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Strong k - Edge Odd Graceful Labeling of Graphs

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Abstract

The edge odd graceful labeling of a graph G that was introduced by Solairaju and Chithra [4]is generalized and developed to a strong k - edge odd graceful labeling. Not all graphs are edge-odd graceful labeling, for example, the complete graph K_4. In this paper, we enlarged this definition and we found the least possible number k so that the graph becomes k- edge odd graceful labeling. we proved that: the tree in which all vertices have an odd degree is 3 - edge odd graceful and the cyclic graph C_{2n} has a strong 2- edge odd graceful labeling. Finally, we proved that the complete bipartite graph $K_{2,n}$ has 2 - edge odd graceful labeling when $n \equiv 2(\; mod \; 4)$ and a **strong** 2 - edge odd graceful labeling when $n \equiv 0 \; (mod \; 8)$ and $n \equiv 2 \; (mod \; 8)$.

Mathematics Subject Classification: 05C78

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1 Introduction

Graph labeling problem can be characterized as follows, a set of numbers from which edge (vertex) labels are chosen, a rule that assigns a value to each edge (vertex) and a condition that these value must satisfy. They were Solairaju and Chithra [4] who introduced the definition of Edge - odd graceful labeling of a graph $G(V(G), \; E(G))$ with p vertices and q edges, which is a bijection f from the set of edges $E(G)$ to the set $\{1, 3, ..., 2q-1\}$ such that, the induced map $f^* : \; V(G) \rightarrow \{0, 1, 2, ..., 2q-1\}$ given by $f^*(v) =$

$(\sum_{u\in V(G)} f(uv)\ )\ mod\ (2q)$, is an injective. A graph which admits edge odd graceful labeling is called an edge odd graceful graph. Following this first paper there has been great progress in the research work of edge odd graceful graph. Lots of results introduced by many others [1, 3]. It should be noted that a given graph may have several distinct edge odd graceful labeling , also not all graphs are edge odd graceful labeling.

1.1 Definitions

Definition 1.1. ***A** k **-edge odd graceful labeling** of a graph $G(V(G),\ E(G))$ with $p = |V(G)|$ vertices and $q = |E(G)|$ edges is an injective mapping f of the edge set $E(G)$ into the set $\{1, 3, 5, \cdots, 2q + 2k - 3\}$ such that the induced mapping f^* from $V(G)$ to $\{0, 1, 2, \cdots, 2q + 2k - 3\}$, given by:*
$f^*(v) = (\sum_{u\in V(G)} f(uv)\ )\ mod\ (2q + 2k - 2)$, is an injective.

Definition 1.2. *A k -edge odd graceful labeling of a graph G is called **A strong** k **-edge odd graceful labeling** if the induced mapping f^* maps from $V(G)$ to $\{k - 1, k, k + 1, \cdots, 2q + 2k - 3\}$.*

Definition 1.3. *If f is a K-edge odd graceful labeling of a graph $G(V(G),\ E(G))$ with $p = |V(G)|$ vertices and $q = |E(G)|$ edges then the labeling $f^{\oplus}$ define by $f^{\oplus}(uv) = (2q + 2k - 2) - f(uv)$ for all $(uv) \in E(G)$ is again a K-edge odd graceful labeling of G and called the **complementary** labeling to f .*

It is clear that the standard definition of edge odd graceful labeling corresponds to a 1- edge odd graceful labeling .

Example 1.4. *In Figure 1 , a strong* 6 *- edge odd graceful labeling and its complementary of a* wheel graph W_5 *are shown.*

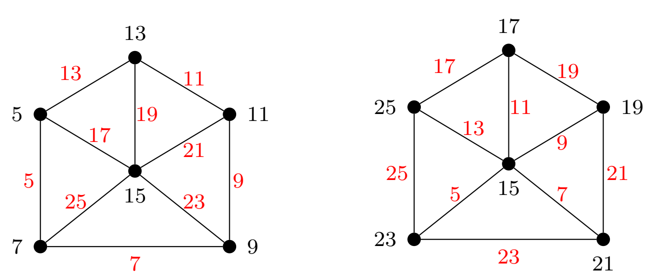

Figure 1: A strong 6 - edge odd graceful labeling of W_5 and its complementary.

