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On Symmetric Bi-Generalized Derivations of Incline Algebras

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Abstract

In this paper, we introduce the notion of symmetric bi-generalized derivation of incline algebras and investigated some related properties. Also, we introduce the notion of joinitive symmetric mapping and obtain some interesting results.

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1 Introduction

Z. Q. Cao, K. H. Kim and F. W. Roush [2] introduced the notion of incline algebras in their book. Some authors studied incline algebras and application. N. O. Alshehri [1] introduced the notion of derivation in incline algebras. In this paper, we introduce the concept of a symmetric bi-generalized derivation in incline algebras and give some properties of incline algebras. Also, In this paper, we introduce the notion of symmetric bi-generalized derivation of incline algebras and investigated some related properties. Also, we introduce the notion of joinitive symmetric mapping and obtain some interesting results.

2 Preliminary

An *incline algebra* is a set K with two binary operations denoted by “ $+$ ” and “ $*$ ” satisfying the following axioms, for all $x, y, z \in K$,

(K1) $x + y = y + x$,

(K2) $x + (y + z) = (x + y) + z$,

(K3) $x * (y * z) = (x * y) * z$,

(K4) $x * (y + z) = (x * y) + (x * z)$,

(K5) $(y + z) * x = (y * x) + (z * x)$,

(K6) $x + x = x$,

(K7) $x + (x * y) = x$,

(K8) $y + (x * y) = y$.

For convenience, we pronounce “ $+$ ” (resp. “ $*$ ”) as addition (resp. multiplication). Every distributive lattice is an incline algebra. An incline algebra is a distributive lattice if and only if $x * x = x$ for all $x \in K$. Note that $x \leq y \Leftrightarrow x + y = y$ for all $x, y \in K$. It is easy to see that “ $\leq$ ” is a partial order on K and that for any $x, y \in K$, the element $x + y$ is the least upper bound of $\{x, y\}$. We say that $\leq$ is induced by operation $+$.

In an incline algebra K, the following properties hold, for all $x, y, a, b \in K$,

(K9) $x * y \leq x$ and $y * x \leq x$ for all $x, y \in K$,

(K10) $y \leq z$ implies $x * y \leq x * z$ and $y * x \leq z * x$, for all $x, y, z \in K$,

(K11) If $x \leq y$ and $a \leq b$, then $x + a \leq y + b$, and $x * a \leq y * b$.

Furthermore, an incline algebra K is said to be *commutative* if $x*y = y*x$ for all $x, y \in K$. A map f is *isotone* if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in K$.

A *subincline* of an incline algebra K is a non-empty subset M of K which is closed under the addition and multiplication. A subincline M is said to be an *ideal* if $x \in M$ and $y \leq x$ then $y \in M$. An element “0” in an incline algebra K is a *zero element* if $x + 0 = x = 0 + x$ and $x * 0 = 0 = 0 * x$ for any $x \in K$. An non-zero element “1” is called a *multiplicative identity* if $x * 1 = 1 * x = x$ for any $x \in K$. A non-zero element $a \in K$ is said to be a *left* (resp. *right*) *zero divisor* if there exists a non-zero $b \in K$ such hat $a * b = 0$ (resp. $b * a = 0$) A zero divisor is an element of K which is both a left zero divisor and a right zero divisor. An incline algebra K with multiplicative identity 1 and zero element 0 is called an *integral incline* if it has no zero divisors. By a *homomorphism* of inclines, we mean a mapping f from an incline algebra K into an incline algebra L such that $f(x+y) = f(x) + f(y)$ and $f(x * y) = f(x) * f(y)$ for all $x, y \in K$. A map $f : K \to K$ is *regular* if $f(0) = 0$. A subincline I of an incline algebra K is said to be *k-ideal* if $x + y \in I$ and $y \in I$, then $x \in I$. Let K be an incline algebra. An element $a \in K$ is called a *additively cancellative* if for all $a, b \in K$, $a + b = a + c \Rightarrow b = c$. If every element of K is additively cancellative, it is called *additively cancellative*.

