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On Right f-Derivations of Γ-Incline Algebras

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Abstract

In this paper, we introduce the concept of a right f-derivation associated with a function f in Γ-incline algebras and give some properties of Γ-incline algebras. Also, the concept of k-ideal is introduced in a Γ-incline algebra with respect to right f-derivations.

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1 Introduction

Incline algebra is a generalization of both Boolean and fuzzy algebra and it is a special type of semiring. It has both a semiring structure and a poset structure. It can also be used to represent automata and other mathematical systems, to study inequalities for non-negative matrices of polynomials. Z. Q. Cao, K. H. Kim and F. W. Roush [4] introduced the notion of incline algebras in their book and later it was developed by some authors [1, 2, 3, 5]. Ahn et al. [1] introduced the notion of quotient incline and obtained the structure of incline algebras. N. O. Alshehri [3] introduced the notion of derivation in incline algebra. Kim [5, 6] studied right derivation and generalized derivation of incline algebras and obtained some results. M. K. Rao etc introduced the concept of generalized right derivation of Γ-incline and obtain some results. In this paper, we introduce the concept of a right f-derivation associated with a function f in Γ-incline algebras and give some properties of Γ-incline algebras.

Also, the concept of k-ideal is introduced in a Γ-incline algebra with respect to right f-derivations.

2 Preliminaries

An *incline algebra* is a set K with two binary operations denoted by "$+$" and "$*$" satisfying the following axioms, for all $x, y, z \in K$,

(K1) $x + y = y + x$,

(K2) $x + (y + z) = (x + y) + z$,

(K3) $x * (y * z) = (x * y) * z$,

(K4) $x * (y + z) = (x * y) + (x * z)$,

(K5) $(y + z) * x = (y * x) + (z * x)$,

(K6) $x + x = x$,

(K7) $x + (x * y) = x$,

(K8) $y + (x * y) = y$.

Definition 2.1. *Let $(K, +)$ and $(\Gamma, +)$ be commutative semigroups. If there exists a mapping $K \times \Gamma \times K \to K((x, \alpha, y) = x\alpha y)$ such that it satisfies the following axioms, for all $x, y \in K$ and $\alpha, \beta \in \Gamma$,*

(K9) $x\alpha(y + z) = x\alpha y + x\alpha z$

(K10) $(x + y)\alpha z = x\alpha z + y\alpha z$

(K11) $x(\alpha + \beta)y = x\alpha y + x\beta y$

(K12) $x\alpha(y\beta z) = (x\alpha y)\beta z$

(K13) $x + x = x$

(K14) $x + x\alpha y = x$

(K15) $y + x\alpha y = y$

Then K is called a Γ-incline algebra(see [7]).

Example 2.2. *Let $K = [0, 1]$ and $\Gamma = N$. Define $+$ by $x + y = \max\{x, y\}$ and ternary operation is defined as $x\alpha y = \min\{x, \alpha, y\}$ for all $x, y \in K$ and $\alpha \in \Gamma$. Then K is an Γ-incline algebra*(see [7]).

Note that $x \leq y \Leftrightarrow x + y = y$ for all $x, y \in K$. It is easy to see that " $\leq$ " is a partial order on K and that for any $x, y \in K$, the element $x + y$ is the least upper bound of $\{x, y\}$. We say that $\leq$ is induced by operation $+$.

In an Γ-incline algebra K, the following properties hold.

(K16) $x\alpha y \leq x$ and $y\alpha x \leq x$ for all $x, y \in K$ and $\alpha \in \Gamma$

(K17) $y \leq z$ implies $x\alpha y \leq x\alpha z$ and $y\alpha x \leq z\alpha x$, for all $x, y, z \in K$ and $\alpha \in \Gamma$

(K18) If $x \leq y$ and $a \leq b$, then $x+a \leq y+b$, and $x\alpha a \leq y\alpha b$ for all $x, y, a, b \in K$ and $\alpha \in \Gamma$.

Furthermore, an Γ-incline algebra K is said to be *commutative* if $x\alpha y = y\alpha x$ for all $x, y \in K$ for all $\alpha \in \Gamma$.