Lemma 1.5. *The complete graph K_4 is* 2 *- edge odd graceful labeling which is always a strong .*

Proof. To show that K_4 is 2 - edge odd graceful labeling we try to find a mapping $f : E(G) \longrightarrow \{1, 3, \cdots, , 2q+2k-3\}$. Here $q = 6$ *and* $k = 2$ i.e., $f : E(G) \longrightarrow \{1, 3, \cdots, , 13\}$ and $f^* : V(G) \rightarrow \{0, 1, 2, \cdots, 13\}$ where $f^*(v) = (\sum_{u \in V(G)} f(uv)) \ mod\ (14)$ is an injective. Because K_4 is a cubic graph , $f^*(v)$ is a summation of three odd numbers which is odd, then the labeling of all vertices is odd and $f^* : V(G) \rightarrow \{1, 3, \cdots, 13\}$. Then the map $f^* : V(G) \rightarrow \{k-1, k, k+1, \cdots, 2q+2k-3\}$ is always a strong 2 - edge odd graceful labeling . This map is shown in Figure 2 and its complementary, which is also a strong 2 - edge odd graceful labeling ☐

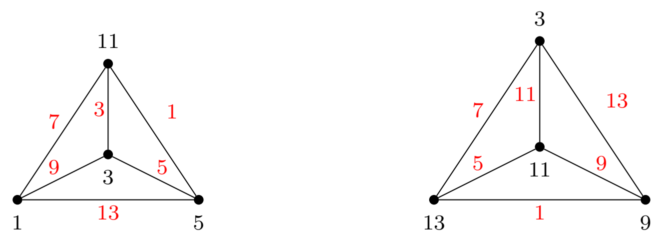

Figure 2: a strong 2 - edge odd graceful for K_4 and its complement

2 Strong k edge odd graceful labeling of odd tree

Seoud and Salim [2] proved that the tree in which all vertices have odd degrees is not edge -odd graceful. So we want to ask what is the least number k such that these trees are k - edge odd graceful labeling.

The tree in Figure 3 (a) is neither edge odd graceful labeling nor 2 - edge odd graceful, but it is 3 - edge odd graceful labeling. This can be shown as follows: consider the following labeling of the tree $f : E(G) \longrightarrow \{1, 3, \cdots, , 2q+2k-3\}$ (Figure 3 (b)), one can think that it is 2-edge odd graceful labeling since the greatest number in the labeling is 11 $= 2q+2k-3 = 2k+7$ and the first number k which satisfy this is $k = 2$. This is not 2-edge odd graceful labeling because the vertex x in this case have $f^*(x) = (\sum_{u \in V(G)} f(ux)) \ mod\ (12) = 7$ which is the labeling of another vertex. But if we think that it is 3 -edge odd graceful labeling , then the greatest number in labeling is 13 $= 2q+2k-3$. In this case $f^*(x) = (\sum_{u \in V(G)} f(ux)) \ mod\ (14) = 5$ and $f^*(y) = 13$. So f^* is injective and hence it is 3 - edge odd graceful labeling. It should be noted that this labeling is strong 3- edge odd graceful labeling since f^* maps from $V(G)$ to $\{k-1, k, k+1, \cdots, 2q+2k-3\} = \{2, 3, \cdots, 13\}$.

Example 2.1. *The tree in Figure 3 has a 3 - edge odd graceful labeling as shown in Figure 4 , since* $q = 5$ *and* $k = 3$ *, so* $f : E(G) \longrightarrow \{1, 3, \cdots, , 13\}$

Figure 3: (a) a tree which is not edge odd graceful (b) a **strong** 3- edge odd graceful of the tree

and $f^* : V(G) \longrightarrow \{0, 1, 2, \cdots, 13\}$ *where* $f^*(v) = (\sum_{u \in V(G)} f(uv)) \bmod (14)$ *is an injective. That labeling is not strong since there is a vertex have label* 1 *which does not belongs to* $\{k-1, k, k+1, \cdots, 2q+2k-3\} = \{2, 3, \cdots, 13\}$.

