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Homomorphism Theorems of Ordered Semirings

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Abstract

In this paper, homomorphism theorems for ordered semirings are studied. We obtain some characterizations of the smallest compatible quasiorder or order-congruence generated by H. A homomorphism theorem and two isomorphism theorems of ordered semirings based on compatible quasiorders and order-congruences have been given. Finally, we have a characterization for the direct product of ordered semirings by using compatible quasiorders on an ordered semiring.

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1 Introduction

Homomorphism theorems for ordered algebras have been investigated by various authors, both from the point of view of ordered algebra and ordered semigroup. Our research is closely related to those which was carried out in [1, 2, 3, 4, 5, 7]. They defined order-congruence, compatible quasiorder on those structures and investigated further results of homomorphism theorems. Similarly, we have the following definitions. For undefined notions and notation we refer to the book [6].

For a semiring R, by R^0 we denote the semigroup obtained from the semigroup $(R, +)$ by adjoining an additive identity if necessary. Similarly, by R^1 we denote the semigroup obtained from the semigroup $(R, \bullet)$ by adjoining a multiplicative identity if necessary.

An *ordered semiring* is a partially ordered set $(R, \leq)$ and a semiring $(R, +, \bullet)$ such that for all $a, a' \in R, a \leq a'$ implies $r + a \leq r + a', a + r \leq a' + r, ra \leq ra'$ and $ar \leq a'r$ for all $r \in R$.

A *homomorphism* $f\colon R \to S$ of ordered semirings is a monotone operation-preserving map from an ordered semiring R to an ordered semiring S. We call a homomorphism f an *order-embedding* if additionally $f(a) \leq_S f(a')$ implies $a \leq_R a'$ for all $a, a' \in R$.

We call a quasiorder σ on an ordered semiring R a *compatible quasiorder* if it is compatible with operations and extends the order of R.

Given a quasiorder σ on a poset $(R, \leq)$ and $a, a' \in R$, we write

$$a \underset{\sigma}{\leq} a' \Leftrightarrow (\exists n \in \mathbb{N})(\exists a_1, a_2, \cdots a_n \in R)(a \leq a_1 \sigma a_2 \leq a_3 \sigma \cdots \sigma a_n \leq a').$$

An *order-congruence* on an ordered semiring $R = (R, +, \bullet, \leq)$ is a congruence θ of the semiring $(R, +, \bullet)$ such that the following condition is satisfied:

$$(\forall a, a' \in R)(a \underset{\theta}{\leq} a' \underset{\theta}{\leq} a \Rightarrow a \theta a').$$

If σ is a compatible quasiorder on R then $\overline{\sigma} = \sigma \cap \sigma^{-1}$ is an order-congruence. Let us denote the sets of all order-congruences and compatible quasiorders of R by $Con(R)$ and $Cqu(R)$, respectively.

2 Preliminaries

For every subset $H \subseteq R \times R$ there is a smallest order-congruence Θ_H (resp. compatible quasiorder Σ_H) on R containing H. This relation Θ_H (resp. Σ_H) is called *the order-congruence generated by* H (resp. *the compatible quasiorder generated by* H). We have the following characterizations of the smallest order-congruence and compatible quasiorder generated by H.

Lemma 2.1. *Let R be an ordered semiring, $H \subseteq R \times R$. Then for any $c, c' \in R, c \underset{\Theta_H}{\leq} c'$ if and only if $c \leq c'$ or there exist $x_i, y_i \in R^1, u_i, v_i \in R^0$ and $(a_i, a_i') \in H \cup H^{-1}$ for $i \in \{1, 2, \cdots, n\}$ such that we have one sequence*

$$c \leq p_1(a_1) \quad p_2(a_2) \leq p_3(a_3) \quad \cdots \qquad p_n(a_n) \leq c',$$

$$p_1(a_1') \leq p_2(a_2') \cdots \quad p_{n-1}(a_{n-1}') \leq p_n(a_n')$$

for $p_i(a_i) = u_i + x_i a_i y_i + v_i$, $p_i(a_i') = u_i + x_i a_i' y_i + v_i$ from c to c'.

