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Quasi-Bigraduations of Modules, Slow Analytic Independence

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Abstract

When $f = \left(I_{(m,n)}\right)_{(m,n)\in(\mathbb{Z}^2)\cup\{\infty\}}$ is a $^+$quasi-bigraduation on a ring $\mathcal{R}$, a f^+-quasi-bigraduation of an $\mathcal{R}-$module $\mathcal{M}$ is a family $g = (\mathbb{G}_{(m,n)})_{(m,n)\in(\mathbb{Z}^2)\cup\{\infty\}}$ of subgroups of $\mathcal{M}$ such that $G_\infty = (0)$ and $I_{(m,n)}G_{(p,q)} \subseteq G_{(m+p,n+q)}$, for all (m,n) and $(p,q) \in (\mathbb{N}\times\mathbb{N}) \cup \{\infty\}$.

We will show that r elements of $\mathcal{R}$ are slowly $J-$independent of order k with respect to a $^+$quasi-bigraduation g on an $\mathcal{R}-$module $\mathcal{M}$ if and only if the two property which follow hold:

they are $J-$independent of order k with respect to the $^+$quasi-bigraduation $f_2\left(\mathcal{A}, I\right)$ and there exists a relation of compatibility between the $^+$quasi-graduation on $\mathcal{R}$ deduced from g and g_I where I is the sub-$\mathcal{A}-$module of $\mathcal{R}$ constructed by these elements.

We give criteria of slow $J-$independence of $^+$quasi-bigraduations on an $\mathcal{R}-$module in term of isomorphisms of graded algebras.

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Introduction

All rings are supposed to be commutative and unitary.

In 1954, D. G. Northcott and D. Rees [7] developed a theory of integral closure and reductions of ideals in a noetherian local ring $(A, \mathfrak{M})$. In particular, they introduced two notions of analytic independence with respect to an ideal in a local ring and they proved that the reduction of an ideal in such a ring is minimal if and only if it has an analytically independent generating set.

In 1970 one notion of independence is generalized by Valla [9] in a noetherian commutative ring. He showed that the maximum number of independent elements in an ideal is bounded from above by its height.

Let $f = (I_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ be a filtration of an arbitrary commutative ring A and

$$R(A, f) = \bigoplus_{n\in\mathbb{N}} I_n X^n \quad \text{and} \quad \Re(A, f) = \bigoplus_{n\in\mathbb{Z}} I_n X^n$$

be its Rees rings. Let k be a positive integer which may be equal to $+\infty$ and let J be an ideal of A such that $J + I_k \neq A$. Take $u = X^{-1}$. Then the following numbers are known in the literature to be extensions to filtrations of the analytic spread

- the maximum number $\ell_J(f, k)$ of elements of the ideal J which are $J-$independent of order k with respect to f and

- the maximum number $\ell_J^a(f, k)$ of elements of the ideal J which are regularly $J-$independent of order k with respect to f.

That work generalized results of Okon [8] concerning the analytic spread of noetherian filtrations, $\sup\left\{\dim \dfrac{\Re(A, f)}{(u, \mathfrak{M})\,\Re(A, f)}\ , \mathfrak{M} \in Max\ A\right\}$ and established comparisons of several extensions.

In [4] we studied theses notions for a ${}^{+}$quasi-graduation of a ring $\mathcal{R}$.

We say that the family (G_n) of subgroups of $\mathcal{R}$ is a quasi-graduation (resp. ${}^{+}$quasi-graduation) of $\mathcal{R}$ if G_0 is a subring of $\mathcal{R}$, $G_{+\infty} = (0)$ and $G_p G_q \subseteq G_{p+q}$ $\forall p, q \in \mathbb{Z}$ (resp. $\mathbb{N}$).

Here we need the following concept of compatibility of a family of subgroups of $\mathcal{R}$ with a given quasi-graduation (resp. ${}^{+}$quasi-graduation) f of $\mathcal{R}$ and we extend this concept to quasi-bigraduations:

Definition 1. *Let $\mathcal{R}$ be a ring.*

1) Let $f = (I_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ be a family of subgroups of $\mathcal{R}$.

We say that f is a quasi-graduation (resp. ${}^{+}$quasi-graduation) of $\mathcal{R}$ if I_0 is a subring of $\mathcal{R}$, $I_\infty = (0)$ and $I_p I_q \subseteq I_{p+q}$ for all p and $q \in \mathbb{Z}$ (resp. $\mathbb{N}$).

*2) Let $f = (I_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ be a quasi-graduation (resp. ${}^{+}$quasi-graduation) of $\mathcal{R}$, $\mathcal{M}$ be an $\mathcal{R}-$module and $g = (G_i)_{i\in\mathbb{Z}\cup\{+\infty\}}$ be a family of subgroups of $\mathcal{M}$. We say that g is an f^{+}-**quasi-graduation** of $\mathcal{M}$ or that g is a ${}^{+}$quasi-graduation of $\mathcal{M}$ compatible with f if $G_\infty = (0)$ and $I_p G_q \subseteq G_{p+q}$ for each p and $q \in \mathbb{N}$.*

For a ring R, the construction of rings of polynomials $R[X_1, \ldots, X_n]$ of n indeterminates with coefficients in R have a critical importance, since geometrical objects (curves, surfaces, etc.) are described by equations in several variables.

Otherwise, in Database, stored values must be accessible concurrently but consistently by multiple users. This proves the importance of the Cartesian product in relational Algebra used in Relational Database. Hence there is an interest to replace $\mathbb{N}$ by $\mathbb{N}\times\mathbb{N}$ as set of indices.

We define the compatible $^{+}$quasi-bigraduation of a ring as follow:

Definition 2. *Let $\mathcal{R}$ be a ring. Let $f = \big(I_{(m,n)}\big)_{(m,n)\in(\mathbb{Z}\times\mathbb{Z})\cup\{\infty\}}$ be a family of subgroups of $\mathcal{R}$ with the convention that $I_{(p,\infty)}$, $I_{(\infty,q)}$ and $I_{(\infty,\infty)}$ mean the same subgroup, denoted I_∞. Let us construct the family $(S_m)_{m\in\mathbb{Z}\cup\{+\infty\}}$ as following :*

$$S_m = \mathcal{A}\ \forall m \le 0;\quad S_\infty = (0)\quad \text{and}\quad S_m = \sum_{-m\le n\le 2m} I_{(n,m-n)}\ \forall m\ge 0.$$

*1) We say that f is a **quasi-bigraduation** (resp. $^{+}$**quasi-bigraduation**) of $\mathcal{R}$ if $I_{(0,0)}$ is a subring of $\mathcal{R}$, $I_\infty = (0)$ and $I_{(p,q)}I_{(r,s)} \subseteq I_{(p+r,q+s)}$ $\forall(p,q)$ and $(r,s)\in\mathbb{Z}^2$ (resp. $\mathbb{N}^2$).*

*2) If f is a quasi-bigraduation (resp. $^{+}$quasi-bigraduation) of $\mathcal{R}$ then the family $(S_m)_{m\in\mathbb{Z}\cup\{+\infty\}}$ is a $^{+}$quasi-graduation of $\mathcal{R}$; it is called the $^{+}$**quasi-graduation of $\mathcal{R}$ deduced from** f.*

In [6] we studied the notion of generalized analytic independence for a $^{+}$quasi-bigraduation of a ring $\mathcal{R}$.

In this paper we have two objectives :

- To give a slow concept of generalized analytic independence of compatible $^{+}$quasi-bigraduations of a module,

- to establish some characterizations of generalized analytic independences of compatible $^{+}$quasi-bigraduations of a module by the mean of isomorphisms of graded algebras.

1. Compatible quasi-bigraduations of module

We define the compatible $^{+}$quasi-bigraduation of a module as follow.

Let Δ be the abelian monoid $\mathbb{Z}^2\cup\{\infty\}$ (resp. $\mathbb{N}^2\cup\{\infty\} = \Delta^+$).

Let $\mathcal{R}$ be a ring, $\mathcal{A}$ be a subring of $\mathcal{R}$ and $\mathcal{M}$ be an $\mathcal{R}$-module.

1.1. Compatible quasi-bigraduations of module.

Definition 3. *Let $f = \big(I_{(m,n)}\big)_{(m,n)\in\Delta}$ be a $^{+}$quasi-bigraduation of $\mathcal{R}$.*

*Let $\mathcal{H} = \big(\mathbb{G}_{(i,j)}\big)_{(i,j)\in\Delta}$ be a family of subgroups of $\mathcal{M}$ with the convention that $\mathbb{G}_{(p,\infty)}$, $\mathbb{G}_{(\infty,q)}$ and $\mathbb{G}_{(\infty,\infty)}$ mean the same subgroup, denoted $\mathbb{G}_\infty$. We say that $\mathcal{H}$ is an **$f-$quasi-bigraduation** (resp. **f^{+}-quasi- bigraduation**) of $\mathcal{M}$ or that $\mathcal{H}$ is **a quasi-graduation (resp. a $^{+}$quasi-graduation) over Δ of $\mathcal{M}$ compatible with** f if $\mathbb{G}_\infty = (0)$ and $I_{(p,q)}\mathbb{G}_{(m,n)} \subseteq \mathbb{G}_{(p+m,q+n)}$ for each (m,n) and $(p,q)\in\Delta$.*

1.2. **Globally compatible quasi-bigraduations of module.**

Let $\mathcal{H} = \left(\mathbb{G}_{(i,j)}\right)_{(i,j)\in\Delta}$ be a family of sub-$\mathcal{A}$-modules of $\mathcal{M}$ such that $\mathbb{G}_\infty = (0)$.

Let us construct the family $(\mathcal{N}_m)_{m\in\mathbb{Z}\cup\{+\infty\}}$ of sub-$\mathcal{A}$-modules of $\mathcal{M}$ as following:

$$\mathcal{N}_m = \mathbb{G}_{(0,0)}\ \forall m \leq 0;\ \mathcal{N}_\infty = (0) \text{ and } \mathcal{N}_m = \sum_{-m\leq n\leq 2m} \mathbb{G}_{(n,m-n)}\ \forall m \geq 0.$$

Definition 4. *Let $f = (I_{(s,t)})_{(s,t)\in\Delta}$ be a $^+$quasi-bigraduation of $\mathcal{M}$.*

Let $\mathcal{A} = I_{(0,0)}$ and $S = (S_m)$ be the quasi-graduation deduced from f (see Definition 2). Let $\mathcal{H} = \left(\mathbb{G}_{(i,j)}\right)_{(i,j)\in\Delta}$ be a family of sub-$\mathcal{A}$-modules of $\mathcal{R}$.

