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Composite Numbers with Large Prime Factors

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Abstract

Let $k \geq 3$ an arbitrary but fixed positive integer. In this note we examine the number of composite numbers not exceeding x such that all their prime factors are large, for example, greater than $x^{1/k}$.

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1 Preliminary Notes and Main Results

We shall see that the numbers such that all their prime factors are "large" have zero density. That is, the number of these numbers not exceeding x is $o(x)$. Therefore, we are interested in to obtain more precise formulae.

We need the following well-known Mertens's theorem.

Lemma 1.1 *The following asymptotic formula holds*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} + O\left(\frac{1}{\log^2 x}\right)$$

Proof. See, for example, [1].

Let $f(x)$ be a positive, continuous, strictly increasing function such that

$$\lim_{x \to \infty} f(x) = \infty$$

and $f(x) \leq \log x$. Now, let $A(x)$ be the number of positive integers not exceeding x such that all their prime factors are greater than $f(x)$. We have the following theorem.

Theorem 1.2 *The following asymptotic formula holds.*

$$A(x) = e^{-\gamma}\frac{x}{\log(f(x))} + O\left(\frac{x}{\log^2(f(x))}\right)$$

Therefore $A(x) = o(x)$.

Proof. It is an immediate consequence of the inclusion-exclusion principle and Lemma 1.1.

$$\begin{aligned} A(x) &= \lfloor x \rfloor - \sum_{p \leq f(x)} \left\lfloor \frac{x}{p} \right\rfloor + \cdots = x \prod_{p \leq f(x)} \left(1 - \frac{1}{p}\right) + E(x) = e^{-\gamma}\frac{x}{\log(f(x))} \\ &+ O\left(\frac{x}{\log^2(f(x))}\right) \end{aligned}$$

Since

$$\begin{aligned} |E(x)| &\leq \binom{\lfloor f(x) \rfloor}{0} + \binom{\lfloor f(x) \rfloor}{1} + \cdots + \binom{\lfloor f(x) \rfloor}{\lfloor f(x) \rfloor} = 2^{\lfloor f(x) \rfloor} \\ &\leq 2^{f(x)} \leq 2^{\log x} = x^{\log 2} \end{aligned}$$

The theorem is proved.

If we put $f(x) = \log x$ then we obtain the following corollary.

Corollary 1.3 *The following asymptotic formula holds.*

$$A(x) = e^{-\gamma}\frac{x}{\log\log x} + O\left(\frac{x}{(\log\log x)^2}\right)$$

Let $\pi_1(x) = \pi(x)$ be the number of primes not exceeding x. It is well-known the following equation (prime number theorem)

$$\pi_1(x) = \pi(x) = \frac{x}{\log x} + f(x)\frac{x}{\log x} \tag{1}$$

where $\lim_{x\to\infty} f(x) = 0$.

Let $\pi_m(x)$ be the number of numbers not exceeding x with exactly $m \geq 1$ prime factors in their prime factorization. It is well-known the following equation (Landau's theorem)

$$\pi_m(x) \sim \frac{x(\log\log x)^{m-1}}{(m-1)!\log x} \tag{2}$$

If $m = 1$ then equation (2) becomes equation (1).

The following equation is well-known (Mertens's theorem)

$$\sum_{p \le x} \frac{1}{p} = \log\log x + M + o(1) \tag{3}$$

where M is Mertens's constant.

Lemma 1.4 *If $0 < \alpha < \beta < 1$ then the following limit holds.*

$$\lim_{x\to\infty} \left(\sum_{x^\alpha < p < x^\beta} \frac{1}{p} \frac{1}{1 - \frac{\log p}{\log x}} \right) = \log\left(\frac{\beta}{1-\beta}\right) - \log\left(\frac{\alpha}{1-\alpha}\right) \tag{4}$$

Proof. Let us consider the partition $\alpha = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n = \beta$, where

$$c_n = \alpha_{i+1} - \alpha_i = \frac{\beta - \alpha}{n} \qquad (i = 0, 1, \ldots, n-1) \tag{5}$$

