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Simulation as an Alternative to Linear Programming of Human T-cell Lymphotropic Virus I (HTLV-I) Model Infection of CD4+ T-Cells

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Abstract

Simulation is a tradition of operating a process or system in the real world. The act of simulating something first requires the development of a model; this model represents the main characteristics of the selected behaviors, processes, and functions. The model represents the system itself, while the simulation represents the operation of the system over time. In this paper, the optimal solutions of fractional human T-cell lymphotropic virus model infection of CD4+ T-cells will selected by using an efficient technique, called the LHAM, which is a series solution method based on the HAM and Laplace transform in obtaining the solutions for a wide class of problem. The Pade approximation technique to enlarge region of convergence for the solutions. Results obtained using the shame presented here are in good agreement with the numerical results obtained before. Our work confirms the efficiency of LHAM as a tool for solving linear and nonlinear fractional differential equations. The numerical method proposed in this thesis can be utilized to solve other problems in field of nonlinear fractional differential equations.

Keywords: Laplace homotopy analysis method, Initial value problems, Simulation, linear programming

1 Introduction

Human T-cell lymphotropic virus type I (HTLV-I) infection is associated with variety of human diseases. Human T-cell lymphotropic virus (HTLV) is a member of the exogeneous human retroviruses that have atropism for T lymphocytes. HTLV-I belongs to the delta–type retroviruses, which also include bovine leukemia virus; human T-cell leukemia virus type II (HTLV-II) katri and simian T–cell leukemia [10]. Infection with HTLV-I is now a global epidemic, affecting 10 million to 20 million people. This virus has been linked to life – threatening, incurable diseases: Adult T-cell leukemia (ATL) and HTLV-I associated myelopathy - tropical spastic paraparesis (HAM/TSP). These syndromes are important causes of mortality and morbidity in the areas where HTLV-I is endemic, mainly in the tropics and subtropics [18]. There are large endemic areas in southern Jaban, the Caribbean, central and West Africa, the Middle East, and equatorial regions of Africa. In Europe and North America, the virus is found chiefly in immigrants from the endemic areas and in some communities of intravenous drug users. There is neither vaccine against the virus, nor a satisfactory treatment for the malignancy or the inflammatory syndromes, HTLV-I is transmitted via three major routes: transmission from mother to child by breast feeding, transmission from male to female (more frequent than from female to male) by sexual contact and transmission by infected blood, either by blood transfusion or by the contaminated needles among drug abusers.

Like HIV, HTLV-I target CD4+ T-cells, the most abundant white cells in the immune system, decreasing the body's ability to fight infection. Primary infection leads to chronic infection, the proviral load of which can be extremely high, approximately 30-50%. However, only a small percentage of infected individuals develop the disease and 2–5% percent of HTLV-I carriers develop symptoms of ATL [10]. Also, there is very little cell–free virus in the plasma. Almost all viral genetic material resides in DNA form integrated within the host genome of infected cells. HTLV-I infection is achieved primarily through cell–to–cell contact. There has been an enormous effort made in the mathematical modeling of HTLV-I since 1990s. In [10] the authors proposed a modified model that describes that T-cell dynamics of human T-cell lymphotropic virus I (HTLV-I) infection and the development of adult T-cell leukemia (ATL). The HTLV-I model is

$$\begin{aligned}
\frac{dT}{dt} &= \lambda - \mu_T T - kVT, \\
\frac{dI}{dt} &= k_1 VT - (\mu_L + \gamma)I, \\
\frac{dV}{dt} &= \gamma I - (\mu_A + \rho)V, \\
\frac{dL}{dt} &= \rho V + \beta L\left(1 - \frac{L}{L_{\max}}\right) - \mu_M L.
\end{aligned} \tag{1}$$

under the initial value

$$T(0) = C_1,\ \mathrm{I}(0) = \mathrm{C}_2,\ \mathrm{V}(0) = \mathrm{C}_3 \text{ and } L(0) = C_4, \tag{2}$$

where T, I and V denote the numbers of uninfected, latent infected, actively infected CD4+ cells, and L the number of leukemia cells, respectively. The parameters λ, μ_T, k and k_1, are the source of CD4+ T-cells from precursors, the natural death rate of CD4+ T-cells, the rate at which uninfected cells are contacted by actively infected cells, the rate of infection of T-cells with virus from actively infected cells, respectively. μ_L, μ_A, μ_M are blanket death terms for latently infected, actively infected and leukemic cells. Additionally, γ and ρ represent the rates at which latently infected and actively infected cells become actively infected and leukemic, respectively. The rate β determines the speed at which the saturation level for leukemia cells is reached. L_{max} is the maximal value that adult T-cell leukemia can reach. Now we introduce the fractional–order into the model of HTLV-I infection of CD4+ T-cells [10]. The new system is described by the following set of fractional differential equation