*We say that $\mathcal{H}$ is a $^+$**quasi-bigraduation** of $\mathcal{M}$ **globally compatible with** f or that $\mathcal{H}$ is a **global** f^+-**quasi-bigraduation** of $\mathcal{M}$ if $\mathbb{G}_\infty = (0)$ and $S_p\mathcal{N}_q \subseteq \mathcal{N}_{p+q}$ for each p and $q \in \mathbb{N}$.*

Remark that if $\mathcal{H}$ is an f^+-quasi-bigraduation of $\mathcal{M}$ then $\mathcal{H}$ is a $^+$quasi-bigraduation of $\mathcal{M}$ globally compatible with f.

If $\mathcal{H}$ is a global f^+-quasi-bigraduation of $\mathcal{M}$ then one denotes $\mathsf{QG}(\mathcal{H}) = (\mathcal{N}_m)$ which is called the S^+**-quasi-graduation of $\mathcal{M}$ deduced from $\mathcal{H}$**.

2. Compatible quasi-bigraduations of module and generalized Analytic Independence

2.1. **Slowly generalized analytic independence.**

Let $\mathcal{R}$ be a ring, $\mathcal{A}$ be a subring of $\mathcal{R}$ and $\mathcal{M}$ be an $\mathcal{R}$-module.

Definition 5. *Let $a_1, \ldots, a_r$ be elements of $\mathcal{R}$ and let I be the sub-$\mathcal{A}$-module of $\mathcal{R}$ that they generate. Put $f_2(\mathcal{A}, I)$ the $^+$quasi-bigraduation $(\mathcal{I}_{(m,n)})$ of $\mathcal{R}$ such that*

$$\begin{cases} \mathcal{I}_{(j_1,j_2)} = \mathcal{A} & \text{if } j_1 + j_2 \leq 0 \\ \mathcal{I}_\infty = (0) & \text{and} \\ \mathcal{I}_{(j_1,j_2)} = I^d & \text{if } j_1 + j_2 = d \geq 0. \end{cases}$$

Suppose that $\mathcal{H} = \left(\mathbb{G}_{(i,j)}\right)$ is a global $(f_2(\mathcal{A}, I))^+$-quasi-bigraduation of $\mathcal{M}$ (i.e., for all $m \in \mathbb{N}$, $\mathcal{N}_m$ is a submodule of the $\mathcal{A}-$module $\mathcal{M}$ and $I(\mathcal{N}_m) \subseteq \mathcal{N}_{m+1}$).

Put $\mathcal{G}_m = \left[\mathcal{N}_m : \mathbb{G}_{(0,0)}\right]_{\mathcal{R}}$ for all $m \in \mathbb{N}$. $(\mathcal{G}_m)$ is an S^+-quasi-graduation of $\mathcal{R}$.

*One denotes $\mathsf{QGT}(\mathcal{H}) = (\mathcal{G}_m)_{m\in\mathbb{Z}\cup\{+\infty\}}$ which is called the S^+**-quasi-graduation of $\mathcal{R}$ transported from $\mathcal{H}$**.*

Let $k \in \overline{\mathbb{N}^*}$ and J be an ideal of $\mathcal{A}$.

*1) If $J\mathbb{G}_{(0,0)} + \mathcal{N}_k \cap \mathbb{G}_{(0,0)} \neq \mathbb{G}_{(0,0)}$, the elements $a_1, \ldots, a_r$ of $\mathcal{R}$ are said to be $J-$**independent of order** k **with respect to** $\mathcal{H}$ if for any homogeneous polynomial $\mathcal{F}$ of degree d in r indeterminates with coefficients in $\mathbb{G}_{(0,0)}$, the* *relation* $\mathcal{F}(a_1,\ldots,a_r) \in J\mathcal{N}_d + \mathcal{N}_{d+k}$ *implies that* $\mathcal{F}$ *has all of its coefficients in* $J\mathbb{G}_{(0,0)} + \mathcal{N}_k$.

2) If $J + \mathcal{G}_k \cap \mathcal{A} \neq \mathcal{A}$, *the elements* $a_1,\ldots,a_r$ *of* $\mathcal{R}$ *are said to be* ***slowly*** $(J,\mathcal{A})$ ***−independent of order*** k ***with respect to*** $\mathcal{H}$ *if for any homogeneous polynomial* $\mathcal{F}$ *of degree* d *in* r *indeterminates with coefficients in* $\mathcal{A}$, *the relation* $\mathcal{F}(a_1,\ldots,a_r) \in J\mathcal{G}_d + \mathcal{G}_{d+k}$ *implies that* $\mathcal{F}$ *has all of its coefficients in* $J + \mathcal{G}_k \cap \mathcal{A}$.

3) If $k = +\infty$ *the term of order* $+\infty$ *can be omit.*

Remarks 2.1.1. *1) Elements* $a_1,\ldots,a_r$ *of* $\mathcal{R}$ *are* $J-$ *independent of order* k *with respect to* $\mathcal{H}$ *iff they are* $J-$ *independent of order* k *with respect to the deduced* S^+*-quasi-graduation* $(\mathcal{N}_m)_{m\in\mathbb{Z}\cup\{+\infty\}}$ *defined in section 1.2.*

2) Elements $a_1,\ldots,a_r$ *of* $\mathcal{R}$ *are slowly* $(J,\mathcal{A})$ *−independent of order* k *with respect to* $\mathcal{H}$ *iff they are* $J-$*independent of order* k *with respect to the transported* S^+*-quasi-graduation* $(\mathcal{G}_m)_{m\in\mathbb{Z}\cup\{+\infty\}}$.

Proposition 2.1.2. *Let* $a_1,\ldots,a_r$ *be elements of* $\mathcal{R}$ *and* I *be the sub-*$\mathcal{A}$*-module of* $\mathcal{R}$ *that they generate. Let* $\mathcal{H} = \left(\mathbb{G}_{(m,n)}\right)_{(m,n)\in\Delta}$ *be a global* $(f_2(\mathcal{A},I))^+$*-quasi-bigraduation of* $\mathcal{M}$. *Let* $k \in \overline{\mathbb{N}^*}$ *and* J *be an ideal of* $\mathcal{A}$ *such that* $J + \mathcal{G}_k \cap \mathcal{A} \neq \mathcal{A}$. *Suppose that* $a_1,\ldots,a_r$ *are slowly* $(J,\mathcal{A})$ *−independent of order* k *with respect to* $\mathcal{H}$.

If $J \supseteq \mathcal{G}_k \cap \mathcal{A}$ *then the elements* $a_1,\ldots,a_r$ *are* $J-$*independent (resp. slowly* $(J,\mathcal{A})$ *−independent) (of order* $+\infty$*) with respect to* $\mathcal{H}$ *and to* $f_2(\mathcal{A},I) = \left(\mathcal{I}_{(m,n)}\right)$.

Proof. Let $x = F(a_1,\ldots,a_r)$ where F is an homogeneous polynomial of degree s in r indeterminates and with coefficients in $\mathcal{A}$.

Suppose that J contains $\mathcal{G}_k \cap \mathcal{A}$. Put $I_m = I^m$. We have
$[x \equiv 0\,(J\mathcal{G}_s)$ or $x \equiv 0\ (JI_s)] \Rightarrow x \in J\mathcal{G}_s + \mathcal{G}_{s+k} \Rightarrow F \in (J + \mathcal{G}_k \cap \mathcal{A})[X_1,\ldots,X_r]$.
Furthermore, $J + \mathcal{G}_k \cap \mathcal{A} = J$
thus the elements $a_1,\ldots,a_r$ are slowly $(J,\mathcal{A})$ −independent (of order $+\infty$) with respect to $\mathcal{H}$ and with respect to $f_2(\mathcal{A},I)$. □

Proposition 2.1.3. *Let* $k \in \overline{\mathbb{N}^*}$ *and* J *be an ideal of* $\mathcal{A}$, $a_1,\ldots,a_r$ *be elements of* $\mathcal{R}$ *and* I *be the sub-*$\mathcal{A}$*-module of* $\mathcal{R}$ *that they generate.*

Let $\mathcal{H} = \left(\mathbb{G}_{(i,j)}\right)_{(i,j)\in\Delta}$ *be a global* $(f_2(\mathcal{A},I))^+$*-quasi-bigraduation of* $\mathcal{M}$.

Suppose that $J + \mathcal{G}_k \cap \mathcal{A} \neq \mathcal{A}$, $(\mathcal{G}_{i+k})_{i\geq 0}$ *(or* $(\mathcal{N}_{i+k})_{i\geq 0}$*) is decreasing and that* $a_1,\ldots,a_r$ *are slowly* $(J,\mathcal{A})$ *−independent of order* k *with respect to* $\mathcal{H}$.

Let $p \geq 1$ *be an integer. If* $\mathcal{G}_k \cap \mathcal{A} \subseteq J + \mathcal{G}_{pk} \cap \mathcal{A}$ *then the elements* $a_1^p,\ldots,a_r^p$ *are slowly* $(J,\mathcal{A})$ *−independent of order* k *with respect to the global* $f_2(\mathcal{A},I^p)^+$*-quasi-bigraduation* $\mathcal{H}^{(p)} = (\mathbb{G}_{(pm,pn)})_{(m,n)\in\Delta}$.

Proof. This is the consequence of the fact that under the hypotheses we have

$$\begin{cases} \forall n \geq 0 \quad J\mathcal{G}_{pn} + \mathcal{G}_{p(n+k)} \subseteq J\mathcal{G}_{np} + \mathcal{G}_{np+k} \text{ and} \\ \qquad J + \mathcal{G}_k \cap \mathcal{A} \cap \mathcal{A} \subseteq J + \mathcal{G}_{pk} \cap \mathcal{A}. \end{cases}$$

Indeed, let F be an homogeneous polynomial of degree n in r indeterminates and with coefficients in $\mathcal{A}$.

$$F\left(a_1^p, \ldots, a_r^p\right) \in \left(J\mathcal{G}_{pn} + \mathcal{G}_{p(n+k)}\right) \Rightarrow G\left(a_1, \ldots, a_r\right) \in \left(J\mathcal{G}_{pn} + \mathcal{G}_{p(n+k)}\right)$$

where $G\left(X_1, \ldots, X_r\right) = F\left(X_1^p, \ldots, X_r^p\right)$ is homogeneous of degree np.