Equation (3) gives

$$\sum_{x^{\alpha_i} < p < x^{\alpha_{i+1}}} \frac{1}{p} = \log \alpha_{i+1} - \log \alpha_i + o(1) \qquad (i = 0, 1, \ldots, n-1) \tag{6}$$

Now, the function $\log x$ has strictly decreasing derivative $\frac{1}{x}$, consequently by the mean value theorem we have the inequality

$$\frac{\alpha_{i+1} - \alpha_i}{\alpha_{i+1}} < \log \alpha_{i+1} - \log \alpha_i < \frac{\alpha_{i+1} - \alpha_i}{\alpha_i} \qquad (i = 0, 1, \ldots, n-1) \tag{7}$$

Let us consider the inequality

$$x^{\alpha_i} < p < x^{\alpha_{i+1}} \qquad (i = 0, 1, \ldots, n-1). \tag{8}$$

This inequality implies the inequality

$$\frac{1}{1-\alpha_i} < \frac{1}{1 - \frac{\log p}{\log x}} < \frac{1}{1 - \alpha_{i+1}} \qquad (i = 0, 1, \ldots, n-1) \tag{9}$$

By integration theory we have

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{\alpha_i} \frac{1}{1-\alpha_i} (\alpha_{i+1} - \alpha_i) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{\alpha_{i+1}} \frac{1}{1-\alpha_{i+1}} (\alpha_{i+1} - \alpha_i)$$

$$= \int_\alpha^\beta \frac{1}{(1-x)x}\, dx = A \tag{10}$$

Let $\epsilon > 0$. There exists n, sufficiently large and depending of ϵ, such that

$$1+\frac{c_n}{\beta} < \frac{\alpha_{i+1}}{\alpha_i} = \frac{\alpha_i + c_n}{\alpha_i} = 1 + \frac{c_n}{\alpha_i} < 1 + \frac{c_n}{\alpha} < 1 + \epsilon \tag{11}$$

$$1-\epsilon < \frac{1}{1+\frac{c_n}{\alpha}} < \frac{\alpha_i}{\alpha_{i+1}} < \frac{1}{1+\frac{c_n}{\beta}} \tag{12}$$

$$\sum_{i=0}^{n-1} \frac{1}{\alpha_i}\frac{1}{1-\alpha_i}(\alpha_{i+1}-\alpha_i) = A + b \tag{13}$$

where $|b| < \epsilon$

$$\sum_{i=0}^{n-1} \frac{1}{\alpha_{i+1}}\frac{1}{1-\alpha_{i+1}}(\alpha_{i+1}-\alpha_i) = A + a \tag{14}$$

where $|a| < \epsilon$.

Equations (9), (6) and (7) give

$$\begin{aligned} & \frac{1}{1-\alpha_i}\frac{1}{\alpha_{i+1}}(\alpha_{i+1}-\alpha_i) + o(1) \\ < & \sum_{x^{\alpha_i} < p < x^{\alpha_{i+1}}} \frac{1}{p}\frac{1}{1-\frac{\log p}{\log x}} < \frac{1}{1-\alpha_{i+1}}\frac{1}{\alpha_i}(\alpha_{i+1}-\alpha_i) + o(1) \end{aligned} \tag{15}$$

Equations (15) and (11) give

$$\begin{aligned} & \sum_{x^{\alpha_i} < p < x^{\alpha_{i+1}}} \frac{1}{p}\frac{1}{1-\frac{\log p}{\log x}} < \frac{1}{1-\alpha_{i+1}}\frac{1}{\alpha_i}(\alpha_{i+1}-\alpha_i) + o(1) \\ = & \frac{1}{1-\alpha_{i+1}}\frac{1}{\alpha_{i+1}}\frac{\alpha_{i+1}}{\alpha_i}(\alpha_{i+1}-\alpha_i) + o(1) \leq (1+\epsilon)\frac{1}{1-\alpha_{i+1}}\frac{1}{\alpha_{i+1}}(\alpha_{i+1}-\alpha_i) \\ + & o(1) \end{aligned} \tag{16}$$

and equations (15) and (12) give

$$\sum_{x^{\alpha_i} < p < x^{\alpha_{i+1}}} \frac{1}{p}\frac{1}{1-\frac{\log p}{\log x}} > (1-\epsilon)\frac{1}{1-\alpha_i}\frac{1}{\alpha_i}(\alpha_{i+1}-\alpha_i) + o(1) \tag{17}$$