Definition 2.1. *Let K be an incline algebra. A mapping $D(.,.) : K \times K \to K$ is called symmetric if $D(x, y) = D(y, x)$ holds for all $x, y \in K$.*

Definition 2.2. *Let K be an incline algebra and $x \in K$. A mapping $d(x) = D(x, x)$ is called trace of $D(.,.)$, where $D(.,.) : K \times K \to K$ is a symmetric mapping.*

Definition 2.3. *Let K be an incline algebra and let $D : K \times K \to K$ be a symmetric mapping. We call D a symmetric bi-derivation on K if it satisfies the following condition*

$$D(x * y, z) = D(x, z) * y + x * D(y, z)$$

for all $x, y, z \in K$.

Lemma 2.4. *Let K be an incline algebra and let $D : K \times K \to K$ be a symmetric bi-derivation of K. Then $D(0, x) = D(x, 0) = 0$ for all $x \in K$.*

3 Symmetric bi-generalized derivations of incline algebras

In what follows, let K denote an incline algebra with a zero element 0 unless otherwise specified.

Definition 3.1. *Let K be an incline algebra. A symmetric map $F : K \times K \to K$ is called a symmetric bi-generalized derivation of K if there exists a symmetric bi-derivation D such that*

$$F(x * y, z) = F(x, z) * y + x * D(y, z)$$

for all $x, y, z \in K$.

Example 3.2. *Let $K = \{0, a, b, 1\}$ be a set in which "+" and "$*$" is defined by*

$+$	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

$*$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

Then it is easy to check that $(K, +, *)$ is an incline algebra. Define a map $D : K \times K \to K$ by

$$D(x, y) = \begin{cases} 0 & \textit{if } (x, y) = (0, 0), (0, a), (a, 0), (0, b), (b, 0), (0, 1), (1, 0) \\ a & \textit{if } (x, y) = (a, a), (a, b), (b, a), (a, 1), (1, a) \\ b & \textit{if } (x, y) = (b, b), (1, 1), (b, 1), (1, b) \end{cases}$$

Then it is easy to prove that D is a symmetric bi-derivation of K. Also, define $F : K \to K$ by

$$F(x, y) = \begin{cases} 0 & \textit{if } (x, y) = (0, 0), (0, a), (a, 0, (0, b), (b, 0), (0, 1), (1, 0) \\ a & \textit{if } (x, y) = (a, a), (a, b), (b, a), (a, 1), (1, a) \\ b & \textit{if } (x, y) = (b, b), (b, 1), (1, b) \\ 1 & \textit{if } (x, y) = (1, 1) \end{cases}$$

Then it is easily checked that F is a symmetric bi-generalized derivation associated with D of K.

Proposition 3.3. *Let D be a symmetric bi-derivation of K. If F is a symmetric bi-generalized derivation associated with D of K, then $F(0, 0) = 0$.*

Proof. Let F be a symmetric bi-generalized derivation associated with D of K. Then we have

$$\begin{aligned} F(0, 0) &= F(0 * 0, 0) \\ &= F(0, 0) * 0 + 0 * D(0, 0) \\ &= 0 + 0 = 0 \end{aligned}$$

□

Proposition 3.4. *Let D be a symmetric bi-derivation of K. If F is a symmetric bi-generalized derivation associated with D of K, then Then $F(0, x) = F(x, 0) = 0$ for all $x \in K$.*

Proof. Let F be a symmetric bi-generalized derivation associated with D of K. Then we have

$$\begin{aligned} F(0, x) &= F(0 * 0, x) \\ &= F(0, x) * 0 + 0 * D(0, x) \\ &= 0 + 0 = 0 \end{aligned}$$

for every $x \in K$. Similarly, $F(x, 0) = 0$ for every $x \in K$.

