}{v}^j \frac{1}{\sin\theta}(-\cos\theta\underset{1}{v}^h + t^h)(-\frac{\cos\theta}{\sin^2\theta})(\dot{\nabla}_k\theta) + \\
&-r\varepsilon_{acb}x^a_j x^c_h n^b \underset{1}{v}^j \frac{1}{\sin^2\theta}[\sin\theta(\dot{\nabla}_k\theta)\underset{1}{v}^h - \cos\theta(\dot{\nabla}_k\underset{1}{v}^h) + (\dot{\nabla}_k)t^h] + \\
&+r^2\varepsilon_{acb}x^a_j x^c_h n^b \underset{1}{v}^j \frac{1}{\sin^2\theta}\omega_{kl}g^{hl}\}\underset{1}{v}^k = 0
\end{aligned} \tag{15}$$

where $\varepsilon_{abc}x^a_j x^b_i x^c_h = 0$, $\varepsilon_{abc}x^a_j n^b n^c = 0$ and $\varepsilon_{abc} = -\varepsilon_{acb}$, or

$$\begin{aligned}&\{\varepsilon_{ji}v^j(-\cos\theta \underset{1}{v}^i + t^i)(-\cos\theta \underset{1}{v}^h + t^h)\omega_{hk} + \varepsilon_{ji}(-\cos\theta \underset{1}{v}^i + t^i)(\dot{\nabla}_k r) - \\ &- r\varepsilon_{ji}\underset{1}{v}^j[\sin\theta(\dot{\nabla}_k\theta)\underset{1}{v}^i - \cos\theta(\dot{\nabla}_k \underset{1}{v}^i) + \dot{\nabla}_k t^i] + r^2\varepsilon_{ji}\underset{1}{v}^j\omega_{kl}g^{il}\}\underset{1}{v}^k = 0\end{aligned} \tag{16}$$

where $\varepsilon_{acb}x^a_j x^b_i n^c = \varepsilon_{ji}$ and i is taken instead of h in the third and fourth terms of the above equation, or

$$\begin{aligned}&\varepsilon_{ij}\underset{1}{v}^j\{-(-\cos\theta \underset{1}{v}^i + t^i)[(-\cos\theta \underset{1}{v}^h + t^h)\omega_{hk} + \dot{\nabla}_k r] + \\ &+ r[\sin\theta(\dot{\nabla}_k\theta)\underset{1}{v}^i - \cos\theta(\dot{\nabla}_k \underset{1}{v}^i) + \dot{\nabla}_k t^i - r\omega_{kl}g^{il}]\}\underset{1}{v}^k = 0\end{aligned} \tag{17}$$

where $\varepsilon_{ji} = -\varepsilon_{ij}$.

(17) is the equation of pseudo lines of curvature in W_2.

Let us consider two special cases:

1) If the congruence be normal, $\theta = \frac{\pi}{2}$, $t = 0$ and $r = 1$. Under this condition, the equation (17) reduces to

$$\varepsilon_{ij}\underset{1}{v}^j\omega_{kl}g^{il}\underset{1}{v}^k = 0. \tag{18}$$

Definition 3.2 *If the prolonged covariant derivative of n^a in the direction of C is parallel to C, then C is called line of curvature.*

Using definition, we can write the following equation [2]:

$$\begin{aligned}\underset{1}{v}^c\dot{\nabla}_c n^a = \underset{1}{v}^k x^c_k \dot{\nabla}_c n^a = \underset{1}{v}^k\dot{\nabla}_k n^a = \lambda \underset{1}{v}^a \\ -\underset{1}{v}^k\omega_{kl}g^{il}x^a_i = \lambda \underset{1}{v}^i x^a_i.\end{aligned} \tag{19}$$

Multiplying (19) by $g_{ab}x^b_j$, we get

$$-\underset{1}{v}^k\omega_{kj} = \lambda \underset{1}{v}^i g_{ij} \tag{20}$$

where $g_{ab}x^a_i x^b_j = g_{ij}$, $\quad g^{il}g_{ij} = \delta^l_j$.

Multiplying (20) by $\underset{1}{v}^j$, we have

$$-\underset{11}{\kappa} = \lambda \tag{21}$$

where $g_{ij}\underset{1}{v}^i\underset{1}{v}^j = 1$, $\quad \omega_{kj}\underset{1}{v}^k\underset{1}{v}^j = \underset{11}{\kappa}$.