$$
\begin{aligned}
D_*^{\alpha} T &= \lambda - \mu_T T - kVT, \\
D_*^{\alpha} I &= k_1 VT - (\mu_L + \gamma) I, \\
D_*^{\alpha} V &= \gamma I - (\mu_A + \rho) V, \\
D_*^{\alpha} L &= \rho V + \beta L \left(1 - \frac{L}{L_{\max}}\right) - \mu_M L.
\end{aligned} \tag{3}
$$

where α is a parameter describing the order of the fractional time–derivative in the Caputo sense and $0 < \alpha < 1$, subject to the same initial conditions given in (2). The general response expression contains a parameter describing the order of the fractional derivatives that can be varied to obtain various responses. Obviously, the integer–order system can be viewed as a special case from the fractional–order system by putting the time–fractional order of the derivative equal to unity. In other words, the ultimate behavior of the fractional system response must converge to the response of the integer order version of the equation.

The homotopy analysis method (HAM) proposed first by Liao [12-16] for solving linear and nonlinear differential and integral equations. This method provides an effective procedure for explicit numerical solutions of a wide and general class of differential systems representing real physical problems [17-23].
The validity of the HAM is independent of whether there exist small parameters or not in the considered equation. The HAM contains a certain auxiliary parameter ħ, auxiliary function H(t) and auxiliary linear operator L which provide us with a simple way to control and adjust the rate of convergence of the series solution. The objective of the present paper is to use the HAM and Laplace transform to provide optimal solutions for a fractional order differential system model of human T-cell lymphotropic virus I (HTLV-I) infection of $CD4^+$ T-cells. However, other category of methods to handle large amount of fractional problem can be found in [24-29].

2 Preliminaries

2.1. Caputo Fractional Derivative

Firstly, we will provide two definitions are important in fractional calculus

Definition

i. Areal function $f(x)$, $x>0$, is said to be in the space C_μ, μ ∈ R if there exists a real number $p>\mu$ such that $f(x)=x^p f_1(x)$ where $f_1(x)$ is continues in [0, ∞). Clearly $C_\mu \subseteq C_\beta$ if $\beta<\mu$.

ii. A function $f(x)\in C_\mu, x>0$ is said to be in the space C_μ^m, $m\in N\cup\{0\}$, if $f^{(m)} \in C\mu$.

The Caputo fractional derivate of $f(t)$ of order $\alpha>0$ is defined as

$$D_*^\alpha f(t)=J^{m-\alpha}D^m f(t)=\begin{cases}\dfrac{1}{\Gamma(\mathrm{m}-\alpha)}\displaystyle\int_0^t \frac{f^m(\tau)}{(t-\tau)^{\alpha+1-m}}d\tau, & \mathrm{m}-1<\alpha<\mathrm{m},\\ \\ \dfrac{d^{\mathrm{m}} f(t)}{dt^m}, & \alpha=\mathrm{m}.\end{cases} \tag{4}$$

Similar to integer-order differentiation, the Caputo's fractional differentiation is a linear operation

$$D_*^\alpha\left[af_1(t)+bf_2(t)\right]=aD_*^\alpha f_1(t)+bD_*^\alpha f_2(t). \tag{5}$$

where a and b are constants.

For $m-1<\alpha\le m$, $f(x)\in C_\alpha^m$ and $\alpha\ge -1$, we have the following properties of the Caputo fractional derivative

1. $D_*^\alpha J^\alpha f(t)=f(t)$.
2. $J^\alpha D_*^\alpha f(t)=f(t)-\sum_{K=0}^{m-1} f^{(k)}(0+)\frac{t^k}{k!}$, $t>0$, $m-1<\alpha\le m$. (6)

For more details of fractional calculus, see [30-37].