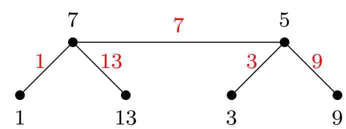

Figure 4: a 3- edge odd graceful of the tree

Theorem 2.2. *The graph* G *obtained from a path* P_n *by attaching exactly one pendant edge to each internal vertex of the path* P_n *denoted by* $T_{n,\ n-2}$ *is* 3 *- edge odd graceful labeling.*

Proof. Let P_n be the path with vertices $\{v_1, v_2, v_3, \cdots, v_n\}$, then $V(G) = \{v_i,\ u_j\ ; 1 \le i \le n,\ \ 2 \le j \le n-1\}$ and $E(G) = \{v_i v_{i+1},\ v_j u_j\ ; 1 \le i \le n-1,\ \ 2 \le j \le n-1\}$. Then $p = 2n-2$ and $q = 2n-3$ and G is a tree of odd degree, see Figure 5.

Since $q = 2n-3$, *and* $k = 3$ then $2q + 2k - 3 = 4n - 3$, so let

i.e., $f : E(G) \to \{1, 3, 5, \cdots, 2n-7, \underline{2n-5}, 2n-3, 2n-1, \underline{2n+1}, 2n+3, \cdots, 4n-3\}$

The labeling map f can be defined as follows:

$$f(v_j u_j) = \begin{cases} 2j-3 & \text{if} \quad j = 2, 3, \cdots, n-2; \\ 2j-1 & \text{if} \quad j = n-1. \end{cases}$$

i.e., $f(v_j u_j) = \{1, 3, 5, \cdots, 2n-9, 2n-7, 2n-3\}$,

$$f(v_i v_{i+1}) = \begin{cases} 4n-2i-1 & \text{if } i = 1, 2, 3, \cdots, n-2; \\ 2n-1 & \text{if } i = n-1. \end{cases}$$

i.e., $f(v_i v_{i+1}) = \{4n-3, 4n-5, \cdots, 2n+5, 2n+3, 2n-1\}$.

Then the induced vertex labels are

Figure 5: $T_{n,\ n-2}$ with ordinary labeling

$$f^*(u_j) \ = \begin{cases} 2j-3 & \text{if } j = 2, 3, \cdots, n-2; \\ 2n-3 & \text{if } j = \ n-1. \end{cases}$$

$$f^*(v_i) \ = \begin{cases} f(v_i v_{i+1}) & \text{if } i = 1, 2, 3, \cdots, n-2 \ ; \\ 2n+1 & \text{if } i = n-1 \ ; \\ 2n-1 & \text{if } i = \ n. \end{cases}$$

Note that each of the pendant vertices takes the label of its edge, so they are odd and different numbers. The label assigned to the vertex v_{n-1} is given by

$$f^*(v_{n-1}) = [(2n-3\,)+(2n+3)+(2n-1)] \, mod \, (2q+2k-2) = (6n-1) mod \, (4n-2) = 2n+1$$

Therefore the edge labels are odd and vertex labels are distinct. Then f is 3 - edge odd graceful labeling. □

Example 2.3. *A* 3 *- edge odd graceful labeling of* $T_{9,7}$ *according to Theorem 2.2 is given in Figure 6 .*

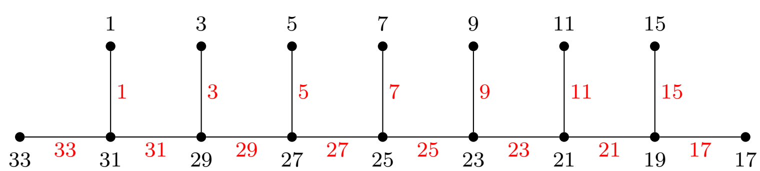

Figure 6: a 3 - edge odd graceful for $T_{9,7}$

3 Strong k edge odd graceful labeling of the cyclic graph C_{2n}

At first, to show that C_4 has a 2 - edge odd graceful labeling , we try to find a mapping $f \ : \ E(C_4) \longrightarrow \{1, 3, \ \cdots \ , , 9\,\}$ and $f^* \ : \ V(C_4) \ \rightarrow \ \{0, 1, 2, \ \cdots \ , 9\,\}$ where $f^*(v) = (\sum_{u \in V(G)} f(uv) \ ) \ mod \ (10)$, is an injective. This map is shown in Figure 7 (a). It should be noted that its complementary is also 2 - edge odd graceful labeling. Also there is a strong 2 - edge odd graceful labeling for C_4 shown in Figure 7 (b) , again its complementary is also strong 2 - edge odd graceful labeling.