Using (21) in (20), we get

$$\underset{1}{v}^k(\omega_{kj} - \underset{11}{\kappa}g_{kj}) = 0 \tag{22}$$

where k is taken instead of i in the right hand side of the equality (20).

Eliminating $\underset{11}{\kappa}$ from (22), we have

$$\begin{aligned} &\underset{1}{v}^1\underset{1}{v}^1(\varepsilon^{12}g_{11}\omega_{12}+\varepsilon^{21}g_{12}\omega_{11})+\\ &+\underset{1}{v}^2\underset{1}{v}^1(\varepsilon^{12}g_{11}\omega_{22}+\varepsilon^{21}g_{22}\omega_{11})+\\ &+\underset{1}{v}^2\underset{1}{v}^2(\varepsilon^{12}g_{21}\omega_{22}+\varepsilon^{21}g_{22}\omega_{21})=0 \end{aligned} \tag{23}$$

or implicitly

$$\varepsilon^{ik}g_{ij}\omega_{kl}\underset{1}{v}^j\underset{1}{v}^l=0 \tag{24}$$

where $\varepsilon^{ik}=e^{ik}/\sqrt{g}$, $\quad e^{12}=1$, $\quad e^{21}=-1$ and $\quad e^{11}=e^{22}=0$.

Since $\varepsilon^{ik}g_{ij}=\varepsilon_{hj}g^{hk}$ [5], from (24), we have

$$\varepsilon_{hj}g^{hk}\omega_{kl}\underset{1}{v}^j\underset{1}{v}^l=0. \tag{25}$$

(22) is equivalent to (25). (25) is the equation of line of curvature in W_2.

From (18) and (25), we can express the following theorem:

Theorem 3.3 *For a normal congruence, the pseudo lines of curvature are lines of curvature.*

2) For a congruence occurred with tangent vector fields, we have $r=0$. Then (17) reduces to

$$\varepsilon_{ij}\underset{1}{v}^j(-\cos\theta\underset{1}{v}^i+t^i)(-\cos\theta\underset{1}{v}^h+t^h)\omega_{hk}\underset{1}{v}^k=0. \tag{26}$$

From (26), we get

$$\varepsilon_{ij}\underset{1}{v}^j(-\cos\theta\underset{1}{v}^i+t^i)=0 \tag{27}$$

or

$$\omega_{hk}(-\cos\theta\underset{1}{v}^h+t^h)\underset{1}{v}^k=0. \tag{28}$$

From (27) and (28):

Theorem 3.4 *For the congruence occurred with tangent vector fields, the pseudo lines of curvature are such that the associate vector field at each point of them is either parallel or conjugate to the vector fields* $\underset{1}{v}$.

Since (17) is an equation of second degree in $\underset{1}{v}^1$ and $\underset{1}{v}^2$, there are two pseudo-lines of curvature.

Theorem 3.5 *There are two pseudo-lines of curvature which pass through each point of* W_2.

Let us denote (17) in the following form:

$$P(\underset{1}{v}^1)^2 + Q\underset{1}{v}^1\underset{1}{v}^2 + R(\underset{1}{v}^2)^2 = 0 \tag{29}$$

where

$$\begin{aligned}
P &= -r[\dot{\nabla}_1 t^2 + (\dot{\nabla}_1\theta)\sin\theta\underset{1}{v}^2 - \cos\theta(\dot{\nabla}_1\underset{1}{v}^2) - r\omega_{1l}g^{2l}]+\\
&\quad + (t^2 - \cos\theta\underset{1}{v}^2)[(t^m - \cos\theta\underset{1}{v}^m)\omega_{m1} + \dot{\nabla}_1 r]\\
Q &= r[\dot{\nabla}_1 t^1 + (\dot{\nabla}_1\theta)\sin\theta\underset{1}{v}^1 - \cos\theta(\dot{\nabla}_1\underset{1}{v}^1) - r\omega_{1l}g^{1l}-\\
&\quad - \dot{\nabla}_2 t^2 - (\dot{\nabla}_2\theta)\sin\theta\underset{1}{v}^2 + \cos\theta(\dot{\nabla}_2\underset{1}{v}^2) + r\omega_{2l}g^{2l}-\\
&\quad - (t^1 - \cos\theta\underset{1}{v}^1)[(t^m - \cos\theta\underset{1}{v}^m)\omega_{m1} + \dot{\nabla}_1 r]+\\
&\quad + (t^2 - \cos\theta\underset{1}{v}^2)[(t^m - \cos\theta\underset{1}{v}^m)\omega_{m2} + \dot{\nabla}_2 r]\\
R &= r[\dot{\nabla}_2 t^1 + (\dot{\nabla}_2\theta)\sin\theta\underset{1}{v}^1 - \cos\theta\dot{\nabla}_2\underset{1}{v}^1 - r\omega_{2l}g^{1l}]-\\
&\quad - (t^1 - \cos\theta\underset{1}{v}^1)[(t^m - \cos\theta\underset{1}{v}^m)\omega_{m2} + \dot{\nabla}_2 r].
\end{aligned} \tag{30}$$