$\square$

Proposition 3.5. *Let D be a symmetric bi-derivation of K and let F be a symmetric bi-generalized derivation associated with D of K. If d is a trace of F, then d is regular.*

Proof. Let d be a trace of F. Then

$$\begin{aligned} d(0) &= F(0,0) = F(x * 0, 0) \\ &= F(x,0) * 0 + x * D(0,x) \\ &= 0 + 0 = 0 \end{aligned}$$

for every $x \in K$.

This completes the proof.

$\square$

Proposition 3.6. *Let D be a symmetric bi-derivation of K and let F be a symmetric bi-generalized derivation associated with D of K. Then $F(x*y, z) \leq F(x, z) + D(y, z)$ for all $x, y, z \in K$.*

Proof. Let F be a symmetric bi-generalized derivation associated with D of K. Then by (K9), we have $F(x, z) * y \leq F(x, z)$ and $x * D(y, z) \leq D(y, z)$ for all $x, y, z \in K$. Hence we obtain, by (K11)

$$\begin{aligned} F(x * y, z) &= F(x, z) * y + x * D(y, z) \\ &\leq F(x, z) + D(y, z) \end{aligned}$$

for all $x, y, z \in K$. This completes the proof.

$\square$

Proposition 3.7. *Let D be a symmetric bi-derivation of integral incline K and let F be a symmetric bi-generalized derivation associated with D of K. Then for all $x, y, z \in K$,*

(1) *$a * F(x, y) = 0$ implies that $a = 0$ or $D = 0$.*

(2) *$F(x, y) * a = 0$ implies that $a = 0$ or $D = 0$.*

Proof. (1) Let $a * F(x, y) = 0$ for every $x, y \in K$. Replacing x by $x * z$ in this relation, we get

$$\begin{aligned} 0 &= a * F(x * z, y) = a * ((F(x, z) * y) + (x * D(y, z))) \\ &= a * (F(x, z) * y) + a * (x * D(y, z)) \\ &= a * (x * D(y, z)). \end{aligned}$$

By putting $x = 1$, we have $a * D(y, z) = 0$ for all $y, z \in K$. Since K is an integral incline, i.e., it has no zero divisors, $a = 0$ or $D(y, z) = 0$ for all $y, z \in K$. Hence $a = 0$ or $D = 0$.

(2) Similarly, we can prove (2).

$\square$

Let K be an incline algebra and let F be a symmetric bi-generalized derivation associated with symmetric bi-derivation D of K. For a fixed element $a \in K$, let us define a map $d_a : K \to K$ such that $d_a(x) = F(x, a)$ for every $x \in K$.

Proposition 3.8. *Let D be a symmetric bi-derivation of K and let F be a symmetric bi-generalized derivation associated with D of K. Then d_a is regular.*

Proof. Let D be a symmetric bi-derivation of K and let F be a symmetric bi-generalized derivation associated with D of K. Then

$$d_a(0) = F(0, a) = F(0 * 0, a) = F(0, a) * 0 + 0 * D(0, a) = 0 + 0 = 0$$

This completes the proof. □

Proposition 3.9. *Let D be a symmetric bi-derivation of K and let F be a symmetric bi-generalized derivation associated with D of K. Then $d_a(x * y) = d_a(x) * y + x * D(y, a)$ for all $x, y \in K$.*

Proof. Let D be a symmetric bi-derivation of K and let F be a symmetric bi-generalized derivation associated with D of K. Then

$$d_a(x * y) = F(x * y, a) = F(x, a) * y + x * D(y, a) = d_a(x) * y + x * D(y, a)$$

for all $x, y \in K$. This completes the proof. □

Proposition 3.10. *Let D be a symmetric bi-derivation of K and let F be a symmetric bi-generalized derivation associated with D of K. Then $d_0(x) = 0$ for all $x \in K$.*

Proof. Let D be a symmetric bi-derivation of K and let F be a symmetric bi-generalized derivation associated with D of K. Then

$$d_0(x) = F(x, 0) = F(0 * 0, x) = F(0, x) * 0 + 0 * D(0, x) = 0 + 0 = 0$$

for all $x \in K$. This completes the proof. □

Let K be an incline algebra and let $F : K \times K \to K$ be a symmetric mapping. We call F a *jointive mapping* if it satisfies

$$F(x + y, z) = F(x, z) + F(y, z)$$

for all $x, y, z \in K$.

