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On the Fractional Matching Number of the Join and Corona of Graphs

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Abstract

A *fractional matching* of a graph $G = (V, E)$ is a function f from E to the interval $[0, 1]$ such that $\sum_{e\in\Gamma(v)} f(e) \leq 1$ for every $v \in V$, where $\Gamma(v)$ is the set of all edges incident to v. The fractional matching number of G, written $\alpha'_*(G)$, is the maximum of $\sum_{e\in E} f(e)$ over all fractional matchings f.

In this paper, we gave the fractional matching number of the join of some graphs, and the corona of some graphs.

Mathematics Subject Classification: 05C70

Keyword: integral matching number, fractional matching number, join, corona

1 Introduction

As mentioned in [2], the study of the chromatic number of a graph may be applied to scheduling problems. It can be used to determine the shortest possible time needed to schedule two or more committee meetings, with some committees having common members, without conflicts. The scheduling problem will be modeled by graph, with committees represented by vertices, and any two vertices are adjacent if the corresponding committees have common members.

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The chromatic number of the resulting graphs is the shortest possible time to schedule the committee meetings without conflicts. However, if the committees are willing to take a break in the middle and then to resume later, then the shortest possible time is the fractional matching number.

There are two different classes of maximum fractional matchings. One was introduced by Uhry [9] and the other was introduced by Muhlbacher et al. [8]. Uhry [9] presented the class of maximum fractional matchings for which the number of cycles in the support is minimized. On the other hand, Muhlbacher et al. [8] presented the class of maximum fractional matchings for which the number of edges assigned the value 1 is maximized. Pulleyblank [3] presented how the Edmonds-Gallai structure theorem for matchings in graphs can be applied to these two different classes of maximum fractional matchings.

Liu et al. [4] characterized graphs for which the fractional matching number equals the matching number, and graphs for which the fractional matching number is equal to half the number of vertices.

Motwani et al. [5] showed that several simple algorithms based on throwing balls into bins deliver a near-perfect fractional matching.

Choi et al. [6] proved that if G is an n-vertex connected graph that is neither K_1 nor K_3, then $\alpha'_f(G) - \alpha'(G) \leq (n-2)/6$ and $\alpha'_f(G)/\alpha'(G) \leq 3n/(2n+2)$. Both inequalities are sharp. They also characterized the infinite family of graphs where equalities hold.

West et al. [7] proved that for a graph G with n vertices, m edges, positive minimum degree d, and maximum degree D, $\alpha'_*(G) \geq max\{m/D, n - m/d, dn/D + d\}$.

The path $P_n = (v_1, v_2, \ldots, v_n)$ is the graph with distinct vertices v_1, v_2, ..., v_n and edges v_1v_2, v_2v_3, ..., $v_{n-1}v_n$. The *cycle* $C_n = [v_1, v_2, \ldots, v_n]$, $n \geq 3$, is the graph with vertices v_1, v_2, ..., v_n and edges v_1v_2, v_2v_3, ..., $v_{n-1}v_n$, v_nv_1. A *complete* graph of order n, denoted by K_n, is the graph in which every pair of distinct vertices are adjacent.

Let X and Y be sets. The *disjoint union* of X and Y, denoted by $X \dot{\cup} Y$, is found by combining the elements of X and Y, treating all elements to be distinct. Thus, $|X \dot{\cup} Y| = |X| + |Y|$. The *join* of two graphs G and H, denoted by $G + H$, is the graph with vertex-set $V(G+H) = V(G) \dot{\cup} V(H)$ and edge-set $E(G+H) = E(G) \dot{\cup} E(H) \dot{\cup} \{uv : u \in V(G), v \in V(H)\}$.

The *complement* of a graph G, denoted by $\overline{G}$, is a graph with the same vertex set as G and where two distinct vertices are adjacent if and only if they are not adjacent in G.

Let G be a graph of order n. The *corona* $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G and n copies of H, and then joining the *ith* vertex of G to every vertex of the ith copy of H.

A *matching* in a graph $G = (V, E)$ is a set of pairwise non-adjacent edges of G. A vertex is *matched* if it is an endpoint of one of the edges in the matching.

A *maximum matching* is a matching that contains the largest possible number of edges. The matching number of a graph, denoted by $\nu(G)$, is the size of a maximum matching. A *perfect matching* is a matching which matches all the vertices of the graph. A *near-perfect matching* is one in which exactly one vertex is unmatched.

A *integral matching* of a simple graph $G = (V, E)$ is a function g from E to the set $\{0, 1, 2, \ldots, k\}$ such that $\sum_{e \in \Gamma(v)} g(e) \leq k$ for every $v \in V$, where $\Gamma(v)$ is the set of all edges incident to v. The integral matching number of G, written $\mu_f(G)$, is the maximum of $\frac{1}{k} \sum_{e \in E} g(e)$ over all integral matchings g.