For sake of simplicity we put

$$A(x) = \sum_{x^{\alpha} < p < x^{\beta}} \frac{1}{p}\frac{1}{1-\frac{\log p}{\log x}} \tag{18}$$

Therefore

$$A(x) = \sum_{x^\alpha < p < x^\beta} \frac{1}{p} \frac{1}{1 - \frac{\log p}{\log x}} = \sum_{i=0}^{n-1} \left(\sum_{x^{\alpha_i} < p < x^{\alpha_{i+1}}} \frac{1}{p} \frac{1}{1 - \frac{\log p}{\log x}} \right) \tag{19}$$

From a certain value of x we have (see below) $|o(1)| < \epsilon$. Equations (19), (16) and (14) give

$$\begin{aligned} & A(x) \leq (1+\epsilon) \sum_{i=0}^{n-1} \frac{1}{\alpha_{i+1}} \frac{1}{1-\alpha_{i+1}} (\alpha_{i+1} - \alpha_i) + o(1) \leq (1+\epsilon)(A+a) \\ + \quad & o(1) \leq (1+\epsilon)(A+\epsilon) + \epsilon = A + \epsilon(A+1) + \epsilon(\epsilon+1) \\ \leq \quad & A + 2\epsilon(A+1) \end{aligned} \tag{20}$$

Equations (19), (17) and (13) give

$$\begin{aligned} & A(x) \geq (1-\epsilon) \sum_{i=0}^{n-1} \frac{1}{\alpha_i} \frac{1}{1-\alpha_i} (\alpha_{i+1} - \alpha_i) + o(1) \geq (1-\epsilon)(A+b) \\ + \quad & o(1) \geq (1-\epsilon)(A-\epsilon) - \epsilon = A - \epsilon(A+1) - \epsilon(1-\epsilon) \\ \geq \quad & A - 2\epsilon(A+1) \end{aligned} \tag{21}$$

Therefore, since ϵ can be arbitrarily small, equations (18), (20) and (21) give equation (4). Note that

$$A = \int_\alpha^\beta \frac{1}{(1-x)x} \, dx = \left[\log\left(\frac{x}{1-x} \right) \right]_\alpha^\beta = \log\left(\frac{\beta}{1-\beta} \right) - \log\left(\frac{\alpha}{1-\alpha} \right)$$

The theorem is proved.

Lemma 1.5 *Let us consider the composite numbers not exceeding x with exactly $t \geq 2$ prime factors in their prime factorization such that each prime factor is greater than $x^{\frac{1}{k}}$ and such that there is in the prime factorization some prime repeated, that is, some prime has multiplicity greater than 1. The number of these composite numbers not exceeding x we denote $\lambda_{t,k}(x)$. The following formula holds.*

$$\lambda_{t,k}(x) = o\left(\frac{x}{\log x} \right) \tag{22}$$

Proof. If $t = 2$ the proof is trivial since $p^2 \leq x$ implies $p \leq \sqrt{x}$ and consequently

$$\lambda_{2,k}(x) \leq \pi\left(\sqrt{x}\right) = \frac{\sqrt{x}}{\log \sqrt{x}}(1 + o(1)) = o\left(\frac{x}{\log x} \right)$$

If $t \geq 3$ we can choose a repeated prime p an therefore we have

$$p_1 \dots p_s p^2 \leq x$$

where $s = t-2 \geq 1$ and the p_i $(i = 1, \dots, s)$ are prime numbers. Consequently

$$p \leq \frac{\sqrt{x}}{\sqrt{p_1 \dots p_s}} \tag{23}$$

Note that since the p_i $(i = 1, \dots, s)$ and p are greater than $x^{\frac{1}{k}}$ we have

$$x^{\frac{s}{k}} < p_1 \dots p_s < x^{1-\frac{2}{k}} \tag{24}$$

and consequently

$$x^{\frac{2}{k}} < \frac{x}{p_1 \dots p_s} < x^{1-\frac{s}{k}} \tag{25}$$

and

$$\frac{1}{1-\frac{s}{k}} < \frac{1}{1-\frac{\log(p_1 \dots p_s)}{\log x}} < \frac{k}{2} \tag{26}$$