Lemma 2.2. *Let R be an ordered semiring, $H \subseteq R \times R$. Then for any $c, c' \in R, c \Theta_H c'$ if and only if $c = c'$ or there exist $x_i, y_i \in R^1, u_i, v_i \in R^0$ and $(a_i, a_i') \in H \cup H^{-1}$ for $i \in \{1, 2, \cdots, n\}$ such that we have one sequence*

$$c \leq p_1(a_1) \quad p_2(a_2) \leq p_3(a_3) \quad \cdots \qquad p_n(a_n) \leq c'.$$

$$p_1(a'_1) \leq p_2(a'_2) \cdots \quad p_{n-1}(a'_{n-1}) \leq p_n(a'_n)$$

for $p_i(a_i) = u_i + x_i a_i y_i + v_i$, $p_i(a'_i) = u_i + x_i a'_i y_i + v_i$ *from* c *to* c' *and another similar sequence from* c' *to* c.

Lemma 2.3. *Let* R *be an ordered semiring,* $H \subseteq R \times R$. *Then for any* $c, c' \in R, c\Sigma_H c'$ *if and only if* $c \leq c'$ *or there exist* $x_i, y_i \in R^1, u_i, v_i \in R^0$ *and* $(a_i, a'_i) \in H \cup H^{-1}$ *for* $i \in \{1, 2, \cdots, n\}$ *such that we have one sequence*

$$c \leq p_1(a_1) \quad p_2(a_2) \leq p_3(a_3) \quad \cdots \qquad p_n(a_n) \leq c'.$$

$$p_1(a'_1) \leq p_2(a'_2) \cdots \quad p_{n-1}(a'_{n-1}) \leq p_n(a'_n)$$

for $p_i(a_i) = u_i + x_i a_i y_i + v_i$, $p_i(a'_i) = u_i + x_i a'_i y_i + v_i$ *from* c *to* c'.

Lemma 2.4. *Let* R *be an ordered semiring,* σ *is a compatible quasiorder on* R. *Let* $\mathcal{R} = \{\mu : \mu$ *is compatible quasiorder on* R *such that* $\sigma \subseteq \mu\}$. *Let* $\mathcal{S}$ *be the set of all compatible quasiorders on* $R/\overline{\sigma}$. *For* $\mu \in \mathcal{R}$, *we define a relation* μ' *on* $R/\overline{\sigma}$ *as follows:*

$$\mu' = \{(x_{\overline{\sigma}}, y_{\overline{\sigma}}) : (\exists a \in x_{\overline{\sigma}}, b \in y_{\overline{\sigma}})(a, b) \in \mu\}.$$

Then the mapping $f : \mathcal{R} \to \mathcal{S}, \mu \mapsto \mu'$ *is a bijection and for* $\mu_1, \mu_2 \in \mathcal{R}$, *we have* $\mu_1 \subseteq \mu_2$ *if and only if* $\mu'_1 \subseteq \mu'_2$.

Definition 2.5. *Let* R *be an ordered semiring,* Σ *is a family of compatible quasiorders on* R. *We say that* Σ *separates the elements of* R *if for each* $x, y \in R, (x, y) \notin \leq$, *there exists* $\sigma \in \Sigma$ *such that* $(x, y) \notin \sigma$.

Theorem 2.6. *Let* R *be an ordered semiring,* Σ *is a family of compatible quasiorders on* R. *If* Σ *separates the elements of* R, *then* $\bigcap\{\sigma : \sigma \in \Sigma\} = \leq$. *Conversely, if* $\bigcap\{\sigma : \sigma \in \Sigma\} \subseteq \leq$, *then* Σ *separates the elements of* R.

Proof. Since Σ is a family of compatible quasiorders on R. Then $\leq \subseteq \bigcap\{\sigma : \sigma \in \Sigma\}$. Let $(x, y) \in \bigcap\{\sigma : \sigma \in \Sigma\}, x \not\leq y$. Since Σ separates the elements of R, there exists $\sigma \in \Sigma$ such that $(x, y) \notin \sigma$. Impossible.