Figure 7: (a) a 2- edge odd graceful for C_4 (b) **a strong** 2- edge odd graceful for C_4

Once more the cyclic graph C_6 has 2 edge - odd graceful labeling, since the mapping $f : E(C_6) \longrightarrow \{1, 3, \cdots, , 13\}$ is shown in Figure 8. It should be noted that its complementary is also 2 - edge odd graceful labeling.

Figure 8: a 2- edge odd graceful for C_6 but not strong and its complement

Theorem 3.1. *The cyclic graph C_{2n} has* ***a strong*** *2 - edge odd graceful labeling.*

Proof. In the cyclic graph C_{2n} , $q = |E(C_{2n})| = 2n$ and we assume that $k = 2$. We try to find a mapping $f : E(G) \longrightarrow \{1, 3, \cdots, , 2q + 2k - 3\} = \{1, 3, \cdots, 4n + 1\}$ and $f^* : V(G) \rightarrow \{1, 2, 3, \cdots, 4n + 1\}$ where $f^*(v) = (\sum_{u \in V(G)} f(uv)) \; mod \; (4n + 2)$, is an injective. There are two cases:

Case (1) : When n is odd. Let the cyclic graph C_{2n} be given as indicated in Figure 9. The edges $e_1, e_2, \cdots, e_n, e_{n+1}, e_{n+2}, \cdots, e_{2n-1}, e_{2n}$ will be labeled as $e_1, e_2, \cdots, e_n, a_1, a_2, \cdots, a_{n-1}, a_n$ where $a_i = e_{n+i}$, $i = 1, \cdots, n$ where a_i is put opposite to e_i

Define the labeling function $f : E(G) \longrightarrow \{1, 3, \cdots, 4n+1\}$ as follows:

$$f(e_i) = 2i - 1; \quad i = 1, 2, \cdots n$$

$$f(a_i) = 4n - 2i + 3; \quad i = 1, 2, \cdots n$$

we see that : $f^*(v_2) = f(e_2) + f(e_1) = 4$

$$f^*(v_n) = f(e_n) + f(a_{n-1}) = (4n + 4) \; mod (4n + 2) = 2$$

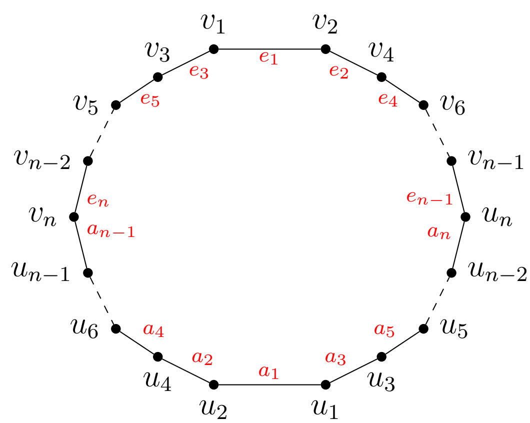

Figure 9: C_{2n} with ordinary labeling when n is odd

$$f^*(v_i) = \begin{cases} (f(e_i) + f(e_{i-2}))\ mod(4n+2) & \text{if} \quad i = 4, 6, \cdots, n-1 \quad ; \\ (f(e_i) + f(e_{i+2}))\ mod(4n+2) & \text{if} \quad i = 1, 3, \cdots, n-2 \quad . \end{cases}$$

$$f^*(u_2) = f(a_2) + f(a_1) = 4n - 2$$

$$f^*(u_n) = f(e_{n-1}) + f(a_n) = 4n$$

$$f^*(u_i) = \begin{cases} (f(a_i) + f(a_{i-2}))\ mod(4n+2) & \text{if} \quad i = 4, 6, \cdots, n-1\ ; \\ (f(a_i) + f(a_{i+2}))\ mod(4n+2) & \text{if} \quad i = 1, 3, \cdots, n-2 \quad . \end{cases}$$

Since the edge labels are odd and distinct, therefore the vertex labels are even and distinct. So the graph has strong 2 - edge odd graceful.