2.2 Laplace Transform

The Laplace transform of a function $f(t)$ is the function $F(s)$ defined by

$$F(s)=\ell\{f(t)\}=\int_0^\infty e^{-st}f(t)dt. \tag{7}$$

Furthermore, the function $f(t)$ is called the inverse transform of $F(s)$ and will be denoted by $\ell^{-1}(F(s))$ that is we shall write

$$f(t) = \ell^{-1}(F(s)). \tag{8}$$

For any function f(t) and g(t) whose Laplace transform exist with

$$\ell\{f(t)\} = F(s) \text{ and } \ell\{g(t)\} = G(s)$$

and any constant a and b, the Laplace transform has the following properties

1. $\ell\{af(t) + bg(t)\} = aF(s) + bG(s).$

(9)

2. $\ell\{f(t) * g(t)\} = F(s)G(s),$

where

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau. \tag{10}$$

the convolution integral of $f(t)$ and $g(t)$.

3. *If* $m-1 < \alpha \le m, m \in N$, then the Laplace transform of the fractional derivative $D_*^{\alpha} f(t)$ is

$$\ell(D_*^{\alpha} f(t)) = s^{\alpha} F(s) - \sum_{k=0}^{m-1} f^{(k)}(0+)s^{\alpha-k-1}, \quad t > 0. \tag{11}$$

3. The methodology of LHAM

In this section, we present a modification of the HAM. This modification is based on the Laplace transform of the fractional derivative $D_*^{\alpha} f(t)$. To illustrate the basic idea, let us consider the following system of fractional differential equations

$$D_*^{\alpha_i} x_i(t) = f_i(t, x_i, ..., x_n), i = 1,2,3,...., n, t \ge 0, 0 < \alpha_i \le 1 \tag{12}$$

subject to the initial conditions

$$x_i(0) = a_i, i = 1, 2, 3, ..., n. \tag{13}$$

Applying Laplace transform to both sides of equations in system (12) and by using linearity of Laplace transform we get

$$\ell(D_*^{\alpha_i} x_i(t)) = \ell(f_i(t, x_1, ..., x_n)) \quad, i = 1, 2, 3, ..., n. \tag{14}$$

Using (11), then we have

$$s^{\alpha} \ell(x_i(t)) - a_i s^{\alpha-1} = \ell(f_i(t, x_1, ..., x_n)), i = 1, 2, 3, ..., n. \tag{15}$$

On simplifying

$$\ell(x_i(t)) - \frac{a_i}{s} - \frac{1}{s^{\alpha}} \ell(f_i(t, x_1, ..., x_n)) = 0, \quad i = 1, 2, 3, ..., n. \tag{16}$$

Define the nonlinear operator

$$N_i(\phi_i(t,q)) = \ell(\phi_i(t,q)) - \frac{a_i}{s} - \frac{1}{s^\alpha}\ell(f_i(t,\phi_1(t,q),...,\phi_n(t,q))),\quad i=1,2,3,....,n. \tag{17}$$

The zeroth–order deformation equations can be constructed as follows

$$(1-q)\ell[\phi_i(t,q)-x_{i0}(t)] = q\hbar_i H_i(s)N_i[\phi_i(t,q)], i=1,2,3,....,n. \tag{18}$$

where $q\in[0,1]$ is an embedding parameter, N_i are nonlinear operators, ℓ is the Laplace transform, $x_{io}(t)$ are initial guesses satisfy the initial conditions (13), $\hbar_i \neq 0$ are auxiliary parameters, $H_i(s)\neq 0$ are auxiliary functions and $\phi_i(t,q)$ are unknown functions. Obviously; when $q=0$, and $q=1$, we get

$$\phi_i(t,o) = x_{io}(t),\qquad \phi_i(t,1) = x_i(t) \tag{19}$$

respectively. Thus, as q increase from 0 to 1, the solution $\phi_i(t,q)$ varies from the initial guesses $x_{io}(t)$ to the solution $x_i(t)$. Expanding $\phi_i(t,q)$ in Taylor series with respect to the embedding parameter q, one has

$$\phi_i(t,q) = x_{io}(t) + \sum_{m=1}^{\infty} x_{im}(t)q^m,\quad i=1,2,3,....,n. \tag{20}$$

where

$$x_{im}(t) = \frac{1}{m!}\frac{\partial^m \phi_i(t,q)}{\partial q^m}\Big|q=0,\quad i=1,2,3,....,n. \tag{21}$$

Assume that the auxiliary parameters $\hbar_i$, the auxiliary function $H_i(s)$ and the initial approximations $x_{i0}(t)$ are properly chosen so that the series (20) converges at $q=1$. Then at $q=1$ and by (19) the series (20) becomes

$$x_i(t) = x_{io}(t) + \sum_{m=1}^{\infty} x_{im}(t),\ i=1,2,3,.....,n. \tag{22}$$