Thus $G\left(a_1, \ldots, a_r\right) \in J\mathcal{G}_{pn} + \mathcal{G}_{(pn+pk)} \subseteq J\mathcal{G}_{np} + \mathcal{G}_{np+k}$

therefore G and F have their coefficients in $J + \mathcal{G}_k \cap \mathcal{A}$.

F has all of its coefficients in

$(J + \mathcal{G}_k) \cap \mathcal{A} = J + \mathcal{G}_k \cap \mathcal{A} \subseteq J + \mathcal{G}_{pk} \cap \mathcal{A}$. □

2.2. Criteria of $J-$independences :

2.2.1. *Preliminaries.*

Let (I_n) and (J_n) be two families of subgroups of $\mathcal{R}$ such that

$$(*) \quad \begin{cases} 1 \in I_0 \\ J_n \subseteq I_n \\ I_n I_m \subseteq I_{n+m} \\ I_n J_m \subseteq J_{n+m} \end{cases} \qquad \forall m, n \in \mathbb{Z} \text{ (resp. } \mathbb{N}).$$

Let $(\mathbb{P}_n)$ and $(\mathbb{K}_n)$ be two families of subgroups of $\mathcal{M}$ such that

$$(**) \quad \begin{cases} \mathbb{K}_n \subseteq \mathbb{P}_n \\ I_n \mathbb{P}_m \subseteq \mathbb{P}_{n+m} \\ J_n \mathbb{P}_m + I_n \mathbb{K}_m \subseteq \mathbb{K}_{n+m} \end{cases} \qquad \forall m, n \in \mathbb{Z} \text{ (resp. } \mathbb{N}).$$

Let $\mathbb{P} = \bigoplus\limits_m \mathbb{P}_m X^m$ and $\mathbb{K} = \bigoplus\limits_m \mathbb{K}_m X^m$. We have

$$\frac{\mathbb{P}}{\mathbb{K}} \simeq \bigoplus_m \frac{\mathbb{P}_m}{\mathbb{K}_m}.$$

Let (K_n) be a family of subgroups of $\mathcal{R}$ such that

$$(***) \quad \begin{cases} K_n \subseteq I_n \\ I_m J_n + I_n K_m \subseteq K_{m+n} \end{cases} \qquad \forall m, n \in \mathbb{Z} \text{ (resp. } \mathbb{N}).$$

Let $P = \bigoplus\limits_m I_m X^m$ and $K = \bigoplus\limits_m K_m X^m$. We have

$$\frac{P}{K} \simeq \bigoplus_m \frac{I_m}{K_m}.$$

2.2.2. *Construction of surjective morphisms relating to elements of the ring.*

Let $f = (I_n)_{n \in \mathbb{Z} \cup \{+\infty\}}$ be a quasi-graduation of $\mathcal{R}$ and $\mathcal{A} = I_0$.

Let J be an ideal of $\mathcal{A}$ and $J_n = I_n \cap (JI_n + I_{n+k})$ for all n.

Denote $f \ltimes f$ the quasi-bigraduation $\left(U_{(i,j)}\right)_{(i,j) \in (\mathbb{Z} \times \mathbb{Z}) \cup \{\infty\}}$ of $\mathcal{R}$ such that $U_{(m,n)} = I_m I_n$ for each $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ and $U_\infty = (0)$.

Let $a_1, \ldots, a_r$ be elements of $\mathcal{R}$ and I be the sub-$\mathcal{A}$-module of $\mathcal{R}$ that they generate. Suppose that $I_n = I^n$ for all $n > 0$ and $I_n = \mathcal{A}$ for all $n \leq 0$.

Put $s_i = a_i + J_1 \;\; \forall i = 1, \ldots, r$. There exists an isomorphism

$$\psi_{1,k} : \bigoplus_{n\geq 0} \frac{I_n}{J_n} \longrightarrow \frac{R(\mathcal{A},I)}{R(\mathcal{A},I)\cap((u^k,J)\,\Re\,(\mathcal{A},I))} \text{ such that}$$

$\psi_{1,k}(s_i) = a_iX + R(\mathcal{A},I)\cap\big((u^k,J)\,\Re\,(\mathcal{A},I)\big)$ and $\psi_{1,k}(\alpha)=\alpha$ for $\alpha \in \dfrac{\mathcal{A}}{J+I^k\cap\mathcal{A}}$.

Furthermore, we have the products :

For all $\alpha\in\mathcal{A}$ and $b_m\in I_m$ if $m=i_1+\cdots+i_r$ then

$$s_1^{i_1}\cdots s_r^{i_r} = \left(a_1^{i_1}\cdots a_r^{i_r}+J_m\right) \text{ and } (b_m+J_m)(\alpha+J_0)=b_m\alpha+J_m.$$

Hence $s_1^{i_1}\cdots s_r^{i_r}(\alpha+J_0)=a_1^{i_1}\cdots a_r^{i_r}\alpha+J_m$.

Let $v_i=\psi_{1,k}(s_i)\ \ \forall i=1,\ldots,r$.

$$v_1^{i_1}\cdots v_r^{i_r} = a_1^{i_1}\cdots a_r^{i_r}X^m+\left(R(\mathcal{A},I)\cap\left((u^k,J)\,\Re\,(\mathcal{A},I)\right)\right) \text{ and}$$

$$v_1^{i_1}\cdots v_r^{i_r}(\alpha+J_0) = a_1^{i_1}\cdots a_r^{i_r}\alpha X^m+\left(R(\mathcal{A},I)\cap\left((u^k,J)\,\Re\,(\mathcal{A},I)\right)\right)$$

and we have the following commutative diagram:

$$\begin{tikzcd}
\dfrac{\mathcal{A}}{J_0}[X_1,\ldots,X_r] \arrow[r, "\varphi_{1,k}"] \arrow[dr, "\theta_{1,k}"'] & S_J(I,k)=\dfrac{\mathcal{A}}{J_0}[s_1,\ldots,s_r]=\displaystyle\bigoplus_{m\geq 0}\frac{I^m}{J_m} \arrow[d, "\wr\ \psi_{1,k}"] \\
& V_J(I,k)=\dfrac{\mathcal{A}}{J_0}[v_1,\ldots,v_r]
\end{tikzcd}$$

2.2.3. *Surjective morphisms relating a global $(f\ltimes f)^+$-quasi-bigraduation.*

I. Let $\mathcal{H}=\left(\mathbb{G}_{(i,j)}\right)_{(i,j)\in(\mathbb{Z}\times\mathbb{Z})\cup\{\infty\}}$ be a global $(f\ltimes f)^+$-quasi-bigraduation of $\mathcal{M}$, $\mathbb{J}_m=I_m\mathbb{G}_{(0,0)}\cap\left(JS_m\mathbb{G}_{(0,0)}+S_{m+k}\mathbb{G}_{(0,0)}\right)\ \forall m\geq 0$ and $\mathbb{K}_m=\left(I_m\mathbb{G}_{(0,0)}\right)\cap\left(J\mathcal{N}_m+\mathcal{N}_{m+k}\right)\ \forall m\geq 0$ where $S=(S_m)_{m\in\mathbb{Z}\cup\{+\infty\}}=\mathsf{QG}(f\ltimes f)$ is the ${}^+$quasi-graduation of $\mathcal{R}$ deduced from $f\ltimes f$ and $(\mathcal{N}_m)_{m\in\mathbb{Z}\cup\{+\infty\}}=\mathsf{QG}(\mathcal{H})$ is the S^+-quasi-graduation of $\mathcal{M}$ deduced from $\mathcal{H}$.

Put $\mathbb{J}=\bigoplus_m\mathbb{J}_mX^m$ and $\mathcal{T}=\bigoplus_d S_d\mathbb{G}_{(0,0)}X^d$.

a) Consider $\mathfrak{N}=\bigoplus_d\mathcal{N}_dX^d$ and $Q_J(\mathcal{H},k)$ the graded $Q_J(f,k)-$module $\sum_{m\geq 0}\dfrac{I_m\mathbb{G}_{(0,0)}}{\mathbb{K}_m}$, where $Q_J(f,k)=\bigoplus_{n\geq 0}\dfrac{I_n}{J_n}$.

$\mathcal{I}=\left(I_m\mathbb{G}_{(0,0)}\right)_{m\in\mathbb{Z}\cup\{+\infty\}}$ is an f^+-quasi-graduation of $\mathcal{R}$ and we have

$$\mathbb{K}_m\subseteq I_m\mathbb{G}_{(0,0)}\subseteq S_m\mathbb{G}_{(0,0)}\subseteq\mathcal{N}_m.$$

b) For each $m\geq 0$, put $K_m=I_m\cap(J\mathcal{G}_m+\mathcal{G}_{m+k})$ where $(\mathcal{G}_m)_{m\in\mathbb{Z}\cup\{+\infty\}}=\mathsf{QGT}(\mathcal{H})$ is the S^+-quasi-graduation of $\mathcal{R}$ transported from $\mathcal{H}$.