Case (2) : When n is even. Let C_{2n} be given as indicated in Figure 10 .

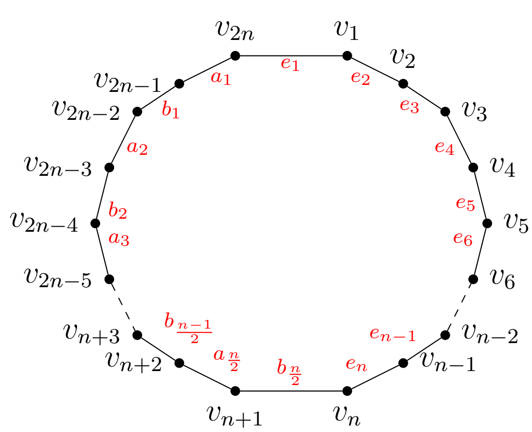

Figure 10: C_{2n} with ordinary labeling when n is even

Define the labeling function $f:\ E(G) \longrightarrow \{1,3,\ \cdots\ ,4n{+}1\}$ as follows:

$$f(e_i) = 2i-1; \quad i = 1,2,\cdots n$$

$$f(a_i) = (4n+2) - f(e_{2i}) = \ 4n - 4i + 3; \quad i = 1,2,\cdots,\frac{n}{2}$$

$$f(b_i) = (4n+2) - f(e_{2i-1}) = \ 4n - 4i + 5; \quad i = 1,2,\cdots,\frac{n}{2}$$

we see that : $\quad f^*(v_i) = \ 4i \ ; \quad i = 1,2,\cdots,n-1$

$$f^*(v_n) = \ (f(e_n) \ + \ f(b_{\frac{n}{2}}))\ mod(4n+2) = \ (\ 4n+4)\ mod(4n+2) \ = 2$$

$$f^*(v_{2n}) = \ (f(e_1) \ + \ f(a_1))\ mod(4n+2) = \ 4n$$

$$f^*(v_{2n-i}) \ = \begin{cases} (f(a_{\frac{i+1}{2}}) + f(b_{\frac{i+1}{2}}))\ mod(4n+2) & \text{if} \quad i = 1,3,5,\cdots,n-1 \ ; \\ (f(a_{\frac{i}{2}+1}) + f(b_{\frac{i}{2}}))\ mod(4n+2) & \text{if} \quad i = 2,4,6,\cdots,n-2 \ . \end{cases}$$

Since the edge labels are odd and distinct, therefore the vertex labels are even and distinct. So the graph has strong 2 - edge odd graceful.

□

Example 3.2. *The graphs $C_{18} = C_{2.9}$ and $C_{16} = C_{2.8}$ labeled according to Theorem 3.1 are presented in Figure 11.*

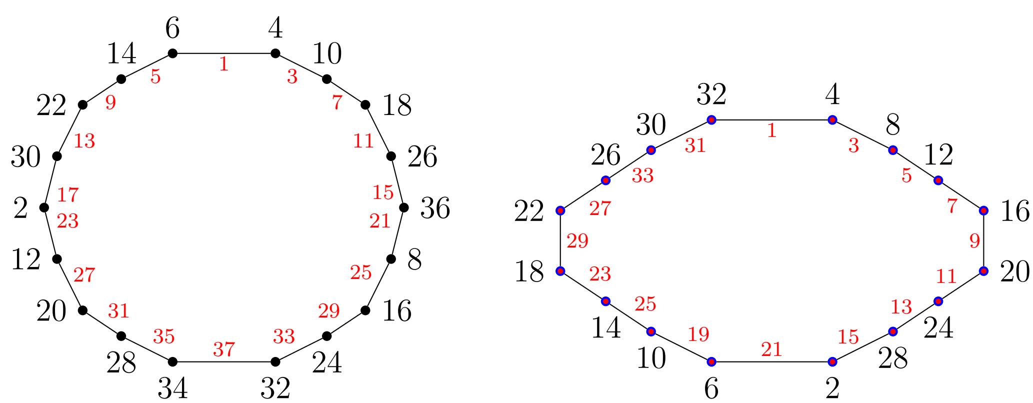

Figure 11: a strong 2 - edge odd graceful for $C_{18} = C_{2.9}$ and $C_{16} = C_{2.8}$

4 Strong k edge odd graceful labeling of $K_{2,n}$

Seoud and Salim [2] proved that the complete bipartite graph $K_{2,n}$ is an edge -odd graceful graph when n is odd. So we want to study the case when n is even and what is the least number k such that these graphs $K_{2,n}$ are k - edge odd graceful labeling. At first the graph $K_{2,2}$ is isomorphic to C_4 which we get a 2 labeling of it.