Define the vector

$$\vec{x}_{im} = \{x_{io}(t), x_i(t),......, x_{im}(t)\},\quad i=1,2,3,.....,n. \tag{23}$$

Differentiating equations (18) m times with respect to the embedding parameter q, then setting $q=0$ and dividing by m! finally using (21), we have the mth–order deformation equations

$$\ell[x_{im}(t) - \chi_m x_{i(m-1)}(t)] = \hbar_i H_i(s) R_{im}(\vec{x}_{i(m-1)}), i=1,2,3,.....,n.$$

where (24)

$$R_{im}(\vec{x}_{i(m-1)}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N_i(\phi_i(t,q))}{\partial q^{m-1}}\Big|_{q=0},\ i=1,2,3,.....,n.$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases} \tag{25}$$

Applying inverse Laplace transform, we have

$$x_{im}(t) = \chi_m \; x_{i(m-1)}(t) + \hbar_i \ell^{-1}(H_i(s) R_{im}(\vec{x}_{i(m-1)})), i = 1, 2, 3, ..., n. \tag{26}$$

The convergence of a series is important. As long as the series solution (22) given by the (LHAM) is convergent, it must be the solution of the considered system of fractional differential equations.

Theorem 1. If the series $\sum_{m=0}^{\infty} x_{im}(t), i = 1, 2, 3, ..., n.$ is convergent, it must be a solution of system (12).

4 Simulation with applications

In this section, the LHAM is applied to the fractional order system given in (3). Applying the Laplace transform to system (3) we have

$$\begin{aligned}
&\ell(D_*^{\alpha} T) = \ell(\lambda - \mu_T T - kVT), \\
&\ell(D_*^{\alpha} I) = \ell(k_1 VT - (\mu_L + \gamma) I), \\
&\ell(D_*^{\alpha} V) = \ell(\gamma I - (\mu_A + \rho) V), \\
&\ell(D_*^{\alpha} L) = \ell(\rho V + \beta L(1 - \frac{L}{L_{\max}}) - \mu_M L).
\end{aligned} \tag{27}$$

Using linearity of Laplace transform and property (11) we get

$$\begin{aligned}
&s^{\alpha} \ell(T) - s^{\alpha-1} T(0) = \frac{\lambda}{s} - \mu_T \ell(T) - k\ell(VT), \\
&s^{\alpha} \ell(I) - s^{\alpha-1} I(0) = k_1 \ell(VT) - (\mu_L + \gamma)\ell(I), \\
&s^{\alpha} \ell(V) - s^{\alpha-1} V(0) = \gamma \, \ell(I) - (\mu_A + \rho)\ell(V), \\
&s^{\alpha} \ell(L) - s^{\alpha-1} L(0) = \rho \ell(V) + (\beta - \mu_M)\ell(L) - \frac{\beta}{L_{\max}} \ell(L^2).
\end{aligned} \tag{28}$$

On simplifying and using (2) we have

$$\ell(T)-\frac{C_1}{s}-\frac{\lambda}{s^{\alpha+1}}+\frac{\mu_T}{s^\alpha}\ell(T)+\frac{k}{s^\alpha}\ell(VT)=0,$$

$$\ell(I)-\frac{C_2}{s}-\frac{k_1}{s^\alpha}\ell(VT)+\frac{\mu_L+\gamma}{s^\alpha}\ell(I)=0,$$

$$\ell(V)-\frac{C_3}{s}-\frac{\gamma}{s^\alpha}\ell(I)+\frac{\mu_A+\rho}{s^\alpha}\ell(V)=0, \tag{29}$$

$$\ell(L)-\frac{C_4}{s}-\frac{\rho}{s^\alpha}\ell(V)-\frac{\beta-\mu_M}{s^\alpha}\ell(L)+\frac{\beta}{L_{\max}s^\alpha}\ell(L^2)=0.$$