Consider $G = \bigoplus_{d\geq 0} \mathcal{G}_d X^d$ and $T_J(\mathcal{H}, k)$ the graded $Q_J(f,k)-$module $\sum_{m\geq 0} \dfrac{I_m}{K_m}$. We have

$$K_m \subseteq I_m \subseteq S_m \subseteq \mathcal{G}_m.$$

Conditions $(*)$, $(**)$ and $(***)$ of 2.2.1 are satisfied for (I_n), (J_n), $(\mathbb{P}_n)$ and $(\mathbb{K}_n)$, (resp. $(*)$ and $(***)$ for (I_n), (J_n) and $(\mathbb{J}_n)$). Hence we have

$$\frac{\mathbb{P}}{\mathbb{K}} = \frac{\bigoplus_m I_m \mathbb{G}_{(0,0)} X^m}{\bigoplus_{m\geq 0} \left[\left(I_m \mathbb{G}_{(0,0)}\right) \cap \left(J\mathcal{N}_m + \mathcal{N}_{m+k}\right)\right] X^m} \simeq \bigoplus_{m\geq 0} \frac{I_m \mathbb{G}_{(0,0)}}{\left(I_m \mathbb{G}_{(0,0)}\right) \cap \left(J\mathcal{N}_m + \mathcal{N}_{m+k}\right)} \text{ and}$$

$$\frac{P}{K} = \frac{\bigoplus_{m\geq 0} I_m X^m}{\bigoplus_{m\geq 0} \left[I_m \cap \left(J\mathcal{G}_m + \mathcal{G}_{m+k}\right)\right] X^m} \simeq \bigoplus_{m\geq 0} \frac{I_m}{I_m \cap \left(J\mathcal{G}_m + \mathcal{G}_{m+k}\right)}.$$

II. Suppose that for each $m \geq 0$, $I_m = I^m$. Thus, for $m \geq 0$ we have
$S_m = I^m$, $\mathbb{J}_m = I^m \mathbb{G}_{(0,0)} \cap \left(JI^m \mathbb{G}_{(0,0)} + I^{m+k}\mathbb{G}_{(0,0)}\right)$, $\mathbb{K}_m = I^m \mathbb{G}_{(0,0)} \cap \left(J\mathcal{N}_m + \mathcal{N}_{m+k}\right)$
and $Q_J(\mathcal{H}, k)$ is the graded $Q_J(f,k)-$module $\sum_{m\geq 0} \dfrac{I^m \mathbb{G}_{(0,0)}}{\mathbb{K}_m}$, where
$(\mathcal{N}_m)_{m\in \mathbb{Z}\cup\{+\infty\}} = \mathsf{QG}(\mathcal{H})$ is the S^+-quasi-graduation of $\mathcal{M}$ deduced from $\mathcal{H}$.

a) Suppose that $J\mathbb{G}_{(0,0)} + \mathcal{N}_k \cap \mathbb{G}_{(0,0)}$ (resp. $J\mathbb{G}_{(0,0)} + \left(I_k \mathbb{G}_{(0,0)}\right) \cap \mathbb{G}_{(0,0)}$) is different from $\mathbb{G}_{(0,0)}$. We have

$$\frac{\bigoplus_{m\geq 0} I^m \mathbb{G}_{(0,0)} X^m}{\bigoplus_{m\geq 0} \left[I^m \mathbb{G}_{(0,0)} \cap \left(J\mathcal{N}_m + \mathcal{N}_{m+k}\right)\right] X^m} \simeq \bigoplus_{m\geq 0} \frac{I^m \mathbb{G}_{(0,0)}}{I^m \mathbb{G}_{(0,0)} \cap \left(J\mathcal{N}_m + \mathcal{N}_{m+k}\right)}$$

(resp.

$$\frac{\bigoplus_{m\geq 0} I^m \mathbb{G}_{(0,0)} X^m}{\bigoplus_{m\geq 0} \left[I^m \mathbb{G}_{(0,0)} \cap \left(JI^m + I^{m+k}\right)\mathbb{G}_{(0,0)}\right] X^m} \simeq \bigoplus_{m\geq 0} \frac{I^m \mathbb{G}_{(0,0)}}{I^m \mathbb{G}_{(0,0)} \cap \left(\left(JI^m + I^{m+k}\right)\mathbb{G}_{(0,0)}\right)} \;)$$

Put $R\left(\mathbb{G}_{(0,0)}, I\right)$ the graded $\mathcal{A}$-module $\bigoplus_{n\geq 0} I^n \mathbb{G}_{(0,0)} X^n$. We have

$$\begin{aligned} R\left(\mathbb{G}_{(0,0)}, I\right) \cap \left[\left(u^k, J\right)\mathfrak{N}\right] &= R\left(\mathbb{G}_{(0,0)}, I\right) \cap \left[\left(u^k, J\right)\mathfrak{N}\right]^+ \\ &= \bigoplus_{d\geq 0} \left[I^d \mathbb{G}_{(0,0)} \cap \left(J\mathcal{N}_d + \mathcal{N}_{d+k}\right)\right] X^d = \bigoplus_{d\geq 0} \mathbb{K}_d X^d. \end{aligned}$$

$$\begin{aligned} \text{(resp.}\quad R\left(\mathbb{G}_{(0,0)}, I\right) \cap \left[\left(u^k, J\right)\Re\left(\mathbb{G}_{(0,0)}, I\right)\right] &= R\left(\mathbb{G}_{(0,0)}, I\right) \cap \left[\left(u^k, J\right)\Re\left(\mathbb{G}_{(0,0)}, I\right)\right]^+ \\ &= \bigoplus_{d\geq 0} \left[I^d \mathbb{G}_{(0,0)} \cap \left(\left(JI^d + I^{d+k}\right)\mathbb{G}_{(0,0)}\right)\right] X^d = \bigoplus_{d\geq 0} \mathbb{J}_d X^d). \end{aligned}$$

We have $J + \mathcal{G}_k \cap \mathcal{A} \neq \mathcal{A}$, $J\mathbb{G}_{(0,0)} + \left(I_k \mathbb{G}_{(0,0)}\right) \cap \mathbb{G}_{(0,0)} \neq \mathbb{G}_{(0,0)}$ and $J + I_k \cap \mathcal{A} \neq \mathcal{A}$ (resp. $J + I_k \cap \mathcal{A} \neq \mathcal{A}$).

b) Suppose that $J + I_k \cap \mathcal{A} \neq \mathcal{A}$ or that $J + \mathcal{G}_k \cap \mathcal{A} \neq \mathcal{A}$. Then we have

$$\frac{\bigoplus_m I^m X^m}{\bigoplus_m I^m \cap (J\mathcal{G}_m + \mathcal{G}_{m+k}) X^m} \simeq \bigoplus_m \frac{I^m}{I^m \cap (J\mathcal{G}_m + \mathcal{G}_{m+k})}.$$

Put $R(\mathcal{A}, I)$ the Rees ring of the ideal I. We have

$$R(\mathcal{A}, I) \cap \left[\left(u^k, J\right) G\right] = \bigoplus_{d \geq 0} \left[I^d \cap (J\mathcal{G}_d + \mathcal{G}_{d+k})\right] X^d = \bigoplus_{d \geq 0} K_d X^d.$$

III. Let us define the products as follows :

a) For all $\alpha \in \mathbb{G}_{(0,0)}$ and $b_m \in I_m$ if $m = i_1 + \cdots + i_r$ then

$$s_1^{i_1} \cdots s_r^{i_r} = \left(a_1^{i_1} \cdots a_r^{i_r} + J_m\right) \text{ and } (b_m + J_m)(\alpha + \mathbb{K}_0) = b_m \alpha + \mathbb{K}_m$$

where $\mathbb{K}_0 = J\mathbb{G}_{(0,0)} + \mathcal{N}_k \cap \mathbb{G}_{(0,0)}$. Hence

$$s_1^{i_1} \cdots s_r^{i_r} (\alpha + \mathbb{K}_0) = a_1^{i_1} \cdots a_r^{i_r} \alpha + \mathbb{K}_m.$$

Furthermore,

$$v_1^{i_1} \cdots v_r^{i_r} = a_1^{i_1} \cdots a_r^{i_r} X^m + R(\mathcal{A}, I) \cap \left(\left(u^k, J\right) \Re(\mathcal{A}, I)\right) \text{ and}$$

$$v_1^{i_1} \cdots v_r^{i_r} (\alpha + \mathbb{J}_0) = a_1^{i_1} \cdots a_r^{i_r} \alpha X^m + R(\mathbb{G}_{(0,0)}, I) \cap \left(\left(u^k, J\right) \Re\left(\mathbb{G}_{(0,0)}, I\right)\right).$$

b) For all $\alpha \in \mathcal{A}$ and $b_m \in I_m$ if $m = i_1 + \cdots + i_r$ then

$$s_1^{i_1} \cdots s_r^{i_r} = \left(a_1^{i_1} \cdots a_r^{i_r} + J_m\right) \text{ and } (b_m + J_m)(\alpha + K_0) = b_m \alpha + K_m$$

where $K_0 = J + \mathcal{G}_k \cap \mathcal{A}$. Hence

$$s_1^{i_1} \cdots s_r^{i_r} (\alpha + K_0) = a_1^{i_1} \cdots a_r^{i_r} \alpha + K_m \quad \text{and Properties}$$

$(*)$, $(**)$ and $(***)$ of 2.2.1 show that the previous products are well defined.

IV. a) Suppose that $\mathcal{N}_k \cap \mathbb{G}_{(0,0)} \subseteq J\mathbb{G}_{(0,0)} + \left(I^k \mathbb{G}_{(0,0)}\right) \cap \mathbb{G}_{(0,0)} \neq \mathbb{G}_{(0,0)}$.
Then $\mathbb{K}_0 = \mathbb{J}_0 \neq \mathbb{G}_{(0,0)}$, $I^m \mathbb{K}_0 X^m = I^m \mathbb{J}_0 X^m \subseteq R(\mathbb{G}_{(0,0)}, I) \cap \left(\left(u^k, J\right) \Re\left(\mathbb{G}_{(0,0)}, I\right)\right)$ and the next product is well defined: For all $\alpha \in \mathbb{G}_{(0,0)}$

$$v_1^{i_1} \cdots v_r^{i_r} (\alpha + \mathbb{K}_0) = a_1^{i_1} \cdots a_r^{i_r} \alpha X^m + R(\mathbb{G}_{(0,0)}, I) \cap \left(\left(u^k, J\right) \Re\left(\mathbb{G}_{(0,0)}, I\right)\right)$$

Put

$$S_J(\mathcal{H}, k) = \frac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0} [s_1, \ldots, s_r] = \left\{ \sum_{i_1, \ldots, i_r} (\alpha_{i_1, \ldots, i_r} + \mathbb{K}_0) s_1^{i_1} \cdots s_r^{i_r} : \alpha_{i_1, \ldots, i_r} \in \mathbb{G}_{(0,0)} \right\}.$$

and $V_J(\mathcal{H}, k) = \dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0} [v_1, \ldots, v_r]$. We have

$$\frac{R\left(\mathbb{G}_{(0,0)}, I\right)}{R\left(\mathbb{G}_{(0,0)}, I\right) \cap (u^k, J) \Re\left(\mathbb{G}_{(0,0)}, I\right)} = \frac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0} [v_1, \ldots, v_r].$$

Let $\widetilde{\varphi}_{1,k} = \widetilde{\varphi}_{1,J}\left(\mathcal{H}, k\right)$ be the graded morphism of graded modules from $\dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\left[X_1, \ldots, X_r\right]$ onto $\dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\left[s_1, \ldots, s_r\right]$ such that $\widetilde{\varphi}_{1,k}(\overline{\alpha}X_i) = \overline{\alpha}s_i$ for each i and $\widetilde{\varphi}_{1,k}(\overline{\alpha}) = \overline{\alpha}$ for $\alpha \in \mathbb{G}_{(0,0)}$ and $\overline{\alpha} = \alpha + \mathbb{K}_0$.