Conversely, suppose $x, y \in R, x \not\leq y$. Then there exists $\sigma \in \Sigma$ such that $(x, y) \notin \sigma$. Suppose $(x, y) \in \sigma$ for any $\sigma \in \Sigma$. Then $(x, y) \in \bigcap\{\sigma : \sigma \in \Sigma\} \subseteq \leq$, thus $x \leq y$. Impossible. □

3 Homomorphism theorems

Given a homomorphism $f : R \to S$ of ordered semirings, the directed kernel $\overrightarrow{ker} f$ of f, defined as $\overrightarrow{ker} f = \{(a, b) \in R \times R : f(a) \leq_S f(b)\}$ is clearly a compatible quasiorder on R. Hence $ker f = (\overrightarrow{ker} f) \bigcap (\overrightarrow{ker} f)^{-1}$ is an order-congruence on R.

Lemma 3.1. *Let R be an ordered semiring, $\sigma \subseteq R \times R$. Then σ is a compatible quasiorder on R if and only if there exists an ordered semiring $(S, \prec)$ and a homomorphism $f\colon R \to S$ such that $\sigma = \overrightarrow{ker}f$.*

Lemma 3.2. *Let R be an ordered semiring, θ is a congruence on R. Then the following statements are equivalent:*

(1) *θ is a order-congruence on R;*

(2) *There exists a compatible quasiorder σ of R such that $\theta = \sigma \cap \sigma^{-1}$;*

(3) *There exists an ordered semiring $(S, \prec)$ and a homomorphism $f\colon R \to S$ such that $\theta = ker f$.*

Theorem 3.3. *(Homomorphism Theorem) Let R, S be ordered semirings, $\varphi\colon R \to S$ is a homomorphism. If σ is a compatible quasiorder on R such that $\sigma \subseteq \overrightarrow{ker}\varphi$, then the mapping $f\colon R/\overline{\sigma} \to S, a_{\overline{\sigma}} \mapsto \varphi(a)$ is the unique homomorphism of $R/\overline{\sigma}$ into S such that the diagram*

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\sigma^*"'] & S \\ R/\overline{\sigma} \arrow[ur, "f"'] & \end{tikzcd}$$

commutes (i.e., $f \circ \sigma^* = \varphi$). Moreover, $Imf = Im\varphi$. Conversely, if σ is a compatible quasiorder on R for which there exists a homomorphism $f\colon R/\overline{\sigma} \to S$ such that the above diagram Commutes, then $\sigma \subseteq \overrightarrow{ker}\varphi$.

Proof. Let $\sigma \in Cqu(R)$, $\sigma \subseteq \overrightarrow{ker}\varphi$, $f\colon R/\overrightarrow{\sigma} \to S, a_{\overline{\sigma}} \mapsto \varphi(a)$. Then we have

$$a_{\overline{\sigma}} = b_{\overline{\sigma}} \Rightarrow (a,b) \in \overline{\sigma} \Rightarrow (a,b),(b,a) \in \sigma \Rightarrow (a,b),(b,a) \in \overrightarrow{ker}\varphi$$

$$\Rightarrow \varphi(a) \leq_S \varphi(b), \varphi(b) \leq_S \varphi(a) \Rightarrow \varphi(a) = \varphi(b).$$

If $a, b \in R$, then

$$\begin{gathered} f(a_{\overline{\sigma}} + b_{\overline{\sigma}}) = f((a+b)_{\overline{\sigma}}) = \varphi(a+b) = \varphi(a) + \varphi(b) = f(a_{\overline{\sigma}}) + f(b_{\overline{\sigma}}); \\ f(a_{\overline{\sigma}} b_{\overline{\sigma}}) = f((ab)_{\overline{\sigma}}) = \varphi(ab) = \varphi(a)\varphi(b) = f(a_{\overline{\sigma}})f(b_{\overline{\sigma}}); \\ a_{\overline{\sigma}} \prec b_{\overline{\sigma}} \Rightarrow (a,b) \in \sigma \subseteq \overrightarrow{ker}\varphi \Rightarrow \varphi(a) \leq_S \varphi(b). \end{gathered}$$

Hence f is a homomorphism.