Define the nonlinear operators

$$N_T\left[\hat{T}(t,q),\hat{I}(t,q),\hat{V}(t,q),\hat{L}(t,q)\right]=\ell\left(\hat{T}(t,q)\right)-\frac{C_1}{s}-\frac{\lambda}{s^{\alpha+1}}+\frac{\mu_T}{s^\alpha}\ell\left(\hat{T}(t,q)\right)+\frac{k}{s^\alpha}\ell\left(\hat{V}(t,q)\hat{T}(t,q)\right),$$

$$N_I\left[\hat{T}(t,q),\hat{I}(t,q),\hat{V}(t,q),\hat{L}(t,q)\right]=\ell\left(\hat{I}(t,q)\right)-\frac{C_2}{s}-\frac{k_1}{s^\alpha}\ell\left(\hat{V}(t,q)\hat{T}(t,q)\right)+\frac{\mu_L+\gamma}{s^\alpha}\ell\left(\hat{I}(t,q)\right),$$

$$N_V\left[\hat{T}(t,q),\hat{I}(t,q),\hat{V}(t,q),\hat{L}(t,q)\right]=\ell\left(\hat{V}(t,q)\right)-\frac{C_3}{s}-\frac{\gamma}{s^\alpha}\ell\left(\hat{I}(t,q)\right)+\frac{\mu_A+\rho}{s^\alpha}\ell\left(\hat{V}(t,q)\right), \tag{30}$$

$$N_L\left[\hat{T}(t,q),\hat{I}(t,q),\hat{V}(t,q),\hat{L}(t,q)\right]=\ell\left(\hat{L}(t,q)\right)-\frac{C_4}{s}-\frac{\rho}{s^\alpha}\ell\left(\hat{V}(t,q)\right)-\frac{\beta-\mu_M}{s^\alpha}\ell\left(\hat{L}(t,q)\right)$$
$$+\frac{\beta}{s^\alpha L_{\max}}\ell\left(\left[\hat{L}(t,q)\right]^2\right) \tag{4.7}$$

In view of the LHAM presented in the previous chapter, the zeroth–order deformation equations are

$$(1-q)\ell[\hat{T}(t,q)-T_0(t)]=q\hbar_1H_1(s)N_T\left[\hat{T},\hat{I},\hat{V},\hat{L}\right],$$

$$(1-q)\ell[\hat{I}(t,q)-I_0(t)]=q\hbar_2H_2(s)N_I\left[\hat{T},\hat{I},\hat{V},\hat{L}\right],$$

$$(1-q)\ell[\hat{V}(t,q)-V_0(t)]=q\hbar_3H_3(s)N_V\left[\hat{T},\hat{I},\hat{V},\hat{L}\right], \tag{31}$$

$$(1-q)\ell[\hat{L}(t,q)-L_0(t)]=q\hbar_4H_4(s)N_L\left[\hat{T},\hat{I},\hat{V},\hat{L}\right].$$

Where $q\in[0,1]$ is an embedding parameter, N_T, N_I, N_V and N_L are nonlinear operators, ℓ is the Laplace transform, $T_o(t)$, $I_o(t)$, $V_o(t)$ and $L_o(t)$

are initial guesses satisfy the initial conditions (4.2), $\hbar_i \neq 0$ are auxiliary parameters, $H_i(s) \neq 0$ are auxiliary functions and $\hat{T}(t,q)$, $\hat{I}(t,q)$, $\hat{V}(t,q)$ and $\hat{L}(t,q)$ are unknown functions. Obviously; when $q=0$, and $q=1$, we get

$$\hat{T}(t,0) = T_0(t), \quad \hat{T}(t,1) = T(t), \quad \hat{I}(t,0) = I_0(t), \quad \hat{I}(t,1) = I(t),$$
$$\hat{V}(t,0) = V_0(t), \quad \hat{V}(t,1) = V(t), \ \hat{L}(t,0) = L_0(t), \quad \hat{L}(t,1) = L(t). \tag{32}$$

respectively. Thus, as q increase from 0 to 1, the solutions $\hat{T}(t,q)$, $\hat{I}(t,q)$, $\hat{V}(t,q)$ and $\hat{L}(t,q)$ varies from the initial guesses $T_o(t),\ I_o(t),\ V_o(t)$ and $L_o(t)$ to the solutions $T(t),\ I(t),\ V(t)$ and $L(t)$, respectively. Expanding $\hat{T}(t,q)$, $\hat{I}(t,q)$, $\hat{V}(t,q)$ and $\hat{L}(t,q)$ in Taylor series with respect to the embedding parameter q, one has

$$\hat{T}(t,q) = T_o(t) + \sum_{m=1}^{\infty} T_m(t)q^m, \ \hat{I}(t,q) = I_o(t) + \sum_{m=1}^{\infty} I_m(t)q^m,$$
$$\hat{V}(t,q) = V_o(t) + \sum_{m=1}^{\infty} V_m(t)q^m, \ \hat{L}(t,q) = L_o(t) + \sum_{m=1}^{\infty} L_m(t)q^m. \tag{33}$$