There exists an isomorphism $\widetilde{\psi}_{1,k}$ of graded modules from $S_J\left(\mathcal{H}, k\right)$ onto $V_J\left(\mathcal{H}, k\right) = \dfrac{R\left(\mathbb{G}_{(0,0)}, I\right)}{R\left(\mathbb{G}_{(0,0)}, I\right) \cap \left[\left(u^k, J\right)\mathfrak{R}\left(\mathbb{G}_{(0,0)}, I\right)\right]}$ such that

$\widetilde{\psi}_{1,k}(\overline{\alpha}s_i) = \overline{\alpha}v_i = \alpha a_i X + R\left(\mathbb{G}_{(0,0)}, I\right) \cap \left[\left(u^k, J\right)\mathfrak{R}\left(\mathbb{G}_{(0,0)}, I\right)\right]$ and $\widetilde{\psi}_{1,k}(\overline{\alpha}) = \overline{\alpha}$ for $\alpha \in \mathbb{G}_{(0,0)}$ and $\overline{\alpha} = \alpha + \mathbb{K}_0$. Put $\widetilde{\theta}_{1,k} = \widetilde{\psi}_{1,k} \circ \widetilde{\varphi}_{1,k}$.

Hence the following diagram commutes:

$$\begin{tikzcd}
\dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\left[X_1, \ldots, X_r\right] \arrow[r, "\widetilde{\varphi}_{1,k}"] \arrow[dr, "\widetilde{\theta}_{1,k}"'] & \dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\left[s_1, \ldots, s_r\right] \arrow[d, "\wr\ \widetilde{\psi}_{1,k}"] \\
 & \dfrac{R(\mathbb{G}_{(0,0)}, I)}{R(\mathbb{G}_{(0,0)}, I) \cap (u^k, J)\mathfrak{R}(\mathbb{G}_{(0,0)}, I)}
\end{tikzcd}$$

b) Suppose that $\mathcal{G}_k \cap \mathcal{A} \subseteq J + I_k \cap \mathcal{A}$ and that $J + I_k \cap \mathcal{A} \neq \mathcal{A}$.

Then $K_0 = J + I_k \cap \mathcal{A} = J_0$ and $I^m K_0 X^m \subseteq R(\mathcal{A}, I) \cap \left(\left(u^k, J\right)\mathfrak{R}\left(\mathcal{A}, I\right)\right)$.

Hence we have the commutative diagram of 2.2.2.

Proposition 2.2.1.

a) If $J\mathbb{G}_{(0,0)} + \left(I^k\mathbb{G}_{(0,0)}\right) \cap \mathbb{G}_{(0,0)} \neq \mathbb{G}_{(0,0)}$ *then the following statements are equivalent:*

(i) $a_1, \ldots, a_r$ *are* $J-$*independent of order* k *with respect to* $f_2\left(\mathbb{G}_{(0,0)}, I\right)$

(ii) $\widetilde{\varphi}_{1,k}$ *is an isomorphism of* $\dfrac{\mathbb{G}_{(0,0)}}{\mathbb{J}_0}\left[X_1, \ldots, X_r\right]$ *over* $\dfrac{\mathbb{G}_{(0,0)}}{\mathbb{J}_0}\left[s_1, \ldots, s_r\right]$

(iii) $\widetilde{\theta}_{1,k}$ *is an isomorphism over* $\dfrac{R\left(\mathbb{G}_{(0,0)}, I\right)}{R\left(\mathbb{G}_{(0,0)}, I\right) \cap \left(u^k, J\right)\mathfrak{R}\left(\mathbb{G}_{(0,0)}, I\right)}$

(iv) The elements $s_1, \ldots, s_r$ *are algebraically independent over* $\dfrac{\mathbb{G}_{(0,0)}}{\mathbb{J}_0}$

(v) The elements $v_1, \ldots, v_r$ *are algebraically independent over* $\dfrac{\mathbb{G}_{(0,0)}}{\mathbb{J}_0}$.

a') If $\mathcal{N}_k \cap \mathbb{G}_{(0,0)} \subseteq J\mathbb{G}_{(0,0)} + \left(I^k\mathbb{G}_{(0,0)}\right) \cap \mathbb{G}_{(0,0)}$ *and* $J\mathbb{G}_{(0,0)} + \left(I^k\mathbb{G}_{(0,0)}\right) \cap \mathbb{G}_{(0,0)}$ *is different from* $\mathbb{G}_{(0,0)}$ *then replacing* $\mathbb{J}_0$ *by* $\mathbb{K}_0$ *we have a result similar to a).*

b) If $\mathcal{G}_k \cap \mathcal{A} \subseteq J + I_k \cap \mathcal{A}$ *then the following statements are equivalent :*

(i) $a_1, \ldots, a_r$ *are slowly* $(J, \mathcal{A})$*-independent of order* k *with respect to* $f_2(\mathcal{A}, I)$ *(resp.* f_I*) (ii)* $\varphi_{1,k}$ *is an isomorphism of* $\dfrac{\mathcal{A}}{K_0}\left[X_1, \ldots, X_r\right]$ *over* $\dfrac{\mathcal{A}}{K_0}\left[s_1, \ldots, s_r\right]$

(iii) $\theta_{1,k}$ *is an isomorphism of* $\dfrac{\mathcal{A}}{K_0}[X_1,\ldots,X_r]$ *over* $\dfrac{R(\mathcal{A},I)}{R(\mathcal{A},I)\cap(u^k,J)\,\Re(\mathcal{A},I)}$

(iv) The elements $s_1,\ldots,s_r$ *are algebraically independent over* $\dfrac{\mathcal{A}}{K_0}$

(v) The elements $v_1,\ldots,v_r$ *are algebraically independent over* $\dfrac{\mathcal{A}}{K_0}$.

Proof. See Theorem 2.4.1 of [6]. □

V. a) With the assumption that $\mathcal{N}_k\cap\mathbb{G}_{(0,0)}\subseteq J\mathbb{G}_{(0,0)}+\left(I^k\mathbb{G}_{(0,0)}\right)\cap\mathbb{G}_{(0,0)}\neq\mathbb{G}_{(0,0)}$ put the canonical morphisms

$$V_J(\mathcal{H},k)=\frac{R\left(\mathbb{G}_{(0,0)},I\right)}{R\left(\mathbb{G}_{(0,0)},I\right)\cap\left[(u^k,J)\,\mathfrak{R}\left(\mathbb{G}_{(0,0)},I\right)\right]}\ \xrightarrow{\widetilde{\theta}_{2,k}}\ \frac{R\left(\mathbb{G}_{(0,0)},I\right)}{R\left(\mathbb{G}_{(0,0)},I\right)\cap\left[(u^k,J)\,\mathfrak{N}\right]}.$$

and $\widetilde{\delta}_k=\widetilde{\theta}_{2,k}\circ\widetilde{\psi}_{1,k}$. We have for $\overline{\alpha}=\alpha+\mathbb{K}_0$

$$v_i\overline{\alpha}=a_i\alpha X+R(\mathbb{G}_{(0,0)},I)\cap\left(\left(u^k,J\right)\Re\left(\mathbb{G}_{(0,0)},I\right)\right)=\widetilde{\psi}_{1,k}(s_i\overline{\alpha})\text{ where }\alpha\in\mathbb{G}_{(0,0)}.$$

Thus

$$\widetilde{\delta}_k\left(s_i\overline{\alpha}\right)=\widetilde{\theta}_{2,k}\left(v_i\overline{\alpha}\right)=a_i\alpha X+R(\mathbb{G}_{(0,0)},I)\cap\left[\left(u^k,J\right)\mathfrak{N}\right]$$

Put $\mathbb{G}_0=\mathbb{G}_{(0,0)}$. The following diagram commutes:

$$\begin{tikzcd}
\dfrac{\mathbb{G}_0}{\mathbb{K}_0}[X_1,\ldots,X_r] \arrow[r, "\widetilde{\varphi}_{1,k}"] \arrow[d, "\widetilde{\theta}_{1,k}"'] \arrow[dr, "\widetilde{\theta}_{1,k}"'] \arrow[drr, "\widetilde{\theta}_k"] & \dfrac{\mathbb{G}_0}{\mathbb{K}_0}[s_1,\ldots,s_r] \arrow[r, "="] \arrow[d, "\wr\ \widetilde{\psi}_{1,k}"] \arrow[dr, "\widetilde{\delta}_k"] & \displaystyle\bigoplus_{m\geq 0}\frac{I^m\mathbb{G}_0}{\mathbb{J}_m} \arrow[d, "\widetilde{\delta}_k"] \\
\dfrac{\mathbb{G}_0}{\mathbb{K}_0}[v_1,\ldots,v_r] \arrow[r, "="'] & \dfrac{R(\mathbb{G}_0,I)}{R(\mathbb{G}_0,I)\cap(u^k,J)\Re(\mathbb{G}_0,I)} \arrow[r, "\widetilde{\theta}_{2,k}"] & \dfrac{R(\mathbb{G}_0,I)}{R(\mathbb{G}_0,I)\cap(u^k,J)\mathfrak{N}}
\end{tikzcd}$$

b) Suppose that $\mathcal{G}_k\cap\mathcal{A}\subseteq J+I_k\cap\mathcal{A}$. Put the canonical morphisms

$$V_J(I,k)=\frac{R(\mathcal{A},I)}{R(\mathcal{A},I)\cap((u^k,J)\,\Re(\mathcal{A},I))}\ \xrightarrow{\theta_{2,k}}\ \frac{R(\mathcal{A},I)}{R(\mathcal{A},I)\cap[(u^k,J)\,G]}$$

and $\delta_k=\theta_{2,k}\circ\psi_{1,k}$.