Theorem 4.1. *The complete bipartite graph* $K_{2,n}$ *has a* 2 *- edge odd graceful labeling when* $n \equiv 2(\ mod\ 4)$.

Proof. Let the complete bipartite graph $K_{2,n}$ be given as indicated in Figure 12. Since $q = 2n$ and we assume that $k = 2$,

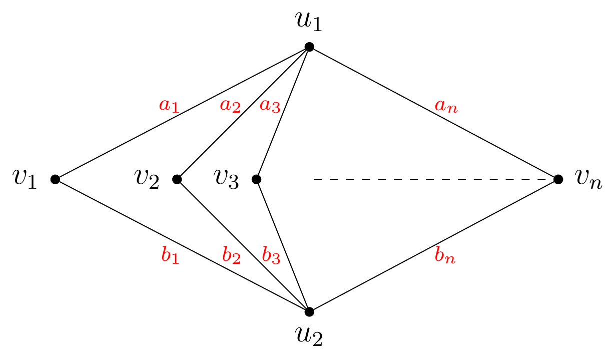

Figure 12: $K_{2,n}$ with ordinary labeling

Define the labeling function $f : E(K_{2,n}) \longrightarrow \{1, 3, \cdots, 4n + 1\}$ as follows:

$$f(a_i) = 4i - 3; \quad i = 1, 2, \cdots n$$

$$f(b_i) = 4i - 1; \quad i = 1, 2, \cdots n$$

we see that :

$$f^*(v_i) = (f(a_i) + f(b_i))\ mod(4n+2) = (\ 8i-4\ )\ mod(4n+2)\ ; \quad i = 1, 2, \cdots n$$

$$f^*(u_1) = (\sum_{i=1}^{n}(4i-3))\ mod(4n+2) = (2n^2 - n)\ mod(4n+2) \equiv 2n+2$$

$$f^*(u_2) = (\sum_{i=1}^{n}(4i-1))\ mod(4n+2) \equiv (2n^2 + n)\ mod(4n+2) \equiv 0$$

This is a 2- edge odd graceful labeling if : $f^*(v_i) \neq f^*(u_1) \neq f^*(u_2)$,i.e.,

$$(\ 8i - 4\ )\ mod(4n+2) \neq 2n+2 \neq 0\ ; \quad i = 1, 2, \cdots n$$

This relations satisfies when $n = 2,\ 6,\ 10,\ 14,\ 18,\ 22,\ \cdots$
i.e., when $n = 4r+2$ for all nonnegative integers r , i.e., $n \equiv 2(\ mod\ 4)$.
□

Example 4.2. *The graph* $K_{2,10}$ *labeled according to Theorem 4.1 is presented in Figure 13.*