Where

$$T_m(t) = \frac{1}{m!}\frac{\partial^m \hat{T}(t,q)}{\partial q^m}\Big| q=0, \quad I_m(t) = \frac{1}{m!}\frac{\partial^m \hat{I}(t,q)}{\partial q^m}\Big| q=0,$$
$$V_m(t) = \frac{1}{m!}\frac{\partial^m \hat{V}(t,q)}{\partial q^m}\Big| q=0, \quad L_m(t) = \frac{1}{m!}\frac{\partial^m \hat{L}(t,q)}{\partial q^m}\Big| q=0. \tag{34}$$

Assume that the auxiliary parameters $\hbar_i$, the auxiliary function $H_i(s)$ and the initial approximations $T_o(t),\ I_o(t),\ V_o(t)$ and $L_o(t)$ are properly chosen so that the series (33) converges at $q=1$.

Then at $q=1$ the series (33) becomes

$$T(t) = T_o(t) + \sum_{m=1}^{\infty} T_m(t),$$
$$I(t) = I_o(t) + \sum_{m=1}^{\infty} I_m(t),$$
$$V(t) = V_o(t) + \sum_{m=1}^{\infty} V_m(t),$$
$$L(t) = L_o(t) + \sum_{m=1}^{\infty} L_m(t). \tag{35}$$

Differentiating equations (31) m times with respect to the embedding parameter q, then setting $q=0$ and dividing by m! finally using (34), we have the mth–order deformation equations

$$\ell[T_m(t)-\chi_m T_{m-1}(t)]=\hbar_1 H_1(s)R_{m,T}(s)$$

$$\ell[I_m(t)-\chi_m I_{m-1}(t)]=\hbar_2 H_2(s)R_{m,I}(s)$$

$$\ell[V_m(t)-\chi_m V_{m-1}(t)]=\hbar_3 H_3(s)R_{m,V}(s) \tag{36}$$

$$\ell[L_m(t)-\chi_m L_{m-1}(t)]=\hbar_4 H_4(s)R_{m,L}(s)$$

where

$$R_{m,T}(s)=\ell(T_{m-1}(t))-\frac{c_1(1-\chi_m)}{s}-\frac{\lambda(1-\chi_m)}{s^{\alpha+1}}+\frac{\mu_T}{s^\alpha}\ell(T_{m-1}(t))+\frac{k}{s^\alpha}\ell\left(\sum_{i=1}^{m-1}V_i(t)T_{m-1-i}(t)\right),$$

$$R_{m,I}(s)=\ell(I_{m-1}(t))-\frac{c_2(1-\chi_m)}{s}-\frac{k_1}{s^\alpha}\ell\left(\sum_{i=1}^{m-1}V_i(t)T_{m-1-i}(t)\right)+\frac{\mu_L+\gamma}{s^\alpha}\ell(I_{m-1}(t)),$$

$$R_{m,V}(s)=\ell(V_{m-1}(t))-\frac{c_3(1-\chi_m)}{s}-\frac{\gamma}{s^\alpha}\ell(I_{m-1}(t))+\frac{\mu_A+\rho}{s^\alpha}\ell(V_{m-1}(t)), \tag{37}$$

$$R_{m,L}(s)=\ell(L_{m-1}(t))-\frac{c_4(1-\chi_m)}{s}-\frac{\rho}{s^\alpha}\ell(V_{m-1}(t))-\frac{\beta-\mu_M}{s^\alpha}\ell(L_{m-1}(t))$$
$$+\frac{\beta}{s^\alpha L_{\max}}\ell\left(\sum_{i=1}^{m-1}L_i(t)L_{m-1-i}(t)\right). \tag{4.15}$$

and $\chi_m=\begin{cases}0, m\le 1\\ 1, m>1\end{cases}$. Applying inverse Laplace transform of (36), we have the form