With $v_i=\psi_{1,k}(s_i)=a_iX+R(\mathcal{A},I)\cap\left(\left(u^k,J\right)\Re(\mathcal{A},I)\right)$ we have $v_i\overline{\alpha}=a_i\alpha X+\left(R(\mathcal{A},I)\cap\left(\left(u^k,J\right)\Re(\mathcal{A},I)\right)\right)=\psi_{1,k}(s_i\overline{\alpha})$ for $\alpha\in\mathcal{A}$ and $\overline{\alpha}=\alpha+K_0$. Thus

$$\delta_k(s_i)=\theta_{2,k}(v_i)=a_iX+R(\mathcal{A},I)\cap\left[\left(u^k,J\right)G\right]\ \text{ and}$$

$$\delta_k(s_i\overline{\alpha})=\theta_{2,k}(v_i\overline{\alpha})=a_i\alpha X+R(\mathcal{A},I)\cap\left[\left(u^k,J\right)G\right]=\left(a_iX+R(\mathcal{A},I)\cap\left[\left(u^k,J\right)G\right]\right)\overline{\alpha}$$

Put $u_i = \theta_{2,k}(v_i)$ and

$$U_J(I,k) = \frac{\mathcal{A}}{K_0}[u_1,\ldots,u_r] = \left\{ \sum_{i_1,\ldots,i_r} (\alpha_{i_1,\ldots,i_r} + K_0)\, u_1^{i_1}\cdots u_r^{i_r} : \alpha_{i_1,\ldots,i_r} \in \mathcal{A} \right\}.$$

We have

$$\frac{R(\mathcal{A},I)}{R(\mathcal{A},I)\cap (u^k,J)\,\Re(\mathcal{A},I)} = \frac{\mathcal{A}}{K_0}[v_1,\ldots,v_r].$$

Put $\varphi_{1,k} = \varphi_{1,J}(I,k)$ and $\theta_{1,k} = \psi_{1,k}\circ\varphi_{1,k}$.

The following diagram commutes:

$$\begin{tikzcd}
\frac{\mathcal{A}}{K_0}[X_1,\ldots,X_r] \arrow[r, "\varphi_{1,k}"] \arrow[d, "\theta_{1,k}"'] \arrow[dr, "\theta_{1,k}"] \arrow[drr, "\theta_k"] & \frac{\mathcal{A}}{K_0}[s_1,\ldots,s_r] \arrow[r, "="] \arrow[d, "\wr\ \psi_{1,k}"] \arrow[dr, "\delta_k"] & \bigoplus_{m\geq 0} \frac{I^m}{J_m} \arrow[d, "\delta_k"] \\
\frac{\mathcal{A}}{K_0}[v_1,\ldots,v_r] \arrow[r, "="'] & \frac{R(\mathcal{A},I)}{R(\mathcal{A},I)\cap (u^k,J)\Re(\mathcal{A},I)} \arrow[r, "\theta_{2,k}"] & \frac{R(\mathcal{A},I)}{R(\mathcal{A},I)\cap (u^k,J)G}
\end{tikzcd}$$

2.2.4. *Properties of independence.*

Under the previous hypotheses we show the following results as in [6]:

Theorem 2.2.2. *Under the notations and hypotheses of 2.2.3 and with the assumption that $\mathcal{G}_k \cap \mathcal{A} \subseteq J + I_k \cap \mathcal{A}$ the following assertions are equivalent :*

(i) $a_1,\ldots,a_r$ are slowly $(J,\mathcal{A})-$independent of order k with respect to $\mathcal{H}$.

(ii) $\begin{cases} a) \ a_1,\ldots,a_r \text{ are } J-\text{independent of order } k \text{ with respect to } f_2(\mathcal{A},I) \text{ and} \\ b) \ I_p \cap (J\mathcal{G}_p + \mathcal{G}_{p+k}) = JI_p + (I_{p+k}\cap I_p) \ \ \forall p \geq 0 \end{cases}$

(iii) $\begin{cases} a) \ \text{The family } \{s_1,\ldots,s_r\} \text{ is algebraically free over } \dfrac{\mathcal{A}}{K_0} \text{ and} \\ b) \ R(\mathcal{A},I)\cap\left[\left(u^k,J\right)G\right] = R(\mathcal{A},I)\cap\left(u^k,J\right)\Re(\mathcal{A},I) \end{cases}$

(iv) $\begin{cases} a) \ \theta_{1,k} \text{ is an isomorphism from } \dfrac{\mathcal{A}}{K_0}[X_1,\ldots,X_r] \text{ onto } \dfrac{\mathrm{R}(\mathcal{A},\mathrm{I})}{\mathrm{R}(\mathcal{A},\mathrm{I})\cap(\mathrm{u^k},\mathrm{J})\Re(\mathcal{A},\mathrm{I})} \text{ and} \\ b) \ \theta_{2,k} \text{ is an isomorphism from } \dfrac{R(\mathcal{A},I)}{R(\mathcal{A},I)\cap(u^k,J)\Re(\mathcal{A},I)} \text{ onto } \dfrac{\mathrm{R}(\mathcal{A},\mathrm{I})}{\mathrm{R}(\mathcal{A},\mathrm{I})\cap[(\mathrm{u^k},\mathrm{J})\,\mathrm{G}]} \end{cases}$

(v) $\begin{cases} a) \ \varphi_{1,k} \text{ is an isomorphism from } \dfrac{\mathcal{A}}{K_0}[X_1,\ldots,X_r] \text{ onto } \dfrac{\mathcal{A}}{K_0}[s_1,\ldots,s_r] \text{ and} \\ b) \ \delta_k \text{ is an isomorphism from } \dfrac{\mathcal{A}}{K_0}[s_1,\ldots,s_r] \text{ onto } \dfrac{\mathrm{R}(\mathcal{A},\mathrm{I})}{\mathrm{R}(\mathcal{A},\mathrm{I})\cap[(\mathrm{u^k},\mathrm{J})\,\mathrm{G}]} \end{cases}$

(vi) $\theta_k = \theta_{2,k}\circ\theta_{1,k}$ is an isomorphism from $\dfrac{\mathcal{A}}{K_0}[X_1,\ldots,X_r]$ *onto* $\dfrac{R(\mathcal{A},I)}{R(\mathcal{A},I)\cap[(u^k,J)\,G]}$.

Proof. (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v)
By Proposition (2.2.1 b)) the elements $a_1, \ldots, a_r$ are $J-$independent of order k with respect to $f_2\left(\mathbb{G}_{(0,0)}, I\right)$ iff the family $\{s_1, \ldots, s_r\}$ is algebraically free over $\dfrac{\mathcal{A}}{J+\mathcal{G}_k\cap\mathcal{A}}$. It is equivalent to the fact that $\theta_{1,k}$ (resp. $\varphi_{1,k}$) is an isomorphism. Moreover,

$I_p \cap (J\mathcal{G}_p + \mathcal{G}_{p+k}) = JI_p + (I_{p+k}\cap I_p)$ $\forall p \geq 0$ if and only if
$R(\mathcal{A}, I) \cap \left[\left(u^k, J\right) G\right] = R(\mathcal{A}, I) \cap \left(u^k, J\right)\Re(\mathcal{A}, I)$ if and only if
$\theta_{2,k}$ (resp. δ_k) is a graded ring isomorphism.

(iv) $\Leftrightarrow$ (vi)
We have: $\theta_{1,k}$ and $\theta_{2,k}$ are surjective and $\theta_k = \theta_{2,k}\circ\theta_{1,k}$. Therefore
θ_k is an isomorphism if and only if both $\theta_{1,k}$ and $\theta_{2,k}$ are isomorphisms.

(i) $\Leftrightarrow$ (vi) As in Theorem 2.4.1 of [6],
[the elements $a_1, \ldots, a_r$ are $J-$independent of order k with respect to $\mathcal{H}$] iff
$\theta_k = \theta_{2,k}\circ\theta_{1,k}$ is an isomorphism from $\dfrac{\mathcal{A}}{K_0}[X_1, \ldots, X_r]$ onto $\dfrac{R(\mathcal{A}, I)}{R(\mathcal{A}, I)\cap[(u^k, J)\,G]}$.
□

Corollary 2.2.3. *If $\mathcal{G}_k\cap\mathcal{A}\subseteq J+I_k\cap\mathcal{A}$ then the following assertions are equivalent:*
(i) $a_1, \ldots, a_r$ are slowly $(J, \mathcal{A})-$independent of order k with respect to $\mathcal{H}$.