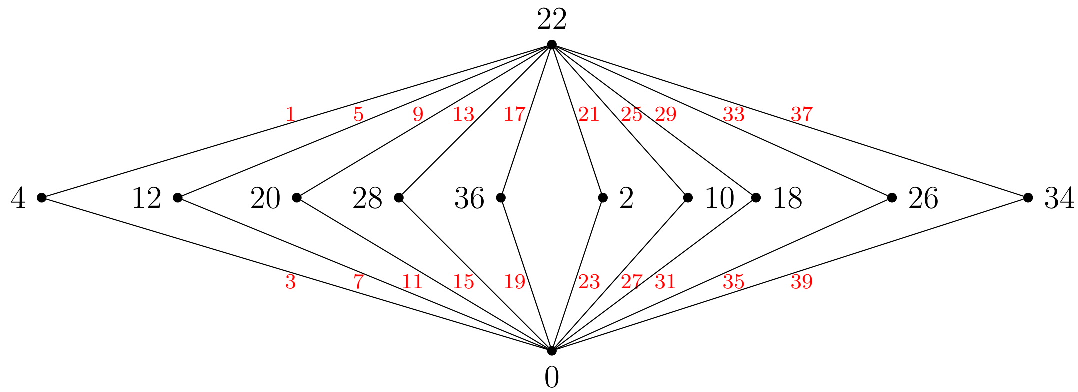

Figure 13: a 2 - edge odd graceful labeling for $K_{2,10}$

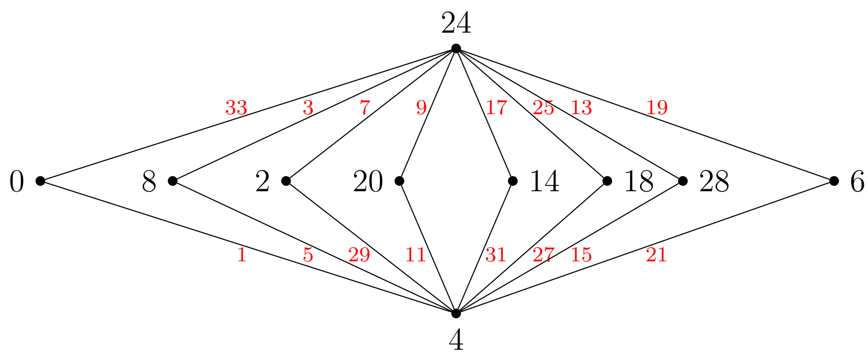

Figure 14: a 2 - edge odd graceful for $K_{2,8}$

Although the technique used in numbering in the previous theory is not achieved in the case when $n \equiv 0(\ mod\ 4)$ i.e., $n = 4r$. We can find a 2 - edge odd graceful for them by another ways. A 2 - edge odd graceful of the graph $K_{2,8}$ is presented in Figure 14.

Lemma 4.3. *The complete bipartite graph $K_{2,4}$ is a 2 - edge odd graceful and* ***a strong*** *2 - edge odd graceful labeling.*

Proof. In Figure 15 (a) we introduce the map $f : E(G) \longrightarrow \{1, 3, \cdots, 17\}$ which is a 2 - edge odd graceful . Also in Figure 15 (b) we introduce the map $f : E(G) \longrightarrow \{1, 3, \cdots, 17\}$ which is a strong 2 - edge odd graceful. □

Theorem 4.4. *The complete bipartite graph $K_{2,n}$ has* ***a strong*** *2 - edge odd graceful labeling when $n \equiv 0\ (mod\ 8)$ and $n \equiv 2\ (mod\ 8)$.*

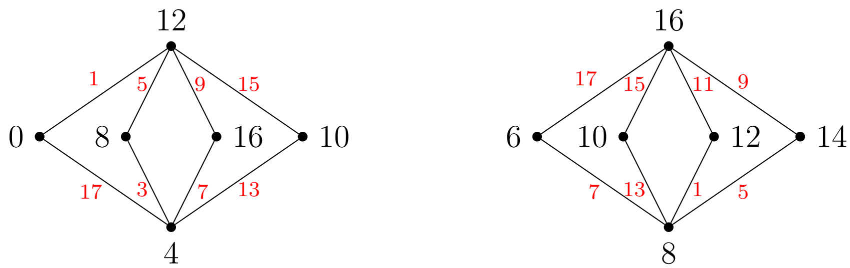

Figure 15: (a) a 2 - edge odd graceful for $K_{2,4}$ (b) a strong 2- edge odd graceful for $K_{2,4}$

Proof. Let the complete bipartite graph $K_{2,n}$ be given as indicated in Figure 12. We try to find a mapping $f : E(G) \longrightarrow \{1, 3, \cdots, 4n+1\}$ where $f^*(v) = (\sum_{u \in V(G)} f(uv))\ mod\ (4n+2)$, is an injective.