For each $a \in R$, $(f \circ \sigma^*)(a) = f(\sigma^*(a)) = f(a_{\overline{\sigma}}) = \varphi(a)$. Then $f \circ \sigma^* = \varphi$. Let $g\colon R/\overline{\sigma} \to S$ be a homomorphism such that $g \circ \sigma^* = \varphi$. For each $a \in R$, we have $f(a_{\overline{\sigma}}) = \varphi(a) = (g \circ \sigma^*)(a) = g(\sigma^*(a)) = g(a_{\overline{\sigma}})$. Then $f = g$. Moreover, $Imf = \{f(a_{\overline{\sigma}} : a \in R\} = \{\varphi(a) : a \in R\} = Im\varphi$.

Conversely, suppose $\sigma \in Cqu(R)$, $f\colon R/\overline{\sigma} \to S$ is a homomorphism, $f \circ \sigma^* = \varphi$. Then we have

$$\begin{gathered} (a,b) \in \sigma \Rightarrow a_{\overline{\sigma}} \prec b_{\overline{\sigma}} \Rightarrow f(a_{\overline{\sigma}}) \leq_S f(b_{\overline{\sigma}}) \Rightarrow f(\sigma^*(a)) \leq_S f(\sigma^*(b)) \Rightarrow \\ (f \circ \sigma^*)(a) \leq_S (f \circ \sigma^*)(b) \Rightarrow \varphi(a) \leq_S \varphi(b) \Rightarrow (a,b) \in \overrightarrow{ker}\varphi. \end{gathered}$$

Therefor $\sigma \subseteq \overrightarrow{ker}\varphi$. □

Corollary 3.4. *Let R, S be ordered semirings, $\varphi: R \to S$ is a homomorphism. Then $R/ker\varphi \cong Im\varphi$.*

Theorem 3.5. *Let R be an ordered semiring, ρ, σ are compatible quasiorders on R, $\rho \subseteq \sigma$. Then σ/ρ is a compatible quasiorder on $R/\overline{\rho}$ and $(R/\overline{\rho})/(\overline{\sigma/\rho}) \cong (R/\overline{\sigma})$.*

Proof. $\leq_\rho \subseteq \sigma/\rho$: If $(a_{\overline{\rho}}, b_{\overline{\rho}}) \in \leq_\rho$, then $(a,b) \in \rho \subseteq \sigma$, $(a_{\overline{\rho}}, b_{\overline{\rho}}) \in \sigma/\rho$. Let $(a_{\overline{\rho}}, b_{\overline{\rho}}) \in \sigma/\rho$, $(b_{\overline{\rho}}, c_{\overline{\rho}}) \in \sigma/\rho$. Then (a,b), $(b,c) \in \sigma$. Thus $(a,c) \in \sigma$, $(a_{\overline{\rho}}, c_{\overline{\rho}}) \in \sigma/\rho$. Let $(a_{\overline{\rho}}, b_{\overline{\rho}}) \in \sigma/\rho$, $c \in R$. Since $(a,b) \in \sigma$, we have $(a+c, b+c) \in \sigma$, $((a+c)_{\overline{\rho}}, (b+c)_{\overline{\rho}}) \in \sigma/\rho$. Thus $(a_{\overline{\rho}} + c_{\overline{\rho}}, b_{\overline{\rho}} + c_{\overline{\rho}}) \in \sigma/\rho$. Similarly, we have $(c_{\overline{\rho}} + a_{\overline{\rho}}, c_{\overline{\rho}} + b_{\overline{\rho}})$, $(a_{\overline{\rho}}c_{\overline{\rho}}, b_{\overline{\rho}}c_{\overline{\rho}})$, $(c_{\overline{\rho}}a_{\overline{\rho}}, c_{\overline{\rho}}b_{\overline{\rho}}) \in \sigma/\rho$. We consider the mapping $f: R/\overline{\rho} \to R/\overline{\sigma}, a_{\overline{\rho}} \mapsto a_{\overline{\sigma}}$. Since

$$a_{\overline{\rho}} = b_{\overline{\rho}} \Rightarrow (a,b) \in \overline{\rho} \Rightarrow (a,b),(b,a) \in \rho \subseteq \sigma \Rightarrow (a,b) \in \overline{\sigma} \Rightarrow a_{\overline{\sigma}} = b_{\overline{\sigma}}.$$