A Γ-*subincline* of an Γ-incline algebra K is a non-empty subset I of K which is closed under the addition and multiplication. An Γ-subincline I is called an *ideal* if $x \in I$ and $y \leq x$ then $y \in I$. An element "0" in an Γ-incline algebra K is a *zero element* if $x + 0 = x = 0 + x$ and $x\alpha 0 = 0 = 0\alpha x$ for any $x \in K$ and $\alpha \in \Gamma$. An non-zero element "1" is called a *multiplicative identity* if $x\alpha 1 = 1\alpha x = x$ for any $x \in K$ and $\alpha \in \Gamma$. A non-zero element $a \in K$ is said to be a *left* (resp. *right*) *zero divisor* if there exists a non-zero $b \in K$ such hat $a\alpha b = 0$ (resp. $b\alpha a = 0$) for all $\alpha \in \Gamma$. A zero divisor is an element of K which is both a left zero divisor and a right zero divisor. An incline algebra K with multiplicative identity 1 and zero element 0 is called an *integral incline* if it has no zero divisors. By a homomorphism of Γ-incline algebras, we mean a mapping f from an Γ-incline algebra K into an Γ-incline algebra L such that $f(x+y) = f(x) + f(y)$ and $f(x\alpha y) = f(x)\alpha f(y)$ for all $x, y \in K$ for all $\alpha \in \Gamma$.

Definition 2.3. *Let K be an Γ-incline algebra. An element $a \in K$ is said to be idempotent of K if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$.*

Let K be an Γ-incline algebra. If every element of K is idempotent, then K is said to be *idempotent* Γ-*incline algebra*. An Γ-incline algebra K with unity 1 and zero element 0 is called an *integral* Γ-*incline* if it has no zero divisors.

Definition 2.4. *Let K be an Γ-incline algebra. By a right derivation of K, we mean a self map d of K satisfying the identities*

$$d(x + y) = d(x) + d(y) \text{ and } d(x\alpha y) = (d(x)\alpha y) + (d(y)\alpha x)$$

for all $x, y \in K$ and $\alpha \in \Gamma$.

3 Right f-derivations of Γ-incline algebras

In what follows, let K denote an Γ-incline algebra with a zero-element unless otherwise specified.

Definition 3.1. *Let K be a Γ-incline algebra and let $f : K \to K$ be a function. By a right f-derivation of K, we mean a self map d of K satisfying the identities*

$$d(x + y) = d(x) + d(y) \text{ and } d(x\alpha y) = (d(x)\alpha f(y)) + (d(y)\alpha f(x))$$

for all $x, y \in K$ and $\alpha \in \Gamma$.

Example 3.2. *Let $K = \{0, a, b, 1\}$ be a set in which "+" and "α" is defined by*

$+$	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

α	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

Then it is easy to check that $(K, +, \alpha)$ is a Γ-incline algebra. Define a map $d, f : K \to K$ by

$$d(x) = \begin{cases} a & \text{if } x = a, b, 1 \\ 0 & \text{if } x = 0 \end{cases} \qquad f(x) = \begin{cases} b & \text{if } x = a, b \\ 1 & \text{if } x = 1 \\ 0 & \text{if } x = 0. \end{cases}$$

Then we can see that d is a right f-derivation associated with a function f of K.

Proposition 3.3. *Let K be a commutative Γ-incline algebra and let f be a endomorphism on K. Then for a fixed $a \in K$, the mapping $d_a : K \to K$ given by $d_a(x) = f(x)\alpha a$, for every $x \in K$ and $\alpha \in \Gamma$, is a right f-derivation of K.*

Proof. Let K be a commutative Γ-incline algebra. Then for a fixed element $a \in K$ and $\alpha \in \Gamma$, we have

$$\begin{aligned} d_a(x\alpha y) &= f(x\alpha y)\alpha a = (f(x\alpha y)\alpha a) + (f(x\alpha y)\alpha a) \\ &= ((f(x)\alpha f(y))\alpha a) + ((f(x)\alpha f(y))\alpha a) \\ &= ((f(x)\alpha a)\alpha f(y)) + ((f(y)\alpha a)\alpha f(x)) \\ &= d_a(x)\alpha f(y) + d_a(y)\alpha f(x) \end{aligned}$$

for all $x, y \in K$ and $\alpha \in \Gamma$. This completes the proof. $\square$

Proposition 3.4. *Let K be a commutative Γ-incline algebra and f be a function of K. Then $d_{a+b} = d_a + d_b$ for all $a, b \in K$.*