Define the edge labeling $f : E(G) \longrightarrow \{1, 3, \cdots, 4n+1\}$ as follows:

$$f(a_i) = 2i - 1; \quad i = 1, 2, \cdots, n$$

$$f(b_i) = 2n + 2i + 1; \quad i = 1, 2, \cdots, n$$

we see that :

$$f^*(v_i) = (f(a_i) + f(b_i))\ mod(4n+2) = (2n+4i)\ mod(4n+2)\ ; \quad i = 1, 2, \cdots, n$$

$$f^*(u_1) = (\sum_{i=1}^{n}(2i-1))\ mod(4n+2) = n^2\ mod(4n+2)$$

$$f^*(u_2) = (\sum_{i=1}^{n}(2n+2i+1))\ mod(4n+2) \equiv (3n^2+2n)\ mod(4n+2)$$

This is a 2- edge odd graceful labeling if : $f^*(v_i) \neq f^*(u_1) \neq f^*(u_2)$,i.e.,

$$(2n+4i)\ mod(4n+2) \neq n^2\ mod(4n+2) \neq (3n^2+2n)\ mod(4n+2)\ ; \quad i = 1, 2, \cdots, n$$

This relations satisfies when $n = 2, 8, 10, 16, 18, 24, 26, \cdots$
i.e., when when $n \equiv 0\ (mod\ 8)$ and $n \equiv 2\ (mod\ 8)$.

□

Example 4.5. *The graph $K_{2,8}$ labeled according to Theorem 4.4 is presented in Figure 16.*

Although the technique used in numbering in the previous theory is not achieved in the case $n = 8r - 2$ and $n = 8r - 4$ for all positive integers r . We can find **a strong** 2 - edge odd graceful for them by another ways. The case $K_{2,4}$ is shown in Lemma 4.3. Also The complete bipartite graph graph $K_{2,6}$ is **a strong** 2 - edge odd graceful labeling. In Figure 17 two different strong 2 - edge odd graceful labeling for $K_{2,6}$ are given.

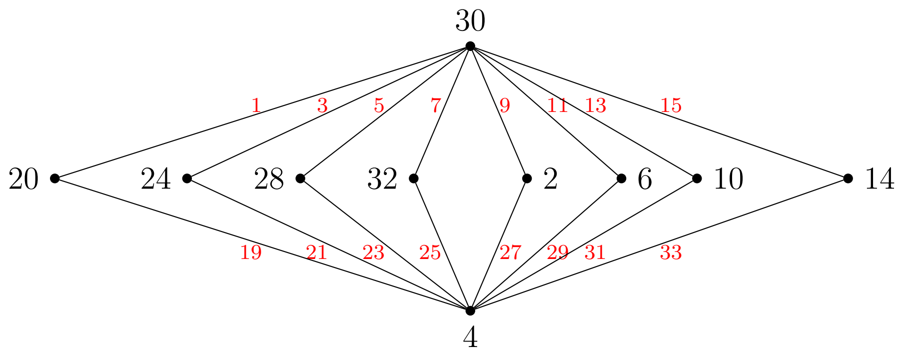

Figure 16: a strong 2 - edge odd graceful for $K_{2,8}$

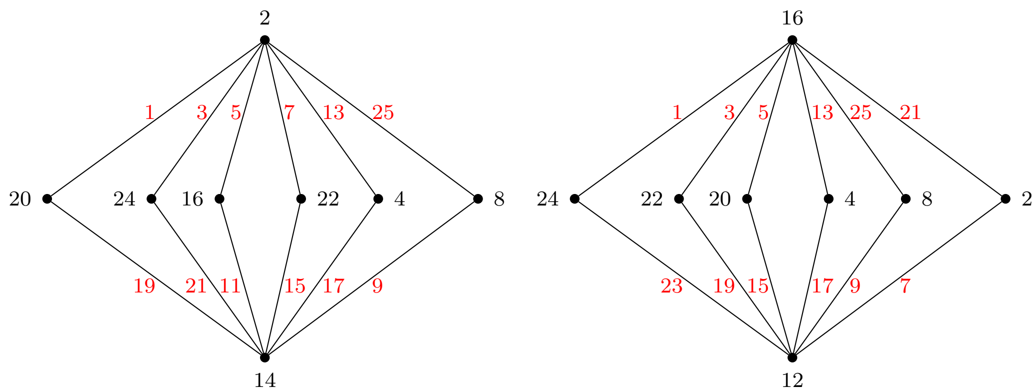

Figure 17: Two different strong 2 - edge odd graceful for $K_{2,6}$

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