Then f is well defined. Let $\circ_\rho, \circ_\sigma$ be the multiplication on $R/\overline{\rho}, R/\overline{\sigma}$ respectively. Then

$$f(a_{\overline{\rho}} \circ_\rho b_{\overline{\rho}}) = f((ab)_{\overline{\rho}}) = (ab)_{\overline{\sigma}} = a_{\overline{\sigma}} \circ_{\overline{\sigma}} (b)_{\overline{\sigma}} = f(a_{\overline{\rho}} \circ_\sigma b_{\overline{\rho}}).$$
$$a_{\overline{\rho}} \leq_\rho b_{\overline{\rho}} \Rightarrow (a,b) \in \rho \subseteq \sigma \Rightarrow a_{\overline{\sigma}} \leq_\sigma b_{\overline{\sigma}}.$$

Hence f is a homomorphism. Thus $(R/\overline{\rho}, \circ_\rho, \leq_\rho), (R/\overline{\sigma}, \circ_\sigma, \leq_\sigma)$ are ordered semirings, the mapping f is a homomorphism. By Corollary 3.4 we have $(R/\overline{\rho})/kerf \cong Imf$. Since $\overrightarrow{ker}f = \{(a_{\overline{\rho}}, b_{\overline{\rho}}) \in R/\overline{\rho} \times R/\overline{\rho} : f(a_{\overline{\rho}}) \leq_\sigma f(b_{\overline{\rho}})\}$. Then we have

$$(a_{\overline{\rho}}, b_{\overline{\rho}}) \in \overrightarrow{ker}f \Leftrightarrow f(a_{\overline{\rho}}) \leq_\sigma f(b_{\overline{\rho}}) \Leftrightarrow a_{\overline{\sigma}} \leq_\sigma b_{\overline{\sigma}} \Leftrightarrow (a,b) \in \sigma \Leftrightarrow (a_{\overline{\rho}}, b_{\overline{\rho}}) \in \sigma/\rho.$$

Hence $\overrightarrow{ker}f = \sigma/\rho, kerf = \overline{\sigma/\rho}$. Then $Imf = \{f(a_{\overline{\rho}}) : a \in R\} = \{a_{\overline{\sigma}} : a \in R\} = R/\overline{\sigma}$. Thus we have $(R/\overline{\rho})/(\overline{\sigma/\rho}) \cong (R/\overline{\sigma})$. □

A homomorphism $f: R \to S$ of ordered semirings is called a *Q-homomorphism* if, for all b, $b' \in S$, $b \leq_S b'$ implies that there exist a, $a' \in R$ such that $b = f(a)$, $a \underset{kerf}{\leq} a'$ and $b' = f(a')$.

Theorem 3.6. *Θ is an order-congruence of ordered semiring R if and only if $\Theta = kerf$ for some epimorphism Q-homomorphism $f: R \to S$.*

Proof. Necessity: Suppose Θ is an order-congruence of ordered semiring R. By Lemma 3.2, there exists a epimorphism homomorphism $f: R \to R/\Theta$ such that $\Theta = kerf$. Let $x, y \in R/\Theta, x \prec y$. There exist a, $b \in R$ such that $x = f(a)$, $y = f(b)$, $(a,b) \in kerf$. Hence $a \underset{kerf}{\leq} b$. Then f is a Q- homomorphism.

Sufficiency: Obviously Θ is an equivalence on R. Let $(a,b) \in \Theta$, $c \in R$. Then $f(a) = f(b)$. Hence Θ is a congruence on R. Let $a \underset{\Theta}{\leq} b \underset{\Theta}{\leq} a$. There exist $a_i, b_i \in R$ for $i \in \{1, 2, \cdots, n\}$ such that

$$a \leq a_1 \Theta a_2 \leq \Theta a_3 \Theta \cdots \Theta a_n \leq b \leq b_1 \Theta b_2 \leq b_3 \Theta \cdots \Theta b_n \leq a.$$

Then

$$\begin{aligned} f(a) \leq f(a_1) = f(a_2) \leq f(a_3) = \cdots = f(a_n) \leq f(b), \\ f(b) \leq f(b_1) = f(b_2) \leq f(b_3) = \cdots = f(b_n) \leq f(a). \end{aligned}$$

Hence $f(a) = f(b)$. Then $a\Theta b$. Therefor Θ is an order-congruence on R. $\square$

Next we investigate two isomorphism theorems of ordered semirings based on compatible quasiorders and order-congruences.