Proof. Let K be a commutative Γ-incline algebra and $a, b \in K$. Then for all $c \in K$, we have

$$d_{a+b}(c) = f(c)\alpha(a + b) = (f(c)\alpha a) + (f(c)\alpha b) = d_a(c) + d_b(c) = (d_a + d_b)(c).$$

□

Proposition 3.5. *Let d be a right f-derivation associated with a function f of K. If $f(0) = 0$, then we have $d(0) = 0$.*

Proof. Let d be a right f-derivation associated with a function f of K. Then we have

$$\begin{aligned} d(0) &= d(0\alpha 0) = d(0)\alpha f(0) + d(0)\alpha f(0) \\ &= d(0)\alpha 0 + d(0)\alpha 0 = 0 + 0 = 0. \end{aligned}$$

for all $\alpha \in \Gamma$. □

Proposition 3.6. *Let K be an idempotent Γ-incline algebra and let d be a right f-derivation associated with a function f of K. Then $d(x) \leq f(x)$ for all $x \in K$ and $\alpha \in \Gamma$.*

Proof. Let K be an idempotent Γ-incline algebra and let d be a right f-derivation associated with a function f of K. Then

$$\begin{aligned} d(x) &= d(x\alpha x) = d(x)\alpha f(x) + d(x)\alpha f(x) \\ &= d(x)\alpha f(x) \leq f(x) \end{aligned}$$

from (K16) for all $x \in K$ and $\alpha \in \Gamma$. □

Proposition 3.7. *Let K be an Γ-incline algebra and let d be a right f-derivation associated with a function f of K. Then we have $d(x\alpha y) \leq d(x+y)$ for all $x, y \in K$ and $\alpha \in \Gamma$.*

Proof. Let $x, y \in K$ and $\alpha \in \Gamma$. By using (K16), we get $d(x)\alpha f(y) \leq d(x)$ and $d(y)\alpha f(x) \leq d(y)$. Thus we get $d(x\alpha y) = (d(x)\alpha f(y)) + (d(y)\alpha f(x)) \leq d(x) + d(y) = d(x + y)$. □

Proposition 3.8. *Let K be an idempotent Γ-incline algebra and let d be a right f-derivation of associated with a function f of K. Define $d^2(x) = d(d(x))$ for all $x \in K$. If $d^2 = d$ and $f \circ d = f$, then $d(x\alpha d(x)) = d(x)$ for all $x \in K$ and $\alpha \in \Gamma$.*

Proof. Let K be an idempotent Γ-incline algebra and d be a right f-derivation of associated with a function f of K.

$$\begin{aligned} d(x\alpha d(x)) &= (d(x)\alpha f(d(x))) + (d^2(x)\alpha f(x)) \\ &= (d(x)\alpha f(x)) + (d^2(x)\alpha f(x)) \\ &= d(x) + (d(x)\alpha f(x)) = d(x) \end{aligned}$$

for all $x \in K$ and $\alpha \in \Gamma$. □

Proposition 3.9. *Let K be an Γ-incline algebra and let d be a right f-derivation associated with a function f of K. Then for all $x, y \in K$ and $\alpha \in \Gamma$, we have $d(x\alpha y) \leq d(x)$ and $d(x\alpha y) \leq d(y)$.*

Proof. Let $x, y \in K$. Then by using (K14), we obtain

$$d(x) = d(x + x\alpha y) = d(x) + d(x\alpha y).$$

Hence we get $d(x\alpha y) \leq d(x)$. Also, $d(y) = d(y + (x\alpha y)) = d(y) + d(x\alpha y)$, and so $d(x\alpha y) \leq d(y)$. □

Definition 3.10. *Let K be an Γ-incline algebra. A mapping f is isotone if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in K$.*

Proposition 3.11. *Let d be a right f-derivation of an Γ-incline algebra K. Then d is isotone.*

Proof. Let $x, y \in K$ be such that $x \leq y$. Then $x + y = y$. Hence we have $d(y) = d(x + y) = d(x) + d(y)$, which implies $d(x) \leq d(y)$. This completes the proof. □

Proposition 3.12. *A sum of two f-right derivations associated with a function f of K is again a right f-derivation associated with a function f of K.*