Therefore (see (23), (24), (25), (1) and (26))

$$\begin{aligned}
\lambda_{s,k}(x) &\leq \sum_{x^{\frac{s}{k}} < p_1 \dots p_s < x^{1-\frac{2}{k}}} \pi\left(\frac{\sqrt{x}}{\sqrt{p_1 \dots p_s}}\right) \\
&\leq \sum_{x^{\frac{s}{k}} < p_1 \dots p_s < x^{1-\frac{2}{k}}} C \frac{\sqrt{x}}{\sqrt{p_1 \dots p_s}} \frac{1}{\log\left(\frac{\sqrt{x}}{\sqrt{p_1 \dots p_s}}\right)} \\
&\leq Ck \frac{\sqrt{x}}{\log x} \sum_{x^{\frac{s}{k}} < p_1 \dots p_s < x^{1-\frac{2}{k}}} \frac{1}{\sqrt{p_1 \dots p_s}} \\
&\leq Ck \frac{\sqrt{x}}{\log x} \sum_{p_1 \dots p_s \leq x} \frac{1}{\sqrt{p_1 \dots p_s}} \qquad (27)
\end{aligned}$$

If we put $f(x) = \frac{1}{\sqrt{x}}$ then we have (see equation (2) and [1])

$$\begin{aligned}
& \sum_{p_1 \dots p_s \leq x} \frac{1}{\sqrt{p_1 \dots p_s}} = \frac{\sqrt{x}(\log\log x)^{s-1}}{(s-1)!\log x} + o\left(\frac{\sqrt{x}(\log\log x)^{s-1}}{(s-1)!\log x}\right) \\
+ \quad & \frac{1}{2}\int_{2^s}^{x} \frac{(\log\log t)^{s-1}}{(s-1)!\sqrt{t}\log t}\, dt + o\left(\int_{2^s}^{x} \frac{(\log\log t)^{s-1}}{\sqrt{t}\log t}\, dt\right) \\
= \quad & \frac{2\sqrt{x}(\log\log x)^{s-1}}{(s-1)!\log x}(1+o(1)) \qquad (28)
\end{aligned}$$

Since (L'Hospital's rule)

$$\lim_{x\to\infty} \frac{\frac{1}{2}\int_{2^s}^{x} \frac{(\log\log t)^{s-1}}{\sqrt{t}\log t}\, dt}{\frac{\sqrt{x}(\log\log x)^{s-1}}{\log x}} = 1$$

Equations (27) and (28) give equation (22). The theorem is proved.

Now, we can prove our main theorems.

Theorem 1.6 *Let $s+1 \geq 2$ an arbitrary but fixed positive integer. Let us consider the composite numbers not exceeding x with exactly $s+1$ different prime factors and such that each prime factor is greater than $x^{\frac{1}{k}}$. The number of these composite numbers not exceeding x we denote $\delta_{s+1,k}(x)$. The following formulae hold. If $s+1 = 2$ then*

$$\delta_{s+1,k}(x) \sim C_{s+1,k}\frac{x}{\log x} \tag{29}$$

where $C_{s+1,k} = \log(k-1)$

If $s+1 \geq 3$ then there exist two positive constants C_1 and C_2 depending of $s+1$ and k such that

$$C_1\frac{x}{\log x} < \delta_{s+1,k}(x) < C_2\frac{x}{\log x} \tag{30}$$

Proof. Note that $k > s+1$. Let us consider a composite number $p_1 \cdots p_s p_{s+1}$ not exceeding x with $s+1$ different prime factors p_i $(i = 1, \ldots, s+1)$ such that each prime factor is greater than $x^{\frac{1}{k}}$. That is,

$$p_1 \cdots p_s p_{s+1} \leq x \qquad (p_i > x^{\frac{1}{k}}) \quad (i = 1, \ldots, s+1) \tag{31}$$