Theorem 3.7. *(First Isomorphism Theorem) Given an ordered semiring R, a subsemiring S of R and a compatible quasiorder σ on R. Define $[S] = \{r \in R : (\exists s \in S)(r, s) \in \theta\}$, where $\theta = \sigma \cap \sigma^{-1}$. Let S' be the subsemiring of R determined by $[S]$. Then the mapping*

$$\alpha \colon S/\ \theta_S \to S'/\theta_{[S]}, [x]_{\theta_S} \mapsto [x]_{\theta_{[S]}}\ ([x] \in [S])$$

is an isomorphism.

Proof. We consider the mapping $\alpha \colon S/\ \theta_S \to S'/\theta_{[S]}, [x]_{\theta_S} \mapsto [x]_{\theta_{[S]}}$ $([x] \in [S])$. Let $[x]_{\theta_S} = [y]_{\theta_S}$. Then $([x], [y]) \in \theta_S$. Because $[x], [y] \in [S]$, we obtain $([x], [y]) \in \theta_{[S]}$. Then $[x]_{\theta_{[S]}} = [y]_{\theta_{[S]}}$. Since we have

$$\begin{gathered} \alpha([x]_{\theta_S} + [y]_{\theta_S}) = \alpha([x+y]_{\theta_S}) = [x+y]_{\theta_{[S]}} = [x]_{\theta_{[S]}} + [y]_{\theta_{[S]}} = \alpha([x]_{\theta_S}) + \alpha([y]_{\theta_S}); \\ \alpha([x]_{\theta_S}[y]_{\theta_S}) = \alpha([xy]_{\theta_S}) = [xy]_{\theta_{[S]}} = [x]_{\theta_{[S]}}[y]_{\theta_{[S]}} = \alpha([x]_{\theta_S})\alpha([y]_{\theta_S}); \\ [x]_{\theta_S} \leq [y]_{\theta_S} \Rightarrow (x, y) \in \theta_S \Rightarrow ([x], [y]) \in \theta_{[S]} \Rightarrow [x]_{\theta_{[S]}} \prec [y]_{\theta_{[S]}}. \end{gathered}$$

Then f is a homomorphism.

Let $[x]_{\theta_{[S]}} = [y]_{\theta_{[S]}}$. Then $[x]_{\theta_{[S]}} \leq [y]_{\theta_{[S]}} \leq [x]_{\theta_{[S]}}$. Hence $(x, y), (y, x) \in \theta_S$. Therefor $[x]_{\theta_S} \leq [y]_{\theta_S} \leq [x]_{\theta_S}$. Then $[x]_{\theta_S} = [y]_{\theta_S}$. Let $[x]_{\theta_{[S]}} \in S'/\theta_{[S]}$. Then $([x], [x]) \in \theta \bigcap ([S] \times [S])$. Therefor $(x, x) \in \theta \bigcap (S \times S)$. Thus $[x]_{\theta_S} \in S/\theta_S$. Then α is an isomorphism. $\square$

Theorem 3.8. *(Second Isomorphism Theorem) Let R be an ordered semiring, σ_1, σ_2 be compatible quasiorders on R with $\sigma_1 \subseteq \sigma_2$, and let $\theta_1 = \sigma_1 \cap \sigma_1^{-1}$, $\theta_2 = \sigma_2 \cap \sigma_2^{-1}$. Then the relation $\overline{\sigma}_2$ on R/σ_1 defined by*

$$(a_{\theta_1}, b_{\theta_1}) \in \overline{\sigma}_2 \Leftrightarrow (a, b) \in \sigma_2$$

is a compatible quasiorder on R/σ_1 and the mapping $\alpha : (R/\sigma_1)/\overline{\sigma}_2 \to R/\sigma_2$, $(a_{\theta_1})_{\overline{\theta}_2} \mapsto a_{\theta_2}$ is an isomorphism, where $\overline{\theta}_2 = \overline{\sigma}_2 \cap \overline{\sigma}_2^{-1}$.