(ii) $\begin{cases} \textit{a) } a_1, \ldots, a_r \textit{ are } J-\textit{ independent of order } k \textit{ with respect to } f_2(\mathcal{A}, I) \\ \textit{b) } \delta_k \textit{ is an isomorphism of } \dfrac{\mathcal{A}}{K_0}[s_1, \ldots, s_r] \textit{ over } \dfrac{R(\mathcal{A}, I)}{R(\mathcal{A}, I)\cap[(u^k, J)\,G]} \end{cases}$

(iii) $\begin{cases} \textit{a) } \varphi_{1,k} \textit{ is an isomorphism of } \dfrac{\mathcal{A}}{K_0}[X_1, \ldots, X_r] \textit{ over } \dfrac{\mathcal{A}}{K_0}[s_1, \ldots, s_r] \\ \textit{b) } \delta_k \textit{ is an isomorphism of } \dfrac{\mathcal{A}}{K_0}[s_1, \ldots, s_r] \textit{ over } \dfrac{R(\mathcal{A}, I)}{R(\mathcal{A}, I)\cap[(u^k, J)\,G]} \end{cases}$

(iv) $\begin{cases} \textit{a) } \theta_{1,k} \textit{ is an isomorphism of } \dfrac{\mathcal{A}}{K_0}[X_1, \ldots, X_r] \textit{ over } \dfrac{R(\mathcal{A}, I)}{R(\mathcal{A}, I)\cap(u^k, J)\,\Re(\mathcal{A}, I)} \\ \textit{b) } \theta_{2,k} \textit{ is an isomorphism of } \dfrac{R(\mathcal{A}, I)}{R(\mathcal{A}, I)\cap(u^k, J)\,\Re(\mathcal{A}, I)} \textit{ over } \dfrac{R(\mathcal{A}, I)}{R(\mathcal{A}, I)\cap[(u^k, J)\,G]} \end{cases}$

(v) $\begin{cases} \textit{a) The elements } s_1, \ldots, s_r \textit{ are algebraically independent over } \dfrac{\mathcal{A}}{K_0} \\ \textit{b) } R(\mathcal{A}, I)\cap\left[\left(u^k, J\right)G\right] = R(\mathcal{A}, I)\cap\left(u^k, J\right)\Re(\mathcal{A}, I) \end{cases}$

(vi) $\begin{cases} \textit{a) The elements } v_1, \ldots, v_r \textit{ are algebraically independent over } \dfrac{\mathcal{A}}{K_0}. \\ \textit{b) } R(\mathcal{A}, I)\cap\left[\left(u^k, J\right)G\right] = R(\mathcal{A}, I)\cap\left(u^k, J\right)\Re(\mathcal{A}, I) \end{cases}$

Applying this result to case $k = +\infty$ we have the following Corollary:

Corollary 2.2.4. *The following assertions are equivalent :*
(i) $a_1, \ldots, a_r$ are slowly $(J, \mathcal{A})-$independent (of order $+\infty$) with respect to $\mathcal{H}$
(ii) The elements $a_1, \ldots, a_r$ are $J-$independent with respect to $f_2(\mathcal{A}, I)$ and $I^p\cap(J\mathcal{G}_p) = JI^p$ for all $p \geq 0$
(iii) The family $\{s_1, \ldots, s_r\}$ (resp. $\{v_1, \ldots, v_r\}$) is algebraically free over $\dfrac{\mathcal{A}}{J}$ and $R(\mathcal{A}, I)\cap JG = JR(\mathcal{A}, I)$.

2.2.5. *$J-$independence and isomorphisms.*

We have the equivalent of Theorem 2.2.2 for independence of a compatible quasi-bigraduation of module:

Theorem 2.2.5. *Under the notations and hypotheses of 2.2.3 and with the assumption that $\mathcal{N}_k \cap \mathbb{G}_0 \subseteq J\mathbb{G}_0 + (I_k\mathbb{G}_0) \cap \mathbb{G}_0$ the following assertions are equivalent:*
(i) The elements $a_1, \ldots, a_r$ are $J-$independent of order k with respect to $\mathcal{H}$

$$\text{(ii)} \begin{cases} \text{a) } a_1, \ldots, a_r \text{ are } J - \text{independent of order } k \text{ with respect to } f_2(\mathbb{G}_0, I) \quad \text{and} \\ \text{b) } I_p\mathbb{G}_0 \cap (J\mathcal{N}_p + \mathcal{N}_{p+k}) = JI_p\mathbb{G}_0 + (I_{p+k} \cap I_p)\,\mathbb{G}_{(0,0)} \ \forall p \geq 0 \end{cases}$$

$$\text{(iii)} \begin{cases} \text{a) } \quad \text{The family } \{s_1, \ldots, s_r\} \text{ is algebraically free over } \dfrac{\mathbb{G}_0}{\mathbb{K}_0} \quad \text{and} \\ \text{b) } R(\mathbb{G}_0, I) \cap \left[\left(u^k, J\right)\mathfrak{N}\right] = R(\mathbb{G}_0, I) \cap \left(u^k, J\right)\Re(\mathbb{G}_0, I). \end{cases}$$

$$\text{(iv)} \begin{cases} \text{a) } \widetilde{\theta}_{1,k} \text{ is an isomorphism over } \dfrac{R(\mathbb{G}_0, I)}{R(\mathbb{G}_0, I) \cap (u^k, J)\Re(\mathbb{G}_0, I)} \quad \text{and} \\ \text{b) } \widetilde{\theta}_{2,k} \text{ is an isomorphism over } \dfrac{R(\mathbb{G}_0, I)}{R(\mathbb{G}_0, I) \cap [(u^k, J)\mathfrak{N}]} \end{cases}$$

$$\text{(v)} \begin{cases} \text{a) } \widetilde{\varphi}_{1,k} \text{ is an isomorphism from } \dfrac{\mathbb{G}_0}{\mathbb{K}_0}[X_1, \ldots, X_r] \text{ onto } \dfrac{\mathbb{G}_0}{\mathbb{K}_0}[s_1, \ldots, s_r] \text{ and} \\ \text{b) } \widetilde{\delta}_k \text{ is an isomorphism from } \dfrac{\mathbb{G}_0}{\mathbb{K}_0}[s_1, \ldots, s_r] \text{ onto } \dfrac{R\left(\mathbb{G}_{(0,0)}, I\right)}{R(\mathbb{G}_0, I) \cap [(u^k, J)\mathfrak{N}]} \end{cases}$$

(vi) $\widetilde{\theta}_k = \widetilde{\theta}_{2,k} \circ \widetilde{\theta}_{1,k}$ *is an isomorphism from* $\dfrac{\mathbb{G}_0}{\mathbb{K}_0}[X_1, \ldots, X_r]$ *onto* $\dfrac{R(\mathbb{G}_0, I)}{R(\mathbb{G}_0, I) \cap [(u^k, J)\mathfrak{N}]}$.

2.2.6. *Other properties of independence.*

With the assumption that $\mathcal{N}_k \cap \mathbb{G}_{(0,0)} \subseteq J\mathbb{G}_{(0,0)} + \left(I^k\mathbb{G}_{(0,0)}\right) \cap \mathbb{G}_{(0,0)} \neq \mathbb{G}_{(0,0)}$ let $t_i = a_i + K_1 = a_i + I \cap (J\ \mathcal{G}_1 + \mathcal{G}_{1+k})$ for $i = 1, \ldots, r$. Put

$$T_J(\mathcal{H}, k) = \frac{\mathcal{A}}{K_0}[t_1, \ldots, t_r] = \left\{ \sum_{i_1, \ldots, i_r} (\alpha_{i_1, \ldots, i_r} + K_0)\, t_1^{i_1} \cdots t_r^{i_r} : \alpha_{i_1, \ldots, i_r} \in \mathcal{A} \right\}.$$

We have $t_i \in \dfrac{I}{I \cap (J\ \mathcal{G}_1 + \mathcal{G}_{1+k})}$ and

$$\frac{R(\mathcal{A}, I)}{R(\mathcal{A}, I) \cap [(u^k, J)\,G]} \simeq \bigoplus_m \frac{I^m}{I^m \cap (J\mathcal{G}_m + \mathcal{G}_{m+k})} = T_J(\mathcal{H}, k)$$

$$S_J(I, k) = \frac{\mathcal{A}}{K_0}[s_1, \ldots, s_r] \quad \xrightarrow{\varphi_{2,k}} \quad T_J(\mathcal{H}, k) = \frac{\mathcal{A}}{K_0}[t_1, \ldots, t_r].$$

Let $\varphi_J(\mathcal{H}, k)$ be the graded morphism from $\dfrac{\mathcal{A}}{K_0}[X_1, \ldots, X_r]$ onto $T_J(\mathcal{H}, k)$ such that $\varphi_J(\mathcal{H}, k)(X_i) = t_i$ for each i and $\varphi_J(\mathcal{H}, k)(\alpha) = \alpha$ for $\alpha \in \dfrac{\mathcal{A}}{K_0}$.

There exists an isomorphism ψ_k from $T_J(\mathcal{H}, k)$ onto $\dfrac{R(\mathcal{A}, I)}{R(\mathcal{A}, I) \cap [(u^k, J)\,G]}$ such that $\psi_k(t_i) = a_iX + R(\mathcal{A}, I) \cap \left[\left(u^k, J\right) G\right]$ and $\psi_k(\alpha) = \alpha$ for $\alpha \in \dfrac{\mathcal{A}}{K_0}$. We have

$u_i = \theta_{2,k}(v_i) = \psi_k(t_i)$ for all i. Put $\varphi_k = \varphi_J(\mathcal{H}, k)$ and $\theta_k = \psi_k \circ \varphi_k$. Thus $\dfrac{R(\mathcal{A}, I)}{R(\mathcal{A}, I) \cap [(u^k, J)G]} = \dfrac{\mathcal{A}}{K_0}[u_1, \ldots, u_r]$, $\theta_{2,k} \circ \psi_{1,k} = \psi_k \circ \varphi_{2,k}$ and $\varphi_k = \varphi_{2,k} \circ \varphi_{1,k}$.

Hence the following diagrams commute:

$$
\begin{tikzcd}
\dfrac{\mathcal{A}}{K_0}[X_1, \ldots, X_r] \arrow[r, "\varphi_k"] \arrow[d, "\theta_k"'] \arrow[dr, "\theta_k"] & \dfrac{\mathcal{A}}{K_0}[t_1, \ldots, t_r] \arrow[d, "\wr\ \psi_k"] \\
\dfrac{\mathcal{A}}{K_0}[u_1, \ldots, u_r] \arrow[r, "="'] & \dfrac{R(\mathcal{A}, I)}{R(\mathcal{A}, I) \cap (u^k, J)G}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\dfrac{\mathcal{A}}{K_0}[X_1, \ldots, X_r] \arrow[r, "\varphi_{1,k}"] \arrow[rr, bend left, "\varphi_k"] \arrow[d, "\theta_{1,k}"'] \arrow[dr, "\theta_{1,k}"] \arrow[drr, "\theta_k"] & \dfrac{\mathcal{A}}{K_0}[s_1, \ldots, s_r] \arrow[r, "\varphi_{2,k}"] \arrow[d, "\wr\ \psi_{1,k}"] \arrow[dr, "\delta_k"] & \dfrac{\mathcal{A}}{K_0}[t_1, \ldots, t_r] \arrow[d, "\wr\ \psi_k"] \\
\dfrac{\mathcal{A}}{K_0}[v_1, \ldots, v_r] \arrow[r, "="'] & \dfrac{R(\mathcal{A}, I)}{R(\mathcal{A}, I) \cap (u^k, J)\Re(\mathcal{A}, I)} \arrow[r, "\theta_{2,k}"] & \dfrac{R(\mathcal{A}, I)}{R(\mathcal{A}, I) \cap (u^k, J)G}
\end{tikzcd}
.
$$