Equation (31) gives

$$x^{\frac{1}{k}} < p_{s+1} \leq \frac{x}{p_1 \cdots p_s} \tag{32}$$

Note that

$$x^{\frac{s}{k}} < p_1 \cdots p_s < x^{1-\frac{1}{k}} \tag{33}$$

Equation (33) gives us

$$x^{\frac{1}{k}} < \frac{x}{p_1 \cdots p_s} < x^{1-\frac{s}{k}} \tag{34}$$

and

$$\frac{1}{1-\frac{s}{k}} < \frac{1}{1-\frac{\log(p_1 \cdots p_s)}{\log x}} < k \tag{35}$$

Equation (35) and equation (3) give

$$\begin{aligned}&\sum_{x^{\frac{s}{k}}<p_1\cdots p_s<x^{1-\frac{1}{k}}}\frac{1}{p_1\cdots p_s}\frac{1}{1-\frac{\log(p_1\cdots p_s)}{\log x}}\leq k\sum_{x^{\frac{s}{k}}<p_1\cdots p_s<x^{1-\frac{1}{k}}}\frac{1}{p_1\cdots p_s}\\ \leq\ & k\left(\sum_{x^{\frac{1}{k}}<p<x^{1-\frac{1}{k}}}\frac{1}{p}\right)^s\leq B \qquad (36)\end{aligned}$$

where B is a positive constant.

We can choose two finite sequences $\alpha_1,\alpha_2,\ldots,\alpha_s$ and $\beta_1,\beta_2,\ldots,\beta_s$ such that

$$\frac{1}{k}<\alpha_1<\beta_1<\alpha_2<\beta_2<\cdots<\alpha_s<\beta_s$$

$$\frac{s}{k}<\alpha_1+\alpha_2+\cdots+\alpha_s<\beta_1+\beta_2+\cdots+\beta_s<1-\frac{1}{k}$$

Therefore equation (35) and equation (3) give

$$\begin{aligned}&\sum_{x^{\frac{s}{k}}<p_1\cdots p_s<x^{1-\frac{1}{k}}}\frac{1}{p_1\cdots p_s}\frac{1}{1-\frac{\log(p_1\cdots p_s)}{\log x}}\geq \frac{1}{1-\frac{s}{k}}\sum_{x^{\frac{s}{k}}<p_1\cdots p_s<x^{1-\frac{1}{k}}}\frac{1}{p_1\cdots p_s}\\ \geq\ & \frac{1}{1-\frac{s}{k}}\prod_{i=1}^{s}\left(\sum_{x^{\alpha_i}<p<x^{\beta_i}}\frac{1}{p}\right)\geq D\end{aligned}$$

where D is a positive constant.

Equation (31), equation (32) and equation (33) give

$$\begin{aligned}&\delta_{s+1,k}(x)\\ =\ & \frac{1}{s+1}\left(\left(\sum_{x^{\frac{s}{k}}<p_1\cdots p_s<x^{1-\frac{1}{k}}}\left(\pi\left(\frac{x}{p_1\cdots p_s}\right)-\pi\left(x^{\frac{1}{k}}\right)\right)\right)-B_{s+1,k}(x)\right) \qquad (37)\end{aligned}$$

where $B_{s+1,k}(x)$ is the number of composite number with $s+1$ prime factors not exceeding x, such that only a prime factor is repeated twice and such that each prime factor is greater than $x^{\frac{1}{k}}$. Therefore, by Lemma 1.5 we have

$$B_{s+1,k}(x)=o\left(\frac{x}{\log x}\right) \qquad (38)$$

On the other hand, equation (1) and equation (2) give

$$\pi\left(x^{\frac{1}{k}}\right)\sum_{x^{\frac{s}{k}}<p_1\cdots p_s<x^{1-\frac{1}{k}}}1\leq\pi_s\left(x^{1-\frac{1}{k}}\right)\pi\left(x^{\frac{1}{k}}\right)=o\left(\frac{x}{\log x}\right) \qquad (39)$$