The directions given by (30) are orthogonal if and only if

$$g_{11}R - g_{12}Q + g_{22}P = 0 \tag{31}$$

is satisfied [8]. Using (30) in (31), we have

$$\begin{aligned}
&- 2r\dot{\nabla}_1 t_2 + 2r\dot{\nabla}_2 t_1 - 2r(\dot{\nabla}_1\theta)\sin\theta\overset{1}{v}_2 + 2r(\dot{\nabla}_2\theta)\sin\theta\overset{1}{v}_1 + 2r\cos\theta\dot{\nabla}_1\overset{1}{v}_2-\\
&- 2r\cos\theta\dot{\nabla}_2\overset{1}{v}_1 + 2(t^m - \cos\theta\underset{1}{v}^m)\omega_{m1}(t_2 - \cos\theta\overset{1}{v}_2)-\\
&- 2(t^m - \cos\theta\underset{1}{v}^m)\omega_{m2}(t_1 - \cos\theta\overset{1}{v}_1) + 2(\dot{\nabla}_1 r)(t_2 - \cos\theta\overset{1}{v}_2)-\\
&- 2(\dot{\nabla}_2 r)(t_1 - \cos\theta\overset{1}{v}_1) + r^2\omega_{11}(g_{22}g^{21} + g_{12}g^{11}) + r^2\omega_{12}(g_{22}g^{22} + g_{12}g^{12})-\\
&- r^2\omega_{21}(g_{12}g^{21} + g_{11}g^{11}) - r^2\omega_{22}(g_{12}g^{22} + g_{11}g^{12}) = 0
\end{aligned} \tag{32}$$

where $g_{11}t^1 = t_1$, $g_{11}\underset{1}{v}^1 = \overset{1}{v}_1$, $g_{12}t^1 = t_2$, $g_{12}\underset{1}{v}^1 = \overset{1}{v}_2$, $g_{12}t^2 = t_1$, $g_{12}\underset{1}{v}^2 = \overset{1}{v}_1$, $g_{22}t^2 = t_2$, $g_{22}\underset{1}{v}^2 = \overset{1}{v}_2$, or

$$\begin{aligned}
&r\{(\dot{\nabla}_2 t_1 - \dot{\nabla}_1 t_2) + \sin\theta[(\dot{\nabla}_2\theta)\overset{1}{v}_1 - (\dot{\nabla}_1\theta)\overset{1}{v}_2] + \cos\theta[\dot{\nabla}_1\overset{1}{v}_2 - \dot{\nabla}_2\overset{1}{v}_1]\}+\\
&+ (t^m - \cos\theta\underset{1}{v}^m)\{\omega_{m1}(t_2 - \cos\theta\overset{1}{v}_2) - \omega_{m2}(t_1 - \cos\theta\overset{1}{v}_1)\}+\\
&+ \dot{\nabla}_1 r(t_2 - \cos\theta\overset{1}{v}_2) - \dot{\nabla}_2 r(t_1 - \cos\theta\overset{1}{v}_1) = 0
\end{aligned} \tag{33}$$

where $g_{ij}g^{jk} = \delta_i^k$

$$g_{11}g^{11} + g_{12}g^{21} = \delta_1^1 = 1$$
$$g_{21}g^{12} + g_{22}g^{22} = \delta_2^2 = 1$$
$$g_{11}g^{12} + g_{12}g^{22} = \delta_1^2 = 0$$
$$g_{21}g^{11} + g_{22}g^{21} = \delta_2^1 = 0.$$

Definition 3.6 *The directions which are tangential to the pseudo-lines of curvature are called pseudo-principal directions.*

From (33):

Theorem 3.7 *The pseudo-principal directions at any point of W_2 are orthogonal if and only if the condition* (33) *is satisfied.*

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