Proof. Obviously $\overline{\sigma}_2$ is a quasiorder on R. Let $(a_{\theta_1}, b_{\theta_1}) \in \overline{\sigma}_2$, $c \in R$. Then $(a,b) \in \sigma_2, c_{\theta_1} \in R/\sigma_1$. Because $\sigma_2 \in Cqu(R)$, we obtain $(c+a, c+b) \in \sigma_2$. Then $((c+a)_{\theta_1}, (c+b)_{\theta_1}) \in \overline{\sigma}_2$. Hence $(c_{\theta_1} + a_{\theta_1}, c_{\theta_1} + b_{\theta_1}) \in \overline{\sigma}_2$. Similarly, we have $(a_{\theta_1} + c_{\theta_1}, b_{\theta_1} + c_{\theta_1}), (c_{\theta_1} a_{\theta_1}, c_{\theta_1} b_{\theta_1}), (a_{\theta_1} c_{\theta_1}, b_{\theta_1} c_{\theta_1}) \in \overline{\sigma}_2$.

Let $a_{\theta_1} \prec_{\sigma_1} b_{\theta_1}$. Then $(a,b) \in \sigma_1$. Because $\sigma_1 \subseteq \sigma_2$, we obtain $(a,b) \in \sigma_2$. Hence $(a_{\theta_1}, b_{\theta_1}) \in \overline{\sigma}_2$. Thus $\overline{\sigma}_2$ is a compatible quasiorder on R.

We consider the mapping $\alpha : (R/\sigma_1)/\overline{\sigma}_2 \to R/\sigma_2$, $(a_{\theta_1})_{\overline{\theta}_2} \mapsto a_{\theta_2}$. Let $(a_{\theta_1})_{\overline{\theta}_2} = (b_{\theta_1})_{\overline{\theta}_2}$. Then $(a_{\theta_1}, b_{\theta_1}), (b_{\theta_1}, a_{\theta_1}) \in \overline{\sigma}_2$. Hence $(a,b), (b,a) \in \sigma_2$. Then $a_{\theta_2} = b_{\theta_2}$. Since we have

$$\alpha((a_{\theta_1})_{\overline{\theta}_2} + (b_{\theta_1})_{\overline{\theta}_2}) = \alpha((a_{\theta_1} + b_{\theta_1})_{\overline{\theta}_2}) = \alpha(((a+b)_{\theta_1 \overline{\theta}_2}) = (a+b)_{\theta_2} =$$
$$a_{\theta_2} + b_{\theta_2} = \alpha((a_{\theta_1})_{\overline{\theta}_2}) + \alpha((b_{\theta_1})_{\overline{\theta}_2});$$
$$\alpha((a_{\theta_1})_{\overline{\theta}_2} (b_{\theta_1})_{\overline{\theta}_2}) = \alpha((a_{\theta_1} b_{\theta_1})_{\overline{\theta}_2}) = \alpha(((ab)_{\theta_1 \overline{\theta}_2}) = (ab)_{\theta_2} = a_{\theta_2} b_{\theta_2} =$$
$$\alpha((a_{\theta_1})_{\overline{\theta}_2}) \alpha((b_{\theta_1})_{\overline{\theta}_2});$$
$$(a_{\theta_1})_{\overline{\theta}_2} \prec_{\overline{\sigma}_2} (b_{\theta_1})_{\overline{\theta}_2} \Rightarrow (a_{\theta_1}, a_{\theta_2}) \in \overline{\sigma}_2 \Rightarrow (a,b) \in \sigma_2 \Rightarrow a_{\theta_2} \prec_{\sigma_2} b_{\theta_2}.$$