$$T_m(t)=\chi_m\ T_{m-1}(t)+\hbar_1\ell^{-1}(H_1(s)R_{m,T}(s)),$$

$$I_m(t)=\chi_m\ I_{m-1}(t)+\hbar_2\ell^{-1}(H_2(s)R_{m,I}(s)),$$

$$V_m(t)=\chi_m\ V_{m-1}(t)+\hbar_3\ell^{-1}(H_3(s)R_{m,V}(s)), \tag{38}$$

$$L_m(t)=\chi_m\ L_{m-1}(t)+\hbar_4\ell^{-1}(H_4(s)R_{m,L}(s)).$$

Finally, we have

$$T(t) = \sum_{m=0}^{\infty} T_m(t),$$

$$I(t) = \sum_{m=0}^{\infty} I_m(t), \tag{39}$$

$$V(t) = \sum_{m=0}^{\infty} V_m(t),$$

$$L(t) = \sum_{m=0}^{\infty} L_m(t).$$

Next, we assumed that all parameters are positive and in mm^3/day as follows

$$\mu_T = 0.6,\ \mu_L = 0.006,\ \mu_A = 0.05,\ \mu_M = 0.0005, \beta = 0.0003,$$
$$\gamma = 0.0004, \rho = 0.00004, L_{\max} = 2200, \lambda = 6\ and\ k = k_1 = 0.1. \tag{40}$$

and the initial conditions

$$T(0) = 1000,\ \mathrm{I}(0) = 250,\ \mathrm{V}(0) = 1.5 \text{ and } L(0) = 0\,. \tag{41}$$

Consider $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha,\ H_1(\mathrm{s}) = H_2(\mathrm{s}) = H_3(\mathrm{s}) = H_4(\mathrm{s}) = 1$, and $\hbar_1 = \hbar_2 = \hbar_3 = \hbar_4 = \hbar = -1$ then following the procedure of the LHAM and with the aid of the computer package Mathematica, we obtain the first few components of the LHAM solution of system (3) as

$$T(t) = 1000 + \frac{1488\hbar t^{\alpha}}{\Gamma(1+\alpha)} + \hbar^2 t^{\alpha}\left(\frac{744}{\Gamma(1+\alpha)} + \frac{555.506\ t^{\alpha}}{\Gamma(1+2\alpha)}\right) + ...$$

$$I(t) = 250 - \frac{296.8\hbar t^{\alpha}}{\Gamma(1+\alpha)} + \hbar^2 t^{\alpha}\left(-\frac{148.4}{\Gamma(1+\alpha)} - \frac{110.056 t^{\alpha}}{\Gamma(1+2\alpha)}\right) + ...$$

$$V(t) = 1.5 - \frac{0.04988\hbar t^{\alpha}}{\Gamma(1+\alpha)} + \hbar^2 t^{\alpha}\left(-\frac{0.02494}{\Gamma(1+\alpha)} + \frac{0.058112 t^{\alpha}}{\Gamma(1+2\alpha)}\right) + ... \tag{42}$$

$$L(t) = -\frac{0.00012\hbar t^{\alpha}}{\Gamma(1+\alpha)} + \hbar^2 t^{\alpha}\left(-\frac{0.00006}{\Gamma(1+\alpha)} + \frac{9.856 * 10^{-7} t^{\alpha}}{\Gamma(1+2\alpha)}\right) + ...$$

Figs. (1)–(4) shows the series solutions obtained using the LHAM at $\alpha = 1,\ \alpha = 0.99 and \alpha = 0.95$