Proof. Let d_1 and d_2 be two right f-derivations associated with a function f of K, respectively. Then we have for all $a, b \in K$ and $\alpha \in \Gamma$,

$$\begin{aligned} (d_1 + d_2)(a\alpha b) &= d_1(a\alpha b) + d_2(a\alpha b) \\ &= d_1(a)\alpha f(b) + d_1(b)\alpha f(a) + d_2(a)\alpha f(b) + d_2(b)\alpha f(a) \\ &= d_1(a)\alpha f(b) + d_2(a)\alpha f(b) + d_1(b)\alpha f(a) + d_2(b)\alpha f(a) \\ &= (d_1 + d_2)(a)\alpha f(b) + (d_1 + d_2)(b)\alpha f(a). \end{aligned}$$

Clearly, $(d_1 + d_2)(a + b) = (d_1 + d_2)(a) + (d_1 + d_2)(b)$ for all $a, b \in K$ and $\alpha \in \Gamma$. This completes the proof. □

Theorem 3.13. *Let K be a commutative Γ-incline algebra and let d_1, d_2 be right f-derivations associated with a function f of K, respectively. Define $(d_1 d_2)(x) = d_1(d_2(x))$ for all $x \in K$. If $d_1 d_2 = 0, d_1 \circ f = f \circ d_1$ and $d_2 \circ f = f \circ d_2$ and $f^2 = f$, then $d_2 d_1$ is a right f-derivation of K.*

Proof. Let K be a commutative Γ-incline algebra and $x, y \in K$ and $\alpha \in \Gamma$. Then we have

$$\begin{aligned}0 &= d_1d_2(x\alpha y) = d_1(d_2(x)\alpha f(y) + d_2(y)\alpha f(x)) \\ &= d_1d_2(x)\alpha f^2(y) + d_1(f(y))\alpha f(d_2(x)) + d_1d_2(y)\alpha f^2(x) + d_1(f(x))\alpha f(d_2(y)) \\ &= d_1(f(y))\alpha f(d_2(x)) + d_1(f(x))\alpha f(d_2(y)) = f(d_1(y))\alpha d_2(f(x)) + f(d_1(x))\alpha d_2(f(y)) \\ &= d_2(f(y))\alpha f(d_1(x)) + d_2(f(x))\alpha f(d_1(y))\end{aligned}$$

Then

$$\begin{aligned}d_2d_1(x\alpha y) &= d_2(d_1(x)\alpha f(y) + d_1(y)\alpha f(x)) \\ &= d_2d_1(x)\alpha f^2(y) + d_2(f(y))\alpha f(d_1(x)) + d_2d_1(y)\alpha f^2(x) + d_2(f(x))\alpha f(d_1(y)) \\ &= d_2d_1(x)\alpha f(y) + d_2d_1(y)\alpha f(x).\end{aligned}$$

Finally, for all $x, y \in K$, we get

$$d_2d_1(x + y) = d_2(d_1(x) + d_1(y)) = d_2d_1(x) + d_2d_1(y).$$

This implies that d_2d_1 is a right f derivation associated with a function f of K. □

Theorem 3.14. *Let K be an integral Γ-incline K and let I be a nonzero ideal of K. If d is a nonzero right f-derivation associated with a function f of K, then d is nonzero on I.*

Proof. Suppose that d is a nonzero right f-derivation associated with a function f but d is zero right f-derivation associated with a function f on I. Let $x \in I$. Then $d(x) = 0$. Also, let $y \in K$ and $\alpha \in \Gamma$. By (K16), we have $x\alpha y \leq x$, which implies $x\alpha y \in I$. Hence $d(x\alpha y) = 0$, which means

$$0 = d(x\alpha y) = (d(x)\alpha f(y)) + (d(y)\alpha f(x)).$$

By hypothesis, K has no nonzero divisors. So, we have $f(x) = 0$ for all $x \in I$ or $d(y) = 0$ for all $y \in K$. Since $f(x)$ is a nonzero function, we get $d(y) = 0$ for all $y \in K$. This contradicts our assumption, that is, d is a nonzero right derivation associated with a function f of K. Hence d is nonzero on I.