Proposition 3.11. *Let K be an incline algebra and let F be a joinitive symmetric bi-generalized derivation associated with symmetric bi-derivation D of K. Then $F(x * y, z) \le F(x, z)$ for all $x, y, z \in K$.*

Proof. Let D be a symmetric bi-derivation of K and let F be a joinitive symmetric bi-generalized derivation associated with D of K. Then

$$F(x, z) = F(x + x * y, z) = F(x, z) + F(x * y, z),$$

which implies that $F(x * y, z) \le F(x, z)$ for all $x, y, z \in K$. This completes the proof.

□

Proposition 3.12. *Let K be an incline algebra and let d be a trace of joinitive symmetric bi-generalized derivation F associated with symmetric bi-derivation D of K. Then $x * D(x, y) \le d(x)$ for all $x, y \in K$.*

Proof. Let d be a trace of joinitive symmetric bi-generalized derivation F associated with symmetric bi-derivation D of K. Then

$$\begin{aligned} d(x) &= F(x, x) = F(x + x * y, x) \\ &= F(x, x) + F(x, x) * y + x * D(x, y) \\ &= d(x) + d(x) * y + x * D(x, y) \\ &= d(x) + x * D(x, y) \end{aligned}$$

for all $x, y \in K$. This implies that $x * D(x, y) \le d(x)$. This completes the proof.

□

Proposition 3.13. *Let K be an incline algebra and let F be a joinitive symmetric bi-generalized derivation associated with symmetric bi-derivation D of K. Then F is an isotone symmetric bi-generalized derivation of K.*

Proof. Let $(x, y) \le (z, t)$ for $x, y, z, t \in K$. Then we have $x + z = z$ and $y + t = t$, and so $(x, y) + (z, t) = (z, t)$. Hence we obtain

$$F(z, t) = F((x, y) + (z, t)) = F(x, y) + F(z, t)$$

for all $x, y, z, t \in K$. This implies that $F(x, y) \le F(z, t)$ for all $x, y, z, t \in K$. This completes the proof.

□

Proposition 3.14. *Let K be an incline algebra and let F be a joinitive symmetric bi-generalized derivation associated with symmetric bi-derivation D of K. Then $d(x + y) = d(x) + d(y) + F(x, y)$ for all $x, y \in K$.*

Proof. Let $x, y \in K$.

$$\begin{aligned} d(x+y) &= F(x+y, x+y) = F(x, x+y) + F(y, x+y) \\ &= F(x,x) + F(x,y) + F(y,x) + F(y,y) \\ &= d(x) + d(y) + F(x,y). \end{aligned}$$

This completes the proof. □

Proposition 3.15. *Let D be a symmetric bi-derivation of K and let F be a joinitive symmetric bi-generalized derivation associated with D of K. If $x \leq y$, then $F(x,z) \leq F(y,z)$ for all $z \in K$.*

Proof. Let $x \leq y$. Then we have $x + y = y$ and

$$F(y,z) = F(x+y, z) = F(x,z) + F(y,z),$$

which implies that $F(x,z) \leq F(y,z)$ for all $z \in K$. This completes the proof. □

Let F be a symmetric bi-generalized bi-derivation associated with symmetric bi-derivation D of K. For fixed element $a \in K$, define a set $Fix_a(K)$ by

$$Fix_a(K) = \{x \in K \mid F(x,a) = x\}.$$

Proposition 3.16. *Let D be a symmetric bi-derivation of K and let F be a joinitive symmetric bi-generalized derivation associated with D of K. Then $Fix_a(K)$ is a subalgebra of K.*

Proof. Let $x, y \in Fix_a(K)$. Then Then

$$\begin{aligned} F(x+y, a) &= F(x,a) + F(y,a) \\ &= x + y, \end{aligned}$$

which implies that $x + y \in Fix_a(K)$. This completes the proof. □

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