For the concepts that were not discussed please refer to [1], [10], [11], [12].

2 Fractional matching Number of Paths, Cycles and Complete Graphs

This section gives the fractional matching number of paths, cycles and complete graphs. Observation 2.1 is found in [7], while Observation 2.4 was stated in [13].

Observation 2.1 *Let G be a graph of order n and f be a fractional matching. Then*

1. $\mu_f(G) \leq n/2$,
2. $\mu_f(G) = n/2$ *if and only if k-regular, and*
3. $\sum_{e \in \Gamma(v)} f(e) = 1$ *if and only if* $\mu_f(G) = n/2$.

Corollary 2.2 *Let C_n be a cycle of order n. Then $\mu_f(C_n) = n/2$.*

Corollary 2.3 *Let K_n be a complete graph of order n. Then $\mu_f(K_n) = n/2$.*

Observation 2.4 *A near-perfect matching is maximum.*

The next Theorem is found in [2].

Theorem 2.5 *If G is a bipartite graph, then $\mu_f(G) = \nu(G)$.*

Theorem 2.6 *Let P_n be a path of order n. Then*

$$\mu_f(P_n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Let $P_n = (u_1, u_2, \ldots, u_n)$ be a path of order n. Define $f : E(P_n) \to \{0, 1, 2\}$ by $f(u_i u_{i+1(\text{mod } n)}) = 1$ if i is odd and 0 otherwise. Then f is a perfect matching if n is even and a near-perfect matching if n is odd. Hence, by Observation 2.4 f is maximum. Since P_n is a bipartite graph, by Theorem 2.5 the assertion follows. ■

3 Fractional matching Number of the Join of Graph

This section gives the fractional matching number of the join of graphs.

Lemma 3.1 Let G be a simple graph. Then $\alpha'_*(G) = \mu_f(G)$.

To see this, let $G = (V, E)$ be a simple graph. Let $\mathscr{A} = \{\frac{1}{k}\sum_{e\in E} g(e) : g \text{ is a } k\text{-int match in } G\}$ and $\mathscr{B} = \{\sum_{e\in E} f(e) : f \text{ is a frac match in } G\}$. It suffices to show that $\mathscr{A} = \mathscr{B}$. Now, let $w \in \mathscr{A}$. Then there exist a k-integral matching $g : E \to \{0, 1, 2, \ldots, k\}$ such that $w = \frac{1}{k}\sum_{e\in E} g(e)$. Define $f : E \to [0, 1]$ by $f(e) = g(e)/k$. Then f is a fractional matching and $\sum_{e\in E} f(e) = \sum_{e\in E} \frac{1}{k} g(e) = \frac{1}{k}\sum_{e\in E} g(e) = w$. Hence, $w \in \mathscr{B}$. This shows that $\mathscr{A} \subseteq \mathscr{B}$.

Next, let $z \in \mathscr{B}$. Then there exist a fractional matching $f : E \to [0, 1]$ such that $z = \sum_{e\in E} f(e)$. Let $f(e_i) = a_i/b_i$ with $(a_i, b_i) = 1$ for $i = 1, 2, \ldots, m$, and k be the least common multiple of the elements of $\{b_i : i = 1, 2, \ldots, m\}$. Note that $k(a_i/b_i) \in \mathbb{N}$ and $k(a_i/b_i) \le k$. Define $f : E \to \{0, 1, \ldots, k\}$ by $g(e) = k(f(e))$. Then $g(e) \le k$ for all $e \in E$, and $\sum_{e\in\Gamma(v)} g(e) = \sum_{e\in\Gamma(v)} k(f(e)) = k\sum_{e\in\Gamma(v)} f(e) \le k$. Thus, g is a k-integral matching, and $\frac{1}{k}\sum_{e\in E} g(e) = \frac{1}{k}\sum_{e\in E} k(f(e)) = \sum_{e\in E} f(e) = z$. Hence, $z \in \mathscr{A}$. This shows that $\mathscr{B} \subseteq \mathscr{A}$.

Accordingly, $\mathscr{A} = \mathscr{B}$. This implies that $\alpha'_*(G) = \mu_f(G)$. ■

By virtue of Lemma 3.1, we may use $\alpha'_*(G)$ and $\mu_f(G)$ interchangeably. Theorem 3.2 presents a sharp upperbound of the fractional matching number of the join of graphs.