Equation (1) gives

$$\begin{aligned}
&\sum_{x^{\frac{s}{k}}<p_1\cdots p_s<x^{1-\frac{1}{k}}} \pi\left(\frac{x}{p_1\cdots p_s}\right) \\
= &\frac{x}{\log x} \sum_{x^{\frac{s}{k}}<p_1\cdots p_s<x^{1-\frac{1}{k}}} \frac{1}{p_1\cdots p_s}\frac{1}{1-\frac{\log(p_1\cdots p_s)}{\log x}} \\
+ &\frac{x}{\log x} \sum_{x^{\frac{s}{k}}<p_1\cdots p_s<x^{1-\frac{1}{k}}} f\left(\frac{x}{p_1\cdots p_s}\right)\frac{1}{p_1\cdots p_s}\frac{1}{1-\frac{\log(p_1\cdots p_s)}{\log x}} \qquad (40)
\end{aligned}$$

Let $\epsilon > 0$. There exists x_ϵ such that if $x \geq x_\epsilon$ (see (34)) we have $\left|f\left(\frac{x}{p_1\cdots p_s}\right)\right| < \epsilon$. Therefore (see (36))

$$\left|\sum_{x^{\frac{s}{k}}<p_1\cdots p_s<x^{1-\frac{1}{k}}} f\left(\frac{x}{p_1\cdots p_s}\right)\frac{1}{p_1\cdots p_s}\frac{1}{1-\frac{\log(p_1\cdots p_s)}{\log x}}\right| \leq \epsilon B$$

and since ϵ can be arbitrarily small we have

$$\lim_{x\to\infty}\left(\sum_{x^{\frac{s}{k}}<p_1\cdots p_s<x^{1-\frac{1}{k}}} f\left(\frac{x}{p_1\cdots p_s}\right)\frac{1}{p_1\cdots p_s}\frac{1}{1-\frac{\log(p_1\cdots p_s)}{\log x}}\right) = 0 \qquad (41)$$

Equations (37), (38), (39), (40) and (41) give

$$\delta_{s+1,k}(x) = \frac{x}{\log x}\left(\frac{1}{s+1}\sum_{x^{\frac{s}{k}}<p_1\cdots p_s<x^{1-\frac{1}{k}}} \frac{1}{p_1\cdots p_s}\frac{1}{1-\frac{\log(p_1\cdots p_s)}{\log x}}\right) + o\left(\frac{x}{\log x}\right) \qquad (42)$$

If $s+1=2$ equation (42) and Lemma 1.4 give us equation (29). If $s+1 \geq 3$ equation (42) and equations (36) give us equation (30). The theorem is proved.

Theorem 1.7 *Let $k \geq 3$ an arbitrary but fixed positive integer. Let us consider the composite numbers not exceeding x such that each prime factor is greater than $x^{\frac{1}{k}}$. The number of these composite numbers not exceeding x we denote $\chi_k(x)$. The following formulae hold. If $k=3$ then*

$$\chi_k(x) \sim (\log 2)\frac{x}{\log x} \qquad (43)$$

If $k \geq 4$ then the order of $\chi_k(x)$ is $\frac{x}{\log x}$. That is, there exist two positive constants D_1 and D_2 depending of k such that

$$D_1\frac{x}{\log x} < \chi_k(x) < D_2\frac{x}{\log x} \qquad (44)$$

Proof. Note that the number of prime factors in these composite numbers does not exceed $k-1$. Therefore the proof is an immediate consequence of Theorem 1.6 and Lemma 1.5. The theorem is proved.

In the following theorem we prove a result stronger (see equations (30) and (44)).

Theorem 1.8 *The following asymptotic formulae hold.*

$$\delta_{s+1,k}(x) \sim \frac{1}{(s+1)!} L \frac{x}{\log x} \tag{45}$$

where $L > 0$ depends of $s+1$ and k and is defined bellow (in the proof)

$$\chi_k(x) \sim D_3 \frac{x}{\log x} \tag{46}$$

where $D_3 > 0$ can be easily expressed in terms of the L's constants.