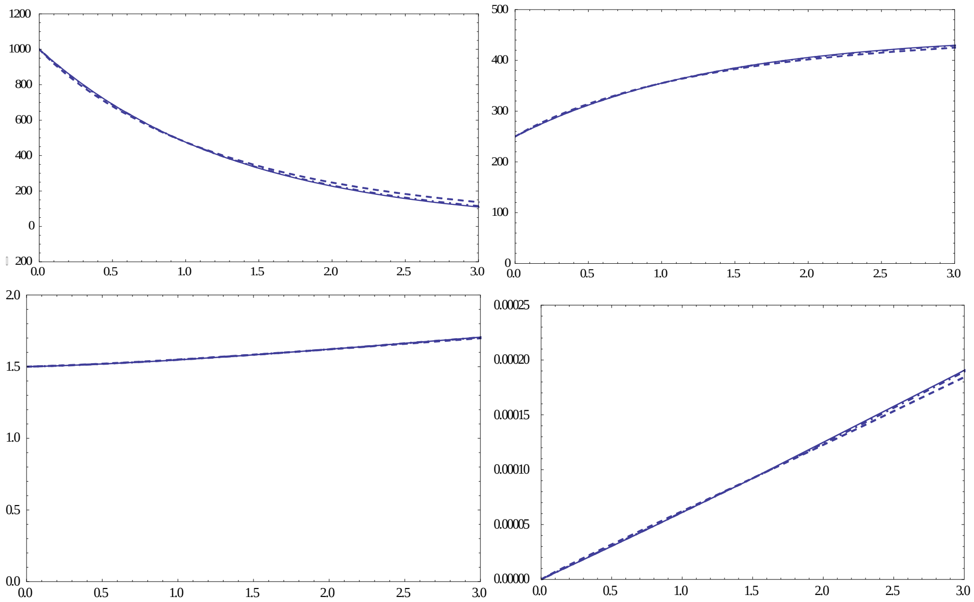

Fig. (1) Plot of T(t), I(t), V(t) and L(t): (Solid line) α=1, (Dashed line) α=0.99, (Dot-dashed line) α=0.95**.**

and this is the same result obtained by using HAM [34]. Figs (5)–(8) shows the series solutions obtained using LHAM combined with pade' approximant at $\alpha = 1, \alpha = 0.95 and \alpha = 0.85$

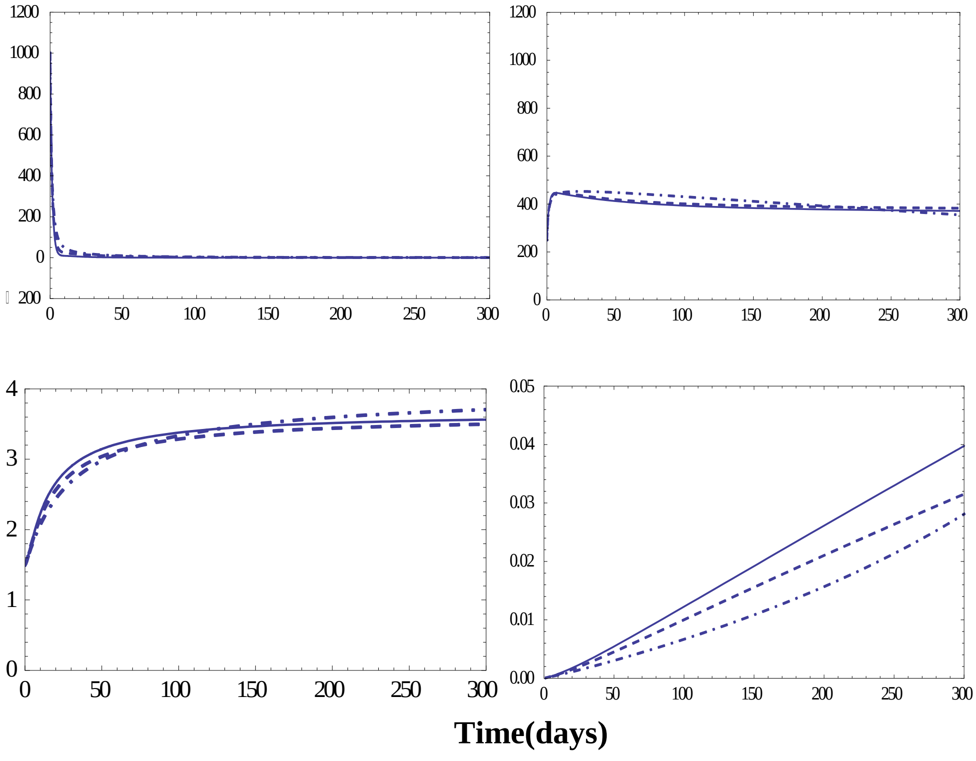

Fig. (2) Plot of T(t), I(t), V(t) and L(t) using LHAM-Pade: (Solid line) α=1, (Dashed line) α=0.95, (Dot-dashed line) α=0.85**.**

The above results are in excellent agreement with the results obtained by using MSGDTM [9], and the result obtained by using GEM [18].

Conclusions

In this paper we presented the LHAM which is an efficient method for solving system of fractional differential equations based on the HAM and Laplace transformation. A fractional order differential system for modeling a human T-cell lymphotropic virus I (HTLV-I) infection of CD4+ T-cells is studied and it is approximate solution is introduced using the LHAM together with Pade approximation technique to enlarge region of convergence for the series solutions. There are important points to make here. First, the LHAM was shown to be a simple, yet powerful analytic scheme for handling for systems of fractional order. Finally generally speaking; the proposed approach can be further implemented to solve other nonlinear problems in fractional calculus field.

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