Theorem 3.2 *Let G and H be graphs of order m and n, respectively. Then*

$$\mu_f(G + H) \ge \frac{mn}{max\,\{m, n\}}.$$

Proof: Let G and H be graphs of order m and n, respectively. Without loss of generality, assume that $m \ge n$. Define $f : E(G + H) \to \{0, 1, 2, \ldots, m\}$ by

$$f(e) = \begin{cases} 0, & \text{if } e \in E(G) \text{ or } e \in E(G) \\ 1, & \text{otherwise.} \end{cases}$$

Then f is an m-integral matching. Hence,

$$\mu_f(G+H) \geq \frac{mn}{m} = \frac{mn}{\max\{m,n\}}.$$

■

Theorem 3.3 *Let G and H be graphs of order n and $n+1$, respectively. Then $\mu_f(G+H) = n$.*

Proof: Let G and H be graphs of order n and $n+1$, respectively. Then $G+H$ has a Hamiltonian path of odd order. This implies that $G+H$ has a near-perfect matching. Hence, by Observation 2.4 $\mu_f(G+H) \geq \nu(G+H) = n$. Therefore, by Observation 2.1, $\mu_f(G+H) = n$. ■

Lemma 3.4 *Let G be a graph of order n. If G is Hamiltonian, then $\mu_f(G) = n/2$.*

Proof: Let $G = (V,E)$ be a Hamiltonian graph and $C_n = [u_1, u_2, \ldots, u_n]$ be a Hamiltonian cycle in G. Define $f : E \to \{0,1,2\}$ by $f(e) = 1$ for all $e \in E(C_n)$ and $f(e) = 0$ otherwise. Then f is a 2-integral matching. Hence, $\mu_f(G) \geq n/2$. By Oservation 2.1, $\mu_f(G) = n/2$. ■

Corollary 3.5 shows that the bound given in Theorem 3.2 is sharp.

Corollary 3.5 *Let G and H be graphs of order m and n, respectively. If $m = n$, then $\mu_f(G+H) = n$.*

Proof: Let G and H be graphs of order m and n, respectively. If $V(G) = \{u_1, u_2, \ldots, u_m\}$ and $V(H) = \{v_1, v_2, \ldots, v_n\}$, then $[u_1, v_1, u_2, v_2, u_3, \ldots, u_n, v_n, u_1]$ is a Hamiltonian cycle. Hence, $G+H$ is Hamiltonian. Therefore, by Lemma 3.4 $\mu_f(G+H) = (m+n)/2 = n$. ■

Corollary 3.6 to Corollary 3.11 also follows from Lemma 3.4 since these graphs are Hamiltonian. These further affirms that the bound given by Theorem 3.2 is sharp.

Corollary 3.6 *Let P_m and P_n be paths of order m and n, respectively. Then $\mu_f(P_m + P_n) = (m+n)/2$.*

Corollary 3.7 *Let P_m be a path of order m and C_n be a cycle of order n. Then $\mu_f(P_m + C_n) = (m+n)/2$.*

Corollary 3.8 *Let C_m and C_n be cycles of order m and n, respectively. Then $\mu_f(C_m + C_n) = (m+n)/2$.*

Corollary 3.9 *Let K_m be a complete graph of order m and P_n be a path of order n. Then $\mu_f(K_m + P_n) = (m+n)/2$.*

Corollary 3.10 *Let K_m be a complete graph of order m and C_n be a cycle of order n. Then $\mu_f(K_m + C_n) = (m+n)/2$.*

Corollary 3.11 *Let K_m and K_n be a complete graph of order m and n, respectively. Then $\mu_f(K_m + K_n) = (m+n)/2$.*

Corollary 3.11 also follows from Corollary 2.3 since the join of complete graphs is a complete graph.

4 Fractional Matching Number of the Corona of Graphs

This section gives the fractional matching number of the corona of graphs. Theorem 4.1 gives a sharp upperbound for the fractional matching number of the corona of two graphs. For example, equality holds for $K_1 \circ P_2$.

Theorem 4.1 *Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be graphs with $|V_1| = n_1$, $|E_1| = m_1$, $|V_2| = n_2$ and $|E_2| = m_2$. Then $\mu_f(G \circ H) \geq (m_1 + n_1 n_2 + n_1 m_2)/(\Delta(G) + n_2)$.*

Proof: Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be graphs with $|V_1| = n_1$, $|E_1| = m_1$, $|V_2| = n_2$ and $|E_2| = m_2$. Define $f : E(G \circ H) \to \{0, 1, 2, \ldots, \Delta(G) + n_2\}$ by

$$f(e) = \begin{cases} 0, & \text{if } e \in E_1 \\ 1, & \text{otherwise.} \end{cases}$$

Then f is a $\Delta(G) + n_2$-integral matching. Hence, $\mu_f(G \circ H) \geq (m_1 + n_1 n_2 + n_1 m_2)/(\Delta(G) + n_2)$. ■

The following observations must be clear.