Then f is a homomorphism. Let $a_{\theta_2} = b_{\theta_2}$. Then $(a,b), (b,a) \in \sigma_2$. Hence $(a_{\theta_1}, b_{\theta_1}), (b_{\theta_1}, a_{\theta_1}) \in \overline{\sigma}_2$. Then $(a_{\theta_1})_{\overline{\theta}_2} = (b_{\theta_1})_{\overline{\theta}_2}$. Let $a_{\theta_2} \in R/\sigma_2$. We have $(a_{\theta_1})_{\overline{\theta}_2} \in (R/\sigma_1)/\overline{\sigma}_2$. Then α is an isomorphism. □

Theorem 3.9. *Let R, R_1, R_2 be ordered semirings. Then $R \cong R_1 \times R_2$ if and only if there exist compatible quasiorders σ_1, σ_2 of R such that*

(1)$\sigma_1 \cap \sigma_2 = \leq$;

(2)$\overline{\sigma}_1 \circ \overline{\sigma}_2 = \overline{\sigma}_2 \circ \overline{\sigma}_1 = R \times R$.

Proof. Necessity: (1) Let $\varphi : R \to R_1 \times R_2$ be an isomorphism. Then $\varphi_i : R \to R_i (i = 1,2)$ are homomorphisms and onto. By lemma 3.1, $\overrightarrow{ker}\varphi_i (i = 1,2)$ are compatible quasiorders on R, then $\leq \subseteq (\overrightarrow{ker}\varphi_1) \cap (\overrightarrow{ker}\varphi_2)$. Furthermore, let $(x,y) \in (\overrightarrow{ker}\varphi_1) \cap (\overrightarrow{ker}\varphi_2)$. Then $\varphi(x) = (\varphi_1(x), \varphi_2(x)) \leq (\varphi_1(y), \varphi_2(y)) = \varphi(y)$. Since φ is an order-embedding, we deduce that $x \leq y$.

(2) Let $(c,d) \in R \times R$ and $k = \varphi_1(c), \varphi_2(d))$. Since φ is an isomorphism, there exists $w \in R$ such that $\varphi(w) = (\varphi_1(c), \varphi_2(d)) = (\varphi_1(w), \varphi_2(w))$. Then $(c,w) \in ker\varphi_1$ and $(w,d) \in ker\varphi_2$. Thus, $(c,d) \in ker\varphi_1 \circ ker\varphi_2$. We conclude that $ker\varphi_1 \circ ker\varphi_2 = R \times R$. Similarly, we can prove $ker\varphi_2 \circ ker\varphi_1 = R \times R$.

Sufficiency: Let σ_1 and σ_2 be compatible quasiorders on R. By lemma 3.2, there exists an order relation $\leq_i$ on $R/\overline{\sigma}_i$ such that $R_i = (R/\overline{\sigma}_i, \circ_i, \leq_i)$ is an ordered semiring, the natural homomorphism $\varphi_i : R \to R/\overline{\sigma}_i, a \mapsto a_{\overline{\sigma}_i}$ is isotone, and $\sigma_i = \{(a,b) \in R \times R : a_{\overline{\sigma}_i} \leq_i b_{\overline{\sigma}_i}$ for $i = 1,2$.

We define a mapping $\varphi : R \to R_1 \times R_2, x \mapsto (\overline{\sigma}_1(x), \overline{\sigma}_2(x))$. Then the semiring homomorphism φ is an ordered semiring isomorphism, i.e., $R \cong R_1 \times R_2$. Indeed, it is easily seen that φ is isotone. If $\varphi(x) \leq \varphi(y)$, then $(x,y) \in \sigma_1 \cap \sigma_2 = \leq$. Thus, φ is an order-embedding.

If $(\overline{x}_1, \overline{x}_2) \in R_1 \times R_2$, there exist $x_1, x_2 \in R$ such that $(x_1)_{\overline{\sigma}_1} = \overline{x}_1$ and $(x_2)_{\overline{\sigma}_2} = \overline{x}_2$. Since $\overline{\sigma}_1 \circ \overline{\sigma}_2 = \overline{\sigma}_2 \circ \overline{\sigma}_1 = R \times R$, there exists $y \in R$ such that $(x_1, y) \in \overline{\sigma}_1$ and $(y, x_2) \in \overline{\sigma}_2$. Thus $\varphi(y) = (\overline{x}_1, \overline{x}_2)$, i.e., φ is onto. $\square$

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