□

Theorem 3.15. *Let K be an integral Γ-incline K and let d be a nonzero right f-derivation associated with a function f of K. If I is a nonzero ideal of K and $a \in K$ and $\alpha \in \Gamma$ such that $a\alpha d(I) = 0$, then $a = 0$.*

Proof. By Theorem 3.14, we know that there is an element $m \in K$ such that $d(m) \neq 0$. Let I be a nonzero ideal of K such that $a\alpha d(I) = 0$, for $a \in K$ and $\alpha \in \Gamma$. Hence for $n \in I$, we have

$$\begin{aligned}0 &= a\alpha d(m\alpha n) = a\alpha(d(m)\alpha f(n) + d(n)\alpha f(m)) \\ &= a\alpha d(m)\alpha f(n) + a\alpha d(n)\alpha f(m) = a\alpha d(m)\alpha f(m).\end{aligned}$$

Since K is an integral Γ-incline, d is a nonzero right f-derivation associated with a function f on K and f is a nonzero function on I, we have $a = 0$.

□

Theorem 3.16. *Let K be an integral Γ-incline and d be a right f-derivation associated with a function f of K and $a \in K$. If $f(1) = 1$, then we have, for all $x \in K$ and $\alpha \in \Gamma$, $a\alpha d(x) = 0$ implies $a = 0$ or $d = 0$*

Proof. Let $a\alpha d(x) = 0$ for all $x \in K$. If we replace x by $x\alpha y$ for all $y \in K$, we get

$$\begin{aligned} 0 = a\alpha d(x) = a\alpha d(x\alpha y) &= a\alpha[(d(x)\alpha f(y)) + (d(y)\alpha f(x))] \\ &= (a\alpha(d(x)\alpha f(y))) + (a\alpha(d(y)\alpha f(x))) = a\alpha(d(y)\alpha f(x)). \end{aligned}$$

In this equation, by taking $x = 1$, we get $a\alpha d(y) = 0$. Since K is an integral Γ-incline algebra, we have $a = 0$ or $d = 0$.

□

Let d be a right f-derivation associated with a function f of K. Define a set $Kerd$ by

$$Kerd := \{x \in K \mid d(x) = 0\}$$

for all $x \in K$.

Proposition 3.17. *Let d be a right f-derivation associated with a function f of K. Then $Kerd$ is a subincline of K.*

Proof. Let $x, y \in Kerd$ and $\alpha \in \Gamma$. Then $d(x) = 0, d(y) = 0$ and

$$\begin{aligned} d(x\alpha y) &= (d(x)\alpha f(y)) + (d(y)\alpha f(x)) \\ &= (0\alpha f(y)) + (0\alpha f(x)) = 0 + 0 = 0, \end{aligned}$$

and

$$d(x + y) = d(x) + d(y) = 0 + 0 = 0.$$

Therefore, $x\alpha y, x + y \in Kerd$. This completes the proof. □

Proposition 3.18. *Let K be an integral Γ-incline algebra K and let d be a right f-derivation associated with a function f of K. If f is an one to one function, then $Kerd$ is an ideal of K.*

Proof. By Proposition 3.17, $Kerd$ is a subincline of K. Now let $x \in K$ and $y \in Kerd$ such that $x \leq y$. Then $d(y) = 0$ and

$$0 = d(y) = d(y + x\alpha y) = d(y) + d(x\alpha y) = 0 + d(x\alpha y),$$

which $d(x\alpha y) = 0$ for all $\alpha \in \Gamma$. Hence we have

$$0 = d(x\alpha y) = (d(x)\alpha f(y)) + (d(y)\alpha f(x)) = d(x)\alpha f(y).$$

Since K has no zero divisors, either $d(x) = 0$ or $f(y) = 0$. If $d(x) = 0$, then $x \in Kerd$. If $f(y) = 0$, then $y = 0$ and so $x \leq y = 0$, i.e., $x = 0$, which implies $x \in Kerd$. □

Definition 3.19. *A subincline I of an Γ-incline algebra K is called a k-ideal if $x+y \in I$ and $y \in I$, then $x \in I$.*

Example 3.20. *In Example 3.2, $I = \{0, a, b\}$ is an k-ideal of K.*

Proposition 3.21. *Let K be an Γ-incline algebra and let d be a right f-derivation associated with a function f of K. Then $Kerd$ is a k-ideal of K.*

Proof. From Proposition 3.17, $Kerd$ is a subincline of K. Let $x+y \in K$ and $y \in Kerd$. Then we have $d(x+y) = 0$ and $d(y) = 0$, and so

$$0 = d(x+y) = d(x) + d(y) = d(x) + 0 = d(x).$$

This implies $x \in Kerd$. □

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