Proof. We shall prove equation (45), since equation (46) is an immediate consequence of equation (45). Let $\alpha = \frac{s}{k}$ and $\beta = 1 - \frac{1}{k}$. We have (see Theorem 1.6 and equation (42))

$$A(x) = \sum_{x^{\frac{s}{k}} < p_1 \cdots p_s < x^{1-\frac{1}{k}}} \frac{1}{p_1 \cdots p_s} \frac{1}{1 - \frac{\log(p_1 \cdots p_s)}{\log x}} \tag{47}$$

Let us consider the partition $\alpha = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{2^n} = \beta$, where $\alpha_{i+1} - \alpha_i = \frac{\beta - \alpha}{2^n}$ $(i = 0, 1, \ldots, 2^{n-1})$.

The inequality

$$x^{\alpha_i} < p_1 \cdots p_s < x^{\alpha_{i+1}} \tag{48}$$

give us

$$\frac{1}{1-\alpha_i} < \frac{1}{1 - \frac{\log(p_1 \cdots p_s)}{\log x}} < \frac{1}{1-\alpha_{i+1}} \tag{49}$$

Let us consider the following two functions depending of n.

$$F_1(2^n, x) = \sum_{i=0}^{2^n-1} \frac{1}{1-\alpha_i} \sum_{x^{\alpha_i} < p_1 \cdots p_s < x^{\alpha_{i+1}}} \frac{1}{p_1 \cdots p_s} \tag{50}$$

$$F_2(2^n, x) = \sum_{i=0}^{2^n-1} \frac{1}{1-\alpha_{i+1}} \sum_{x^{\alpha_i} < p_1 \cdots p_s < x^{\alpha_{i+1}}} \frac{1}{p_1 \cdots p_s} \tag{51}$$

We have (see equation (49))

$$F_1(2^n, x) < A(x) < F_2(2^n, x) \tag{52}$$

Now, $A(x) \leq B$ (see equation (36)) therefore $F_1(2^n, x) \leq B$.

Note that

$$\begin{aligned} 0 &\leq F_2(2^n, x) - F_1(2^n, x) \leq \frac{\beta - \alpha}{2^n} \frac{1}{(1-\beta)^2} \sum_{x^\alpha < p_1 \cdots p_s < x^\beta} \frac{1}{p_1 \cdots p_s} \\ &\leq \frac{\beta - \alpha}{2^n} \frac{1}{(1-\beta)^2} M \end{aligned} \tag{53}$$

since (see equation (3))

$$\sum_{x^\alpha < p_1 \cdots p_s < x^\beta} \frac{1}{p_1 \cdots p_s} < \left(\sum_{x^{\frac{1}{k}} < p < x^\beta} \frac{1}{p} \right)^s < M \tag{54}$$

Equation (3) and integration theory give us

$$\sum_{x^{\alpha_i} < p_1 \cdots p_s < x^{\alpha_{i+1}}} \frac{1}{p_1 \cdots p_s} = \frac{1}{s!} \int_{D_i} \frac{1}{x_1} \frac{1}{x_2} \cdots \frac{1}{x_s} \, dx_1 dx_2 \cdots dx_s + o(1) \tag{55}$$

where the domain D_i is

$$\alpha_i \leq x_1 + x_2 + \cdots + x_s \leq \alpha_{i+1}, \quad x_i \geq \frac{1}{k} \quad (i = 1, 2, \ldots, s) \tag{56}$$

Note that the frontier of D_i has zero content and the set of points $(x_1, x_2, \cdots, x_s)$ in D_i with repeated coordinates also has zero content. We also need, from the integration theory, a limit formula as (10) for functions of s variables. Note also that (mean value theorem)

$$\Pi_{i=1}^s \frac{1}{x_i + \Delta x_i} \Delta x_i < \Pi_{i=1}^s (\log(x_i + \Delta x_i) - \log x_i) < \Pi_{i=1}^s \frac{1}{x_i} \Delta x_i$$