Observation 4.2 *Let G be a connected graphs. If H is a connected subgraph of G, then $\mu_f(H) \leq \mu_f(G)$.*

Observation 4.3 *Let G and H be connected graphs. Then $\mu_f(G \cup H) = \mu_f(G) + \mu_f(H)$.*

Lemma 4.4 follows from Lemma 3.4, and Observations 4.2 and 4.3.

Lemma 4.4 *Let $G = (V, E)$ be a graph of order n. If there is a partition $\{V_1, V_2, \ldots, V_n\}$ of V such that for each $i = 1, 2, \ldots, n$, $\langle V_i \rangle$ is Hamiltonian, then $\mu_f(G) = n/2$.*

Theorem 4.5 *Let G and H be graphs of order m and n, respectively. If H has a Hamiltonian path, then $\mu_f(G \circ H) = m(n+1)/2$.*

Proof: Let G be a graph with vertex set $\{u_1, u_2, \ldots, u_m\}$ and H be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. Let $G \circ H$ be the graph obtained by taking a copy of G and m copies of H (which we denote by $H_n^j = (\{v_1^j, v_2^j, \ldots, v_n^j\}, E_H^j)$ for $j = 1, 2, \ldots, m$) and then joining each jth vertex of G to every vertex in the jth-copy of H. If H has a Hamiltonian path, then there exists a path $P_n = (v_1, v_2, v_3, \ldots, v_n)$ in H that passes through each vertex exactly ones. Consider the partition $\{\{u_i\} \cup \{v_i^j : i = 1, 2, \ldots, n\} : j = 1, 2, \ldots, m\}$ of $V(G \circ H)$. Note that for each $i = 1, 2, \ldots, m$, $\left\langle \{u_i\} \cup \{v_i^j : i = 1, 2, \ldots, n\} \right\rangle$ is Hamiltonian. Hence, by Lemma 4.4 $\mu_f(G \circ H) = m(n+1)/2$. ■

Corollary 4.6 *Let G be a graph of order m and P_n be a path of order n. Then $\mu_f(G \circ P_n) = m(n+1)/2$.*

Corollary 4.7 *Let G be a graph and C_n be a cycle of order n. Then $\mu_f(G \circ C_n) = m(n+1)/2$.*

Corollary 4.8 *Let G be a graph and K_n be a complete graph of order n. Then $\mu_f(G \circ K_n) = m(n+1)/2$.*

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References

[1] J.L. Gross, J.Yellen, *Handbook of Graph Theory*, CRC Press LLC, N.W. Corporate Blvd., Boca Raton, FL, USA, 2003. https://doi.org/10.1201/9780203490204

[2] E.R. Scheinerman, D.H. Ullman, *Fractional Graph Theory: A Rational Approach to the Theory of Graphs*, Wiley & Sons, 2008.

[3] W.R. Pulleyblank, Fractional Matchings and the Edmonds-Gallai Theorem, *Discrete Applied Mathematics*, **16**(1) (1987), 51-58. https://doi.org/10.1016/0166-218x(87)90053-9

[4] Y. Liu, G. Liu, The Fractional Matching Numbers of Graphs, *Networks*, **40**(4) (2002), 228-231. https://doi.org/10.1002/net.10047

[5] R. Motwani, R. Panigrahy, Y. Xu, Fractional Matching Via Balls and Bins, Approximation, *Randomization and Combinatorial Optimization and Techniques, LNCS*, **4110**, 487-498. https://doi.org/10.1007/11830924_44

[6] I. Choi, J. Kim, S. O, The Difference and Ratio of the Fractional Matching Number and the Matching Number of Graphs, arXiv:1512.07595v1 [math.CO] 23 Dec 2015.

[7] R.E. Behrend, O. Suil, D.B. West, Sharp Lower Bounds on the Fractional Matching Number, *Discrete Applied Mathematics*, **186** (2015), 272-274. https://doi.org/10.1016/j.dam.2015.01.014

[8] J. Miihlbacher, F. Steinparz and G. Tinhofer, On certain classes of fractional matchings, preprint (1983).

[9] J.P. Uhry, Sur le problem du couplage maximal, *RAIRO Rech. Op.*, **9** (1975), 13-20. https://doi.org/10.1051/ro/197509v300131

[10] G. Chartrand, *Introductory Graph Theory*, Dover.

[11] F. Harary, *Graph Theory*, Reading, MA: Addison-Wesley.

[12] West, D. B., *Introduction to Graph Theory*, 2nd ed. Englewood Cliffs, NJ: Prentice-Hall.

[13] Wikipedia contributors, 'Matching (graph theory)' Wikipedia, The Free Encyclopedia, Web. 6 April, 2017. https://en.wikipedia.org/wiki/Matching (graph theory)

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