Under these assumptions, we have the following Theorem as in [3]:

Theorem 2.2.6. *(see 1.2.4 of* [3] *). The following statements are equivalent :*

(i) $a_1, \ldots, a_r$ *are slowly* $(J, \mathcal{A})-$*independent of order* k *with respect to* $\mathcal{H}$

(ii) φ_k *is an isomorphism of* $\dfrac{\mathcal{A}}{J + \mathcal{G}_k \cap \mathcal{A}}[X_1, \ldots, X_r]$ *over* $\dfrac{\mathcal{A}}{J + \mathcal{G}_k \cap \mathcal{A}}[t_1, \ldots, t_r]$

(iii) θ_k *is an isomorphism of* $\dfrac{\mathcal{A}}{J + \mathcal{G}_k \cap \mathcal{A}}[X_1, \ldots, X_r]$ *over* $\dfrac{R(\mathcal{A}, I)}{R(\mathcal{A}, I) \cap [(u^k, J)G]}$

(iv) The elements $t_1, \ldots, t_r$ *are algebraically independent over* $\dfrac{\mathcal{A}}{J + \mathcal{G}_k \cap \mathcal{A}}$

(v) The elements $u_1, \ldots, u_r$ *are algebraically independent over* $\dfrac{\mathcal{A}}{J + \mathcal{G}_k \cap \mathcal{A}}$.

2.2.7. *Particular case of global quasi-graduations of a ring.*

Let $\mathcal{R}$ be a ring and $\mathcal{A}$ be a subring of $\mathcal{R}$. Let $a_1, \ldots, a_r$ be elements of $\mathcal{R}$, I be the sub-$\mathcal{A}$-module of $\mathcal{R}$ that they generate, $f = (I^n)_{n \in \mathbb{Z} \cup \{+\infty\}}$ be the quasi-graduation of $\mathcal{R}$ with $I^n = \mathcal{A}$ for all $n \leq 0$ and $J_n = I^n \cap \left(JI^n + I^{n+k}\right)$ for all n where J is an ideal of $\mathcal{A}$. Denote $f \ltimes f$ the quasi-bigraduation $\left(U_{(i,j)}\right)$ of $\mathcal{R}$ such that $U_{(m,n)} = I^m I^n$ for each $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ and $U_\infty = (0)$.

Suppose that $\mathcal{H} = \left(G_{(i,j)}\right)$ is a global $f \ltimes f^+$−quasi-bigraduation of $\mathcal{R}$ and $(\mathcal{N}_m) = \mathsf{QG}(\mathcal{H})$ is the ${}^+$quasi-graduation of $\mathcal{R}$ deduced from $\mathcal{H}$. Hence $\mathcal{G}_m = \mathcal{N}_m$ and for $\mathcal{A} = G_{(0,0)}$, the elements $a_1, \ldots, a_r$ of $\mathcal{R}$ are J−independent of order k with respect to $\mathcal{H}$ iff they are slowly $(J, \mathcal{A})$−independent of order k with respect to $\mathcal{H}$.

Moreover, considering the equalities $\widetilde{f} = f$ everywhere, $\mathcal{G}_i = \mathcal{N}_i$, $\mathbb{K}_0 = K$ and $\mathfrak{N} = G$, Theorems 2.2.2 and 2.2.5 coincide.

Example 2.2.7. *Let $\mathcal{R} = \mathbb{R}[X, Y]$ be the ring of polynomials of two indeterminates X and Y with coefficients in $\mathbb{R}$, $\mathcal{A} = \mathbb{Z}[X, Y]$ and $\mathcal{M} = i\mathbb{Q}[X, Y]$.*

Let $f = \left(I_{(m,n)}\right)$, where $I_{(m,n)} = (X^n Y^m)\,\mathbb{Z}[X, Y]$ for all $m, n \in \mathbb{N}$.

Let $\mathcal{H} = \left(\mathbb{G}_{(p,q)}\right)$ such that $\mathbb{G}_{(p,q)} = (X^p Y^q)\, i\mathbb{Q}[X, Y]$ for all $(p, q) \in \mathbb{N} \times \mathbb{N}$.

f is a quasi-bigraduation of $\mathcal{R}$ and a bifiltration of $\mathcal{R}$.

$\mathcal{H}$ is an f^+-quasi-bigraduation of $\mathcal{M}$.

Let $a_1 = X$ and $a_2 = Y$, $J = (X, Y)\,\mathbb{Z}[X, Y]$. We have

$\mathcal{N}_d = \left(X^d, X^{d-1}Y, X^{d-2}Y^2, \ldots, Y^d\right) i\mathbb{Q}[X, Y]$

$\mathcal{N}_1 = (X, Y)\, i\mathbb{Q}[X, Y]$ *and* $\mathcal{N}_0 = \mathbb{G}_{(0,0)} = i\mathbb{Q}[X, Y]$.

1) Let $\mathcal{F} = \sum\limits_{i+j=d} \alpha_{i,j} X_1^i X_2^j \in \mathcal{A}[X_1, X_2]$ an homogeneous polynomial of degree d

$\mathcal{F}(a_1, a_2) = \sum\limits_{i+j=d} \alpha_{i,j} X^i Y^j \in J\mathcal{N}_d \Rightarrow$

$\sum\limits_{i+j=d} \alpha_{i,j} X^i Y^j \in (X, Y)\left(X^d, X^{d-1}Y, X^{d-2}Y^2, \ldots, Y^d\right) i\mathbb{Q}[X, Y] \Rightarrow$

$\alpha_{i,j} \in \mathcal{N}_1 = (X, Y)\, i\mathbb{Q}[X, Y] = (X, Y)\, i\mathbb{Z}[X, Y]\,\mathbb{Q}[X, Y] = J\mathbb{G}_{(0,0)}$

Thus $\alpha_{i,j} \in (J\mathcal{N}_0) \cap \mathbb{G}_{(0,0)} = J\mathbb{G}_{(0,0)}$.

Therefore, X and Y are J−independent with respect to $\mathcal{H}$. We have:

X^2 and Y^2 are J−independent with respect to $\mathcal{H}$.

2) Put $a_3 = XY$ and $\mathcal{F}(X_1, X_2) = iYX_1 - iX_2 = \alpha_1 X_1 + \alpha_2 X_2$ where

$\alpha_1 = iY$ and $\alpha_2 = -i \in \mathcal{A}$

Let $\mathcal{F} = iYX_1 - iX_2 \in \mathcal{N}_0[X_1, X_2]$.

$\mathcal{F}$ is an homogeneous polynomial of degree 1.

$\mathcal{F}(a_1, a_3) = iYa_1 - ia_3 = iYX - iXY = 0 \in J\mathcal{N}_1$. *But* $\alpha_2 = -i \notin J\mathbb{G}_{(0,0)}$.

Therefore, X and XY aren't J−independent with respect to $\mathcal{H}$.

Example 2.2.8. *Let $\mathcal{R} = \mathbb{R}[X, Y]$ be the ring of polynomials of two indeterminates X and Y with coefficients in $\mathbb{R}$, $\mathcal{A} = \mathbb{Z}[X, Y]$ and $\mathcal{M} = i\mathbb{Q}[X, Y]$*

Let $f = \left(I_{(m,n)}\right)$, where $I_{(m,n)} = (X^n Y^m)\,\mathbb{Z}[X, Y]$ for all $m, n \in \mathbb{N}$

f is a quasi-bigraduation of $\mathcal{R}$ and a bifiltration of $\mathcal{A}$.

Let $\mathcal{H}$ be the family $\left(\mathbb{G}_{(p,q)}\right)$ such that

* $\mathbb{G}_{(p,q)} = (X^p Y^q)\, i\mathbb{Q}[X, Y]$ *for all* $(p, q) \in \mathbb{N} \times \mathbb{N}$
* $\mathbb{G}_{(p,q)} = (Y^{p+q})\, i\mathbb{Q}[X, Y]$ *for all* $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ *with* $-q \leq p < 0$
* $\mathbb{G}_{(p,q)} = (X^{p+q})\, i\mathbb{Q}[X, Y]$ *for all* $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ *with* $-p \leq q < 0$
* $\mathbb{G}_{(p,q)} = i\mathbb{Q}[X, Y]$ *for all* $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ *with* $p \leq 0$ *and* $q \leq 0$.

Let $a_1 = X$ *and* $a_2 = Y$, $J = (X, Y)\mathcal{A}$
$\mathcal{N}_d = \left(X^d, X^{d-1}Y, X^{d-2}Y^2, \ldots, Y^d\right) i\mathbb{Q}[X, Y]$, $\mathcal{N}_1 = (X, Y)\, i\mathbb{Q}[X, Y]$ *and*
$\mathcal{N}_0 = \mathbb{G}_{(0,0)} = i\mathbb{Q}[X, Y]$. $(\mathcal{N}_m)$ *is decreasing.*
$\mathcal{H}$ *is an* f^+-*quasi-bigraduation of* $\mathcal{M}$.
$\alpha_{i,j} \in \mathbb{G}_{(0,0)} = i\mathbb{Q}[X, Y]$. *Put* $k = +\infty$
Put $b_1 = XY$, $b_2 = X^2$ *and* $b_3 = Y^2$. *We have:* $b_1^2 - b_2 b_3 = 0$
Let $\mathcal{F}(X_1, X_2, X_3) = iX_1^2 - iX_2X_3$
$\mathcal{F}(X_1, X_2, X_3) = \alpha_1 X_1^2 + \alpha_2 X_2 X_3 \in \mathbb{G}_{(0,0)}[X_1, X_2, X_3]$ *is an homogeneous polynomial of degree* 2, *where* $\alpha_1 = i$ *and* $\alpha_2 = -i \in \mathcal{A}$
$\mathcal{F}(b_1, b_2, b_3) = 0 \in J\mathcal{N}_1$ *but* $\alpha_1 = i \notin J\mathbb{G}_{(0,0)}$.
Therefore, X^2, XY *and* Y^2 *aren't* $J-$*independent (with respect to* $\mathcal{H}$ *and to* f*).*

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