Equation (50) and equation (55) give

$$\begin{aligned} F_1(2^n, x) &= \frac{1}{s!} \sum_{i=0}^{2^n - 1} \frac{1}{1 - \alpha_i} \int_{D_i} \frac{1}{x_1} \frac{1}{x_2} \cdots \frac{1}{x_s} \, dx_1 dx_2 \cdots dx_s + o(1) \\ &= \frac{1}{s!} S_1(n) + o(1) \end{aligned} \tag{57}$$

where $S_1(n)$ is the sequence

$$S_1(n) = \sum_{i=0}^{2^n - 1} \frac{1}{1 - \alpha_i} \int_{D_i} \frac{1}{x_1} \frac{1}{x_2} \cdots \frac{1}{x_s} \, dx_1 dx_2 \cdots dx_s \tag{58}$$

Note that

$$S_1(n) < S_1(n+1) \tag{59}$$

since the function $\frac{1}{1-x}$ is strictly increasing in the interval $(0,1)$ and

$$\begin{aligned}&\int_{D_i \bigcup D_{i+1}} \frac{1}{x_1}\frac{1}{x_2}\cdots\frac{1}{x_s}\ dx_1dx_2\cdots dx_s \\ = &\int_{D_i} \frac{1}{x_1}\frac{1}{x_2}\cdots\frac{1}{x_s}\ dx_1dx_2\cdots dx_s + \int_{D_{i+1}} \frac{1}{x_1}\frac{1}{x_2}\cdots\frac{1}{x_s}\ dx_1dx_2\cdots dx_s\end{aligned} \tag{60}$$

Since $F_1(2^n, x) \leq B$ (see above) equation (57) implies that $S_1(n)$ is bounded and hence there exists

$$\lim_{n\to\infty} S_1(n) = L_1 > 0 \tag{61}$$

Equation (51) and equation (55) give

$$\begin{aligned}F_2(2^n, x) &= \frac{1}{s!}\sum_{i=0}^{2^n-1} \frac{1}{1-\alpha_{i+1}} \int_{D_i} \frac{1}{x_1}\frac{1}{x_2}\cdots\frac{1}{x_s}\ dx_1dx_2\cdots dx_s + o(1) \\ &= \frac{1}{s!}S_2(n) + o(1)\end{aligned} \tag{62}$$

where $S_2(n)$ is the sequence

$$S_2(n) = \sum_{i=0}^{2^n-1} \frac{1}{1-\alpha_{i+1}} \int_{D_i} \frac{1}{x_1}\frac{1}{x_2}\cdots\frac{1}{x_s}\ dx_1dx_2\cdots dx_s \tag{63}$$

Note that

$$S_2(n) > S_2(n+1) \tag{64}$$

Therefore there exists

$$\lim_{n\to\infty} S_2(n) = L_2 \tag{65}$$

Equations (53), (57) and (62) give us

$$L_1 = L_2 = L > 0 \tag{66}$$

Equations (52), (57) and (62) give

$$\frac{1}{s!}S_1(n) + o(1) \leq A(x) \leq \frac{1}{s!}S_2(n) + o(1) \tag{67}$$

Let $\epsilon > 0$. There exists n_ϵ such that (see equations (61), (65) and (66)) $S_1(n_\epsilon) = L + a$ and $S_2(n_\epsilon) = L + b$, where $|a| < \epsilon$ and $|b| < \epsilon$. On the other hand from a certain value of x we have $|o(1)| < \epsilon$. Consequently equation (67) gives

$$\frac{1}{s!}L - 2\epsilon \leq \frac{1}{s!}(L + a) + o(1) \leq A(x) \leq \frac{1}{s!}(L + b) + o(1) \leq \frac{1}{s!}L + 2\epsilon \tag{68}$$

Now, ϵ can be arbitrarily small. Therefore

$$\lim_{x\to\infty} A(x) = \frac{1}{s!}L \tag{69}$$

Equations (42) and (69) give equation (45). The theorem is